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A Specialisation of the Bump–Friedberg L-function

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Abstract. We study the restriction of Bump–Friedberg integrals to affine lines $\{(s+\alpha, 2s), s \in \mathbb{C}\}$. It has simple theory, very close to that of the Asai L-function. It is an integral representation of the product $L(s+\alpha,\pi)L(2s,\Lambda^2,\pi)$, which we denote by $L^{\mathrm{lin}}(s,\pi,\alpha)$ for this abstract, when π is a cuspidal automorphic representation of $GL(k,\mathbb{A})$ for $\mathbb{A}$ the adeles of a number field. When k is even, we show that the partial L-function $L^{\mathrm{lin},S}(s,\pi,\alpha)$ has a pole at $1/2$ if and only if π admits a (twisted) global period. This gives a more direct proof of a theorem of Jacquet and Friedberg, asserting that π has a twisted global period if and only if $L(\alpha+1/2,\pi)\neq 0$ and $L(1,\Lambda^2,\pi)=\infty$. When k is odd, the partial L-function is holmorphic in a neighbourhood of $\mathrm{Re}(s)\geq 1/2$ when $\mathrm{Re}(\alpha)$ is ≥ 0.

1 Introduction

In this paper, we study the restriction of the integrals of two complex variables (s_1, s_2) defined in [5], and attached to global and local smooth complex representations of $GL(2n)$, to the line $s_2 = 2(s_1-\alpha)$, for $\alpha\in\mathbb{C}$. We actually study slightly more general integrals. It turns out that these integrals have a theory very close to that of Asai L-functions, whose Rankin–Selberg theory, initiated by Flicker, is quite complete now (see [1, 2, 8, 9, 17, 19–21]).

In [5], for π a cuspidal automorphic representation of $GL(n)$ of the adeles $\mathbb{A}$ of a global field, the authors mainly define the global integrals as the integral of a cusp form in π against an Eisenstein series, prove their functional equation, and show that they unravel to the integral of the Whittaker function associated to the cusp form against a function in the space of an induced representation. This allows them to obtain an Euler factorisation; they then compute the local integrals at the unramified places and thus obtain an integral representation of $L(s_1,\pi)L(s_2,\Lambda^2,\pi)$. The location of the possible poles is briefly discussed.

In the first paragraphs of Section 3, we define the L function $L^{\mathrm{lin}}(s,\pi)$ for a generic representation π of $GL(n,F)$ for n even equal to $2m$ (Theorem 3.1), when F is a non-archimedean local field, and show a nonvanishing result. A much more complete study of this non archimedean L-function can be found in [22].

We compute the Rankin–Selberg integrals when π is unramified in Section 3.2.

In the archimedean case (Section 3.3), we prove results of convergence and non-vanishing of the archimedean integrals that we use in the global situation.

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Section 4 is devoted to the global theory. We take π a smooth cuspidal automorphic representation of $GL(2m, \mathbb{A})$, for $\mathbb{A}$ the adele ring of a number field k. We first study the integrals $I(s, \phi, \Phi)$ associated with a cusp form ϕ in the space of π, a Schwartz function Φ on $\mathbb{A}^m$, and a (Hecke) character χ of $GL(m, \mathbb{A}) \times GL(m, \mathbb{A})$ trivial on the diagonal embedding of $Gl(m, \mathbb{A})$, using mirabolic Eisenstein series similar to those in [14], [15], or [8]; this seems to avoid the normalisation by the L-function of the central character of π as in [5]. We thus obtain their meromorphicity, functional equation as well as the location of their possible poles in Theorem 4.2. Then we prove the equality of these integrals (Theorem 4.4) with the Rankin–Selberg integrals $\Psi(s, W_\phi, \chi, \Phi)$ obtained by integrating the Whittaker functions associated with ϕ, and thus get the Euler factorisation in Section 4.2. The proof of this is similar to that of [5], but we use successive partial Fourier expansions (Proposition 4.3), which makes the computations quicker.

In the last part, we define the partial L-function $L^{\mathrm{lin},S}(s, \pi, \chi)$, and show that it is meromorphic. Moreover, when the real part of the idele class character defining χ is non-negative, that it is holomorphic for $\mathrm{Re}(s) > 1/2$, and that it has a pole at $1/2$ if and only if π has a twisted global period (Theorem 4.5). We deduce from this the theorem of Friedberg and Jacquet discussed in the abstract (Theorem 4.7). It seems that the proof of the aforementioned theorem is not in [5] because the local L functions were not really studied in [loc. cit.]. In particular, the nonvanishing results, which are easy (especially in the nonarchimedean case), are absent in [5]. Studying the Bump–Friedberg L-function through its restriction to complex lines of slope 2 (in particular considering it as a function of one complex variable) simplifies the analysis.

In Section 5, we give the results for the odd case. The global Rankin-Selberg integrals are holomorphic this time, and we prove that the partial L-function is holomorphic in a neighbourhood of $\mathrm{Re}(s) \geq 1/2$ with the same assumption on the idele character defining χ.

2 Preliminaries

Let n belong to $\mathbb{N}$. We will use the notations G_n for the algebraic group $GL(n)$, Z_n for its center, P_n for its mirabolic subgroup (the matrices in G_n with last row $(0, \dots, 0, 1)$), B_n for the Borel subgroup of upper triangular matrices in G_n, N_n for its unipotent radical. We will write U_n for the unipotent radical of P_n, and $u(x)$ will be the matrix $\left(\begin{smallmatrix} I_{n-1} & x \\ 0 & 1 \end{smallmatrix}\right)$ in U_n. Let $\mathcal{M}_k$ denote the set of $k \times k$ square matrices and $\mathcal{M}_{a,b}$ the $a \times b$ matrices. For $n > 1$, the map $g \mapsto \left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is an embedding of the group G_{n-1} in G_n. In particular, one has $P_n = G_{n-1} U_n$.

Suppose $n = 2m$ is even. Let $w_n \in G_n$ be the permutation matrix for the permutation given by

$$\left(\begin{array}{cccc|cccc} 1 & 2 & \cdots & m & m+1 & m+2 & \cdots & 2m \\ 1 & 3 & \cdots & 2m-1 & 2 & 4 & \cdots & 2m \end{array}\right).$$

In this case, we denote by M_n the standard Levi of G_n associated with the partition (m, m) of n. Let $H_n = w_n M_n w_n^{-1}$; we write $h(g_1, g_2) = w_n \,\mathrm{diag}(g_1, g_2) w_n^{-1}$ for $\mathrm{diag}(g_1, g_2)$ in M_n.

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Suppose $n = 2m+1$ is odd. In this case, we let w_n be the permutation matrix in G_n associated with the permutation

$$\left(\begin{array}{cccc|ccccc} 1 & 2 & \cdots & m & m+1 & m+2 & \cdots & 2m & 2m+1 \\ 1 & 3 & \cdots & 2m-1 & 2 & 4 & \cdots & 2m & 2m+1 \end{array}\right).$$

so that $w_{2m} = w_{2m+1}|_{G_{2m}}$, and we let $w_{2m+1} = w_{2m+2}|_{GL_{2m+1}}$ so that w_{2m+1} is the permutation matrix corresponding to

$$\left(\begin{array}{cccc|cccc} 1 & 2 & \cdots & m+1 & m+3 & m+4 & \cdots & 2m+1 \\ 1 & 3 & \cdots & 2m+1 & 2 & 4 & \cdots & 2m-2 \end{array}\right).$$

We let M_n denote the standard parabolic associated with the partition $(m+1, m)$ of n and set $H_n = w_n M_n w_n^{-1}$ as in the even case. Again, we write

$$h(g_1, g_2) = w_n \operatorname{diag}(g_1, g_2) w_n^{-1}$$

for $\operatorname{diag}(g_1, g_2)$ in M_n. Note that the H_n are compatible in the sense that $H_n \cap G_{n-1} = H_{n-1}$.

For $C \subset G_n$, we write C^σ for $C \cap H_n$. We will also need the matrix w_n', which is the matrix of the permutation

$$\left(\begin{array}{cccc|c|cccc} 1 & 2 & \cdots & m & m+1 & m+2 & m+3 & \cdots & 2m \\ 1 & 3 & \cdots & 2m-1 & 2m & 2 & 4 & \cdots & 2m-2 \end{array}\right)$$

when $n = 2m$ is even, and of

$$\left(\begin{array}{cccc|c|cccc} 1 & 2 & \cdots & m & m+1 & m+2 & m+3 & \cdots & 2m+1 \\ 2 & 4 & \cdots & 2m & 2m+1 & 1 & 3 & \cdots & 2m-1 \end{array}\right)$$

when $n = 2m+1$ is odd.

In the sequel, F will generally be a local field, whereas $\mathbb{A}$ will be the ring of adeles of a number field k. When G is the points of an algebraic group defined over $\mathbb{Z}$ on F or $\mathbb{A}$, we denote by $Sm(G)$ the category of smooth complex G-modules. Every representation we will consider from now on will be smooth and complex.

We will denote by δ_H the positive character of $N_G(H)$ such that if μ is a right Haar measure on H, and int is the action given by $(\operatorname{int}(n)f)(h) = f(n^{-1}hn)$, of $N_G(H)$ smooth functions f with compact support on H, then $\mu \circ \operatorname{int}(n) = \delta_H^{-1}(n)\mu$ for n in $N_G(H)$.

If $G = G_n(\mathbb{A})$, $H = H_n(\mathbb{A})$, π is a cuspidal representation of G with trivial central character, and χ is a smooth character of H trivial on $H_n(k)Z_n(A)$, then we say that π has an (H, χ)-period if there is a cusp form in the space of π such that the integral (which is convergent by [3, Proposition 1])

$$\int_{Z_n(\mathbb{A})H_n(k)\backslash H_n(\mathbb{A})} \phi(h)\chi^{-1}(h)dh$$

is nonzero.

For $m \in \mathbb{N} - \{0\}$, we will denote by $\mathcal{S}(F^m)$ the Schwartz space of functions (smooth and rapidly decreasing) on F^m when F is archimedean, and by $\mathcal{C}_c^\infty(F^m)$ the Schwartz

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space of smooth functions with compact support on F^m when F is non-archimedean. We denote by $\mathcal{S}(\mathbb{A}^m)$ the space of Schwartz functions on $\mathbb{A}^m$, which is by definition the space of linear combinations of decomposable functions $\Phi = \prod_v \Phi_v$, with Φ_v in $\mathcal{S}(k_v^m)$ when v is an archimedean place, and in $\mathcal{C}_c^\infty(k_v^m)$ when v is non-archimedean, with the extra condition that $\Phi_v = \mathbf{1}_{\mathfrak{O}_v{}^m}$ for almost every non-archimedean place v. On these spaces, there is a natural action of either $G_m(F)$, or $G_m(\mathbb{A})$. In every case, if θ is a nontrivial character of F or $\mathbb{A}/k$, we will denote by $\widehat{\Phi}^\theta$ or $\widehat{\Phi}$ the Fourier transform of a Schwartz function Φ, with respect to a θ-self-dual Haar measure.

If χ is a character of the local field F of characteristic zero, with normalised absolute value $|\cdot|_F$, we denote by $\mathrm{Re}(\chi)$ the real number r such that for all x in F^*, one has $\sqrt{\chi(x)\overline{\chi(x)}} = |x|_F^r$. If χ is a character of $\mathbb{A}^*/k^*$, for k a number field and $\mathbb{A}$ its adele ring, we denote by $\mathrm{Re}(\chi)$ the real number such that for all x in $\mathbb{A}^*$, one has $\sqrt{\chi(x)\overline{\chi(x)}} = |x|^r$ for $|\cdot|$ the norm of $\mathbb{A}^*$.

In the sequel, the equalities of two integrals, involving integration over quotients or subgroups, are valid up to correct normalisation of Haar measures.

3 The Local Theory

We start with the non-archimedean case, which was studied in great detail in [22]. Here we just give the definition of the L-function, a non vanishing property needed for the global case, as well as the unramified computation.

3.1 The Local Non-Archimedean L-function

We let $\mathfrak{O}$ denote the ring of integers of F. Let θ be a nonzero character of F. Let π be a generic representation of G_n, let W belong to the Whittaker model $W(\pi,\theta)$, and let Φ be a function in $C_c^\infty(F^m)$. We denote by χ a character of H_n of the form $h(h_1,h_2) \mapsto \alpha(\det(h_1)/\det(h_2))$, for α a character of F^*, and by δ the character $h(h_1,h_2) \mapsto |h_1|/|h_2|$ of $H_n(F)$.

Denoting by L_m the m-th row of a matrix, we formally define the integral

$$\Psi(s,W,\chi,\Phi) = \int_{N_n\cap H_n\backslash H_n} W(h)\Phi(L_m(h_2))|h|^s\chi(h)\delta(h)^{-1/2}dh.$$

This integral is convergent for $\mathrm{Re}(s)$ large, and defines an element of $\mathbb{C}(q^{-s})$.

Theorem 3.1 *There is a real number $r_{\pi,\chi}$, such that each integral $\Psi(s,W,\chi,\Phi)$ converges for $\mathrm{Re}(s) > r_{\pi,\chi}$. Moreover, when W and Φ vary in $W(\pi,\theta)$ and $C_c^\infty(F^m)$, respectively, they span a fractional ideal of $\mathbb{C}[q^s,q^{-s}]$ in $\mathbb{C}(q^{-s})$ generated by an Euler factor, which we denote $L^{\mathrm{lin}}(s,\pi,\chi)$.*

Proof The convergence for $\mathrm{Re}(s)$ greater than a real $r_{\pi,\chi}$ is classical. It is a consequence of the asymptotic expansion of the restriction of W to the torus A_n, which can be found in [12], for example. The fact that these integrals span a fractional ideal of $\mathbb{C}[q^s,q^{-s}]$ is a consequence of the observation that $\Psi(s,W,\chi,\Phi)$ is multiplied by $|h|^{-s}\chi^{-1}(h)\delta^{1/2}(h)$ when one replaces W and Φ by their right translates under h.

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Denoting by c_π the central character of π, and by K_n the points of H_n on $\mathfrak{O}$, thanks to Iwasawa decomposition we write the integral $\Psi(s, W, \chi, \Phi)$ as

$$\int_{K_n}\int_{N_n^\sigma\backslash P_n^\sigma} W(pk)|p|^{s-1/2}\chi(pk)\bigg(\int_{F^*}\Phi(aL_m(k))c_\pi(a)|a|^{ns}da\bigg)dpdk.$$

As in [12], for any ϕ in $\mathcal{C}_c^\infty(N_n\backslash P_n, \theta)$, there is a W such that $W_{|P_n}$ is ϕ. Such a ϕ is right invariant under an open subgroup U of K_n, which also fixes χ. We then choose Φ to be the characteristic function of $\{L_m(h_2), h_2 \in U\}$; the integral then reduces to a positive multiple of

$$\int_{N_n^\sigma\backslash P_n^\sigma}\phi(p)|p|^{s-1/2}\chi(p)dp.$$

We now see that for ϕ well chosen, this last integral is 1, *i.e.*, $\Psi(s, W, \chi, \Phi)$ is 1. This implies that the generator of the fractional ideal spanned by the $\Psi(s, W, \chi, \Phi)$ can be chosen as an Euler factor. ■

Remark 3.2 There is a notational difference with [22]. What we are calling $L^{\mathrm{lin}}(s, \pi, \chi\delta^{1/2})$ is called $L^{\mathrm{lin}}(s, \pi, \chi)$ there.

We have the following corollary to the previous proof.

Corollary 3.3 *There are $W \in W(\pi, \theta)$, and ϕ the characteristic function of a neighbourhood of $(0, \dots, 0, 1) \in F^n$, such that $\Psi(s, W, \chi, \Phi)$ is equal to 1 in $\mathbb{C}(q^{-s})$.*

3.2 The Unramified Computation

Here we show that the local Rankin–Selberg integrals give the expected L-function at the unramified places.

Let π^0 be an unramified generic representation of $GL(n, F)$, let W^0 be the normalised spherical Whittaker function in $W(\pi^0, \theta)$ (here θ has conductor $\mathfrak{O}$), and let Φ^0 be the characteristic function of $\mathfrak{O}^n$. We will use the notations of [8, Section 3]. We recall that π^0 is a commuting product (in the sense of [4], *i.e.*, corresponding to normalised parabolic induction) $\chi_1 \times \cdots \times \chi_n$ of unramified characters, and we denote $\chi_i(\varpi)$ by z_i. Then it is well known (see [23]) that if λ is an element of $\mathbb{Z}^n$, then $W(\varpi^\lambda)$ is zero unless λ belongs to the set Λ^+ consisting of λ's satisfying $\lambda_1 \geq \cdots \geq \lambda_n$, in which case $W(\varpi^\lambda) = \delta_{B_n}^{1/2}(\varpi^\lambda)s_\lambda(z)$, where $s_\lambda(z) = \det(z_i^{\lambda_j+n-j})/\det(z_i^{n-j})$.

In this case, using Iwasawa decomposition, denoting by Λ^{++} the subset of Λ^+ with $\lambda_n \geq 0$, and writing a' for $(a_1, a_3, \dots, a_{n-1})$ and a'' for $(a_2, a_4, \dots, a_n)$, one has the identities

$$\begin{aligned}\Psi(s, W^0, \chi, \Phi^0) &= \int_{A_n} W^0(a)\delta_{B_n}^{-1}(a')\delta_{B_n}^{-1}(a'')\chi(a)\Phi^0(a_n)\chi_0(a)|a|^s\delta(a)^{-1/2}da \\ &= \int_{A_n} W^0(a)\chi_0(a)\Phi^0(a_n)\delta_{B_n}^{-1/2}(a)\alpha(\det(a'))\alpha^{-1}(\det(a''))|a|^s da \\ &= \sum_{\lambda\in\Lambda^{++}} s_\lambda(z)q^{-s.tr\lambda}\alpha(\varpi)^{\sum_{i=1}^m(\lambda_{2i-1}-\lambda_{2i})} = \sum_{\lambda\in\Lambda^{++}} s_\lambda(q^{-s}z)\alpha(\varpi)^{c(\lambda)},\end{aligned}$$

where $c(\lambda) = \sum_{i=1}^m(\lambda_{2i-1} - \lambda_{2i})$. We now refer to [18, Example 7, p. 78], which asserts that if $x = (x_1, \dots, x_n)$ is a vector with nonzero complex coordinates and t is a complex number, then the sum $\sum_{\lambda\in\Lambda^{++}} s_\lambda(x)t^{c(\lambda)}$ is equal to

$$\prod_i (1-tx_i)\prod_{j<k}(1-x_jx_k).$$

In particular, with $x_i = z_i q^{-s}$ and $t = \alpha(\varpi)$, we obtain that $\Psi(s, W^0, \chi, \Phi^0)$ is equal to

$$\prod_i (1-\alpha(\varpi)z_i q^{-s})\prod_{j<k}(1-z_jz_kq^{-2s}) = L(\alpha\otimes\pi^0, s)L(\pi^0, \Lambda^2, 2s).$$

We end with the archimedean theory.

3.3 Convergence and Non-vanishing of the Archimedean Integrals

Here F is archimedean, θ is a unitary character of F, and π is a generic unitary representation of G_n, as in [15, Section 2], to which we refer concerning this vocabulary. We denote by $W(\pi,\theta)$ its smooth Whittaker model.

We denote by δ the character $h(h_1,h_2)\mapsto |h_1|/|h_2|$ of $H_n(F)$, and by χ the character $h(h_1,h_2)\mapsto \alpha(\det(h_1))/\alpha((\det(h_2)))$, for α a character of F^*.

We now formally define the following integral, for W in $W(\pi,\theta)$, and Φ in $\mathcal{S}(F^n)$:

$$\Psi(s,W,\chi,\Phi) = \int_{N_n\cap H_n\backslash H_n} W(h)\Phi\big(L_m(h_2)\big)\chi(h)|h|^s\delta(h)^{-1/2}\,dh.$$

We first state a proposition concerning the convergence of this integral.

Proposition 3.4 *For* $\mathrm{Re}(\alpha)\geq 0$*, there is a positive real* ϵ *independent of* W *and* Φ*, such that the integral* $\Psi(s,W,\chi,\Phi)$ *is absolutely convergent for* $s\geq 1/2-\epsilon$*. In particular it defines a holomorphic function on this half plane.*

Proof As a consequence of Iwasawa decomposition, it is enough to prove this statement for the integral

$$\int_{A_{n-1}} W(a)|a|^s\delta_{B_n}^{-1}(a')\delta_{B_n}^{-1}(a'')\alpha\big(\det(a')\big)\alpha^{-1}\big(\det(a'')\big)\delta(a)^{-1/2}\,da,$$

where $a' = (a_1,\dots,a_{n-1})$ and $a'' = (a_2,\dots,a_{n-2},1)$. However, we have the equality $\delta_{B_n}^{-1}(a')\delta_{B_n}^{-1}(a'')\delta(a)^{-1/2} = \delta_{B_n}^{-1/2}(a) = \delta_{B_{n-1}}^{-1/2}(a)|a|^{-1/2}$. But according to [16, Section 4], writing t_i for a_i/a_{i+1} there is a finite set X consisting of functions that are products of polynomials in the logarithm of the $|t_i|$'s and a character $\chi(a)=\prod_{i=1}^{n-1}\chi_i(t_i)$ with $\mathrm{Re}(\chi_i)>0$, such that $|W(a)|$ is majorised by a sum of functions of the form $S(t_1,\dots,t_{n-1})\delta_{B_{n-1}}^{1/2}(t)C_\chi(t)$, where S is a Schwartz function on F^{n-1}, and C_χ belongs to X. Hence, we only need to consider the convergence of

$$\int_{A_{n-1}} C_\chi\big(t(a)\big)S\big(t(a)\big)\alpha\big(\det(a')/\det(a'')\big)|a|^{s-1/2}\,da = \int_{A_{n-1}} C_\chi(t)S(t)\prod_{j=1}^{n}\alpha(t_{2j-1})\prod_{i=1}^{n-1}|t_i|^{i(s-1/2)}\,dt.$$

The statement follows, taking $\epsilon=\min(\mathrm{Re}(\chi_j))$ for C_χ in X. ∎

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Now we state our second result about the nonvanishing of our integrals at $1/2$ for good choices of W and Φ. The proof of this proposition, as that of the previous one, will be an easy adaptation of the techniques of [15], though we followed [9] even more closely.

Proposition 3.5 *Suppose that* $\mathrm{Re}(\alpha) \geq 0$*, and let* s *be a complex number with* $\mathrm{Re}(s) \geq 1/2 - \epsilon$*. There is* W *in* $W(\pi, \theta)$*, and* Φ *in* $\mathcal{S}(F^n)$*, such that* $\Psi(s, W, \chi, \Phi)$ *is nonzero. Moreover, one can choose* $\Phi \geq 0$*, so that* $\widehat{\Phi}(0) > 0$.

Proof If not, $\Psi(s, W, \chi, \Phi)$ is zero for every W in $W(\pi, \theta)$, and $\Phi \geq 0$ in $S(F^n)$. We are first going to prove that this implies that

$$\int_{N_{n-1}^{\sigma}\backslash H_{n-1}} W\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \chi(h)|h|^{s-1/2} dh = 0$$

for every W in $W(\pi, \theta)$. Indeed, one has

$$\begin{aligned}
&\int_{N_n^{\sigma}\backslash G_n^{\sigma}} W(h)\Phi\big(L_m(h_2)\big)\chi(h)|h|^{s}\delta(h)^{-1/2} dh \\
&= \int_{N_n^{\sigma}\backslash G_n^{\sigma}} W(h)\Phi\big(L_m(h)\big)\chi(h)|h|^{s-1/2}|h_2| dh \\
&= \int_{P_n^{\sigma}\backslash G_n^{\sigma}} \Big(\int_{N_n^{\sigma}\backslash P_n^{\sigma}} W(ph)\chi(ph)|ph|^{s-1/2} dp \Big) \Phi\big(L_m(h_2)\big) d\bar{h}
\end{aligned}$$

where $|h_2|dh$ is quasi-invariant on $N_n^{\sigma}\backslash G_n^{\sigma}$, and $d\bar{h}$ is quasi-invariant on $P_n^{\sigma}\backslash G_n^{\sigma}$.

But $P_n^{\sigma}\backslash G_n^{\sigma} \simeq F^n - \{0\}$ via $\bar{h} \mapsto L_n(h_2)$, and the Lebesgue measure on $F^n - \{0\}$ corresponds to $d\bar{h}$ via this homeomorphism. Hence, setting

$$G(\bar{h}) = \int_{N_n^{\sigma}\backslash P_n^{\sigma}} W(ph)\chi(ph)|ph|^{s-1/2} dp,$$

one has that for every Φ,

$$\int_{F^n - \{0\}} G(x)\Phi(x) dx = 0.$$

In particular, $G(0, \dots, 0, 1) = 0$ (taking $\Phi \geq 0$ approximating the Dirac measure supported at $(0, \dots, 0, 1)$), hence $\int_{N_n^{\sigma}\backslash P_n^{\sigma}} W(p)\chi(p)|p|^{s-1/2} dp = 0$.

Then, one checks (see [15, Section 2]), that for every $\Phi \in \mathcal{S}(F^{n-1})$, the map $W_{\phi}\colon g \mapsto \int_{F^{n-1}} W(gu^{\sigma}(x))\Phi(x)dx$, where u^{σ} is the natural isomorphism between F^{n-1} and U_n^{σ}, belongs to $W(\pi, \theta)$ again. But $W_{\phi}\left(\begin{smallmatrix} h & \\ & 1 \end{smallmatrix}\right) = W\left(\begin{smallmatrix} h & \\ & 1 \end{smallmatrix}\right)\widehat{\Phi}(L_m(h_1))$ for h in H_{n-1}, hence

$$\int_{N_{n-1}^{\sigma}\backslash H_{n-1}} W\begin{pmatrix} h & \\ & 1 \end{pmatrix} \widehat{\Phi}(L_m(h_1))\chi(h)|h|^{s-1/2} dh$$

is zero for every W and Φ, which in turn implies the equality

$$\int_{N_{n-2}^{\sigma}\backslash H_{n-2}} W\begin{pmatrix} h & 0 \\ 0 & I_2 \end{pmatrix} \chi(h)|h|^{s-1/2} dh = 0$$

for every W in $W(\pi, \theta)$. Continuing the process, we obtain $W(I_n) = 0$ for every W in $W(\pi, \theta)$, a contradiction. We did not check the convergence of our integrals at each step, but it follows from Fubini's theorem. ∎

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4 The Global Theory

4.1 The Eisenstein Series

In the global case, let π be a smooth automorphic cuspidal representation of $G(\mathbb{A})$ with trivial central character, ϕ a cusp form in the space of π, and Φ an element of the Schwartz space $\mathcal{S}(\mathbb{A}^n)$. We denote by χ a character of $H_n(\mathbb{A})$ of the form $h(h_1, h_2) \mapsto \alpha(\det(h_1)/\det(h_2))$ for α a character of $\mathbb{A}^*/k^*$, and by δ again the character $h(h_1, h_2) \mapsto |h_1|/|h_2|$ of $H_n(\mathbb{A})$. Then we define

$$f_{\chi,\Phi}(s,h) = |h|^s \chi(h)\delta^{-1/2}(h) \int_{\mathbb{A}^*} \Phi\big(aL_m(h_2)\big)|a|^{ns} d^*a$$

for h in H, s in the half plane $\mathrm{Re}(s) > 1/n$, where the integral converges absolutely. It is obvious that $f_{\chi,\Phi}(s,h)$ is $Z_n(k)P_n^\sigma(k)$-invariant on the left.
Now we average f on $H_n(k)$ to obtain the following Eisenstein series:

$$E(s,h,\chi,\Phi) = \sum_{\gamma \in Z_n(k)P_n^\sigma(k)\backslash H_n(k)} f_{\chi,\Phi}(s,\gamma h).$$

One can rewrite $E(s,h,\chi,\Phi)$ as

$$|h|^s\chi(h)\delta^{-1/2}(h) \int_{k^*\backslash\mathbb{A}^*} \Theta'_\Phi(a,h)|a|^{ns}d^*a,$$

where $\Theta'_\Phi(a,h) = \sum_{\xi\in k^n-\{0\}} \Phi(a\xi h_2)$.
According to [7, Lemmas 11.5 and 11.6], it is absolutely convergent for $\mathrm{Re}(s) > 1/2$, uniformly on compact subsets of $H_n(k)\backslash H_n(\mathbb{A})$, and of moderate growth with respect to h.

Write $\Theta_\Phi(a,h)$ for $\Theta'_\Phi(a,h) + \Phi(0)$; then the Poisson formula for Θ_Φ gives

$$\Theta_\Phi(a,h) = |a|^{-n}|h_2|^{-1}\Theta_{\hat{\Phi}}(a^{-1}, {}^t h^{-1}).$$

This allows us to write, for c a certain nonzero constant,

$$\begin{aligned} E(s,h,\chi,\Phi) = |h|^s\chi(h)\delta^{-1/2}(h) \int_{|a|\geq 1} \Theta'_\Phi(a,h)|a|^{ns}d^*a \\ + h|^{s-1/2}\chi(h) \int_{|a|\geq 1} \Theta'_{\widehat{\Phi}}(a, {}^t h^{-1})|a|^{n(1-2s)}d^*a + u(s) \end{aligned}$$

with $u(s) = -c\Phi(0)|h|^s\chi(h)\delta^{-1/2}(h)/2s + c\widehat{\Phi}(0)\chi(h)|h|^{s-1/2}/(1-2s)$.

We deduce from this, appealing again to [7, Lemma 11.5], the following proposition.

Proposition 4.1 *$E(s,h,\chi,\Phi)$ admits a meromorphic extension to $\mathbb{C}$, has at most simple poles at 0 and $1/2$, and satisfies the functional equation*

$$E(1/2-s, {}^t h^{-1}, \chi^{-1}\delta^{1/2}, \widehat{\Phi}) = E(s,h,\chi,\Phi).$$

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Then the following integral converges absolutely for $\mathrm{Re}(s) > 1/2$:

$$I(s,\phi,\chi,\Phi) = \int_{Z_n(\mathbb{A})H_n(k)\backslash H_n(\mathbb{A})} E(s,h,\chi,\Phi)\phi(h)dh.$$

Theorem 4.2 *The integral $I(s,\phi,\chi,\Phi)$ extends meromorphically to $\mathbb{C}$, with poles at most simple at 0 and 1/2; moreover, a pole at 1/2 occurs if and only if the global χ^{-1}-period*

$$\int_{Z_n(\mathbb{A})H_n(k)\backslash H_n(\mathbb{A})} \chi(h)\phi(h)dh$$

is not zero, and $\widehat{\Phi}(0) \neq 0$. The integral $I(s,\phi,\chi,\Phi)$ also admits the following functional equation:

$$I(1/2-s,\widetilde{\phi},\chi^{-1}\delta^{1/2},\widehat{\Phi}) = I(s,\phi,\chi,\Phi),$$

where $\widetilde{\phi}\colon g \mapsto \phi({}^t g^{-1})$.

Proof It is clear that the residue of $I(s,\phi,\chi,\Phi)$ at 1/2 is

$$c\widehat{\Phi}(0)\int_{Z_n(\mathbb{A})H_n(k)\backslash H_n(\mathbb{A})} \chi(h)\phi(h)dh,$$

hence the result about periods follows. ∎

4.2 The Euler Factorisation

Let θ be a nontrivial character of $\mathbb{A}/k$; we denote by W_ϕ the Whittaker function on $G_n(\mathbb{A})$ associated with ϕ, and we let

$$\Psi(s,W_\phi,\chi,\Phi) = \int_{N_n^\sigma(\mathbb{A})\backslash H_n(\mathbb{A})} W(h)\Phi(\eta h)\chi(h)|h|^s\delta^{-1/2}(h)dh.$$

This integral converges absolutely for $\mathrm{Re}(s)$ large by classical gauge estimates of [11, Section 13], and is the product of the similar local integrals. We will need the following expansion of cusp forms on the mirabolic subgroup, which can be found in [6, p. 5].

Proposition 4.3 *Let ϕ be a cusp form on $P_l(\mathbb{A})$; then one has the following partial Fourier expansion with uniform convergence for p in compact subsets of $P_l(\mathbb{A})$:*

$$\phi(p) = \sum_{\gamma\in P_{l-1}(k)\backslash G_{l-1}(k)} \left(\int_{y\in(\mathbb{A}/k)^{l-1}} \phi\left(u(y)\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}p\right)\theta^{-1}(y)dy\right)$$

We now unravel the integral $I(s,\phi,\chi,\Phi)$, following the strategy in [5, 16], which is to unravel step by step.

Theorem 4.4 *One has the identity $I(s,\phi,\chi,\Phi) = \Psi(s,W_\phi,\chi,\Phi)$ for $\mathrm{Re}(s)$ large.*

Proof We suppose that s is large enough so that $I(s,\phi,\chi,\Phi)$ is absolutely convergent. Denoting by χ_s the character $\chi\delta^{-1/2}|\cdot|^s$ of $H_n(\mathbb{A})$, we start with

$$\begin{aligned} I(s,\phi,\chi,\Phi) &= \int_{Z_n(\mathbb{A})H_n(k)\backslash H_n(\mathbb{A})} E(s,h,\chi,\Phi)\phi(h)dh \\ &= \int_{Z_n(\mathbb{A})P_n^\sigma(k)\backslash H_n(\mathbb{A})} f_{\chi,\Phi}(s,h)\phi(h)dh \\ &= \int_{P_n^\sigma(k)\backslash H_n(\mathbb{A})} \Phi(Ln(h_2))\phi(h)\chi_s(h)dh. \end{aligned}$$

We denote for the moment $\Phi(L_m(h_2))\chi_s(h)$ by $F(h)$, and for l between 0 and $m-1$, we write

$$I_l = \int_{P_{2l}^\sigma(k)(U_{2l+1}\dots U_n)^\sigma(\mathbb{A})\backslash H_n(\mathbb{A})} F(h)\Bigg(\int_{(U_{2l+1}\dots U_n)(k)\backslash(U_{2l+1}\dots U_n)(\mathbb{A})} \phi(nh)\theta^{-1}(n)dn\Bigg)dh.$$

In particular, $I_0 = \Psi(W_\phi,\Phi,s)$. We also write $I_m = I(s,\phi,\chi,\Phi)$.

To prove the theorem, we only need to prove that $I_l = I_{l+1}$ for $0 \le m-1$, which we do now. We will see that the absolute convergence of I_l (*i.e.*, the fact that

$$F(h)\Bigg(\int_{(U_{2l+1}\dots U_n)(k)\backslash(U_{2l+1}\dots U_n)(\mathbb{A})} \phi(nh)\theta^{-1}(n)dn\Bigg)$$

is absolutely integrable over the quotient $P_{2l}^\sigma(k)(U_{2l+1}\dots U_n)^\sigma(\mathbb{A})\backslash H_n(\mathbb{A}))$ implies that of I_{l+1} during the process. We will tacitly use several times the fact that if G is a unimodular locally compact group, $K < H$ closed unimodular subgroups of G, and A is a continuous integrable function on $K\backslash G$, then $B(x) = \int_{K\backslash H} A(hx)dh$ is absolutely convergent for all $x \in G$, integrable over $H\backslash G$, and one has $\int_{H\backslash G} B(h)dh = \int_{K\backslash G} A(g)dg$.

We thus suppose that I_l is absolutely convergent, so one can write I_l as the absolutely convergent sum

$$\begin{aligned} I_l = \sum_\gamma &\int_{P_{2l+1}^\sigma(k)(U_{2l+1}\dots U_n)^\sigma(\mathbb{A})\backslash H_n(\mathbb{A})} \\ &F(h)\Bigg(\int_{(U_{2l+1}\dots U_n)(k)\backslash(U_{2l+1}\dots U_n)(\mathbb{A})} \phi(n\gamma h)\theta^{-1}(n)dn\Bigg)dh \\ = &\int_{P_{2l+1}^\sigma(k)(U_{2l+1}\dots U_n)^\sigma(\mathbb{A})\backslash H_n(\mathbb{A})} \\ &F(h)\Bigg(\sum_\gamma \int_{(U_{2l+1}\dots U_n)(k)\backslash(U_{2l+1}\dots U_n)(\mathbb{A})} \phi(n\gamma h)\theta^{-1}(n)dn\Bigg)dh, \end{aligned}$$

where the sum is over $\gamma \in P_{2l}^\sigma U_{2l+1}^\sigma(k)\backslash P_{2l+1}^\sigma(k)$. Now, a system of representatives of $P_{2l}^\sigma(k)U_{2l+1}^\sigma(k)\backslash P_{2l+1}^\sigma(k)$ is given by the elements $w'_{2l+1}(\gamma)$, for γ in $P_l(k)\backslash G_l(k)$.

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We now apply Proposition 4.3 at $p=1$ to the cusp form

$$p\in P_{l+1}\longmapsto \int_{(U_{2l+2}\dots U_n)(k)\backslash(U_{2l+2}\dots U_n)(\mathbb{A})}\phi(nw'_{2l+1}(p)h)\theta^{-1}(n)dn.$$

The sum being absolutely convergent by Proposition 4.3, we thus obtain the relation

$$\sum_{\gamma\in P_l(k)\backslash G_l(k)}\left(\int_{(U_{2l+1}U_{2l+2}\dots U_n)(k)\backslash(U_{2l+1}U_{2l+2}\dots U_n)(\mathbb{A})}\phi(nw'_{2l+1}(\gamma)h)\theta^{-1}(n)dn\right)=$$
$$\int_{u\in U^{\sigma}_{2l+1}(k)\backslash U^{\sigma}_{2l+1}(\mathbb{A})}\left(\int_{n\in(U_{2l+2}\dots U_n)(k)\backslash(U_{2l+2}\dots U_n)(\mathbb{A})}\phi(nuh)\theta^{-1}(n)dn\right)du.$$

Replacing in I_l, with J_l absolutely convergent, one obtains the equality

$$I_l=J_l=\int_{P^{\sigma}_{2l+1}(k)(U_{2l+2}\dots U_n)^{\sigma}(\mathbb{A})\backslash H_n(\mathbb{A})} F(h)\left(\int_{(U_{2l+2}\dots U_n)(k)\backslash(U_{2l+2}\dots U_n)(\mathbb{A})}\phi(nh)\theta^{-1}(n)dn\right)dh.$$

Again, we have

$$J_l=\sum_{\gamma}\int_{P^{\sigma}_{2l+2}(k)(U_{2l+2}\dots U_n)^{\sigma}(\mathbb{A})\backslash H_n(\mathbb{A})} F(h)\left(\int_{(U_{2l+2}\dots U_n)(k)\backslash(U_{2l+2}\dots U_n)(\mathbb{A})}\phi(n\gamma h)\theta^{-1}(n)dn\right)dh,$$

where the sum is now over $\gamma\in P^{\sigma}_{2l+1}(k)U^{\sigma}_{2l+2}(k)\backslash P^{\sigma}_{2l+2}(k)$, a system of representatives of which is given by the elements $w'_{2l+2}(\gamma)$, for γ in $P_{l+1}(k)\backslash G_{l+1}(k)$. Applying Proposition 4.3 at $p=1$ again, this time to the cusp form:

$$p\in P_{l+2}\longmapsto \int_{(U_{2l+3}\dots U_n)(k)\backslash(U_{2l+3}\dots U_n)(\mathbb{A})}\phi(nw'_{2l+2}(p)h)\theta^{-1}(n)dn,$$

one has

$$\sum_{\gamma\in P_{l+1}(k)\backslash G_{l+1}(k)}\left(\int_{(U_{2l+2}U_{2l+3}\dots U_n)(k)\backslash(U_{2l+2}U_{2l+3}\dots U_n)(\mathbb{A})}\phi(nw'_{2l+2}(\gamma)h)\theta^{-1}(n)dn\right)=$$
$$\int_{u\in U^{\sigma}_{2l+2}(k)\backslash U^{\sigma}_{2l+2}(\mathbb{A})}\left(\int_{n\in(U_{2l+3}\dots U_n)(k)\backslash(U_{2l+3}\dots U_n)(\mathbb{A})}\phi(nuh)\theta^{-1}(n)dn\right)du.$$

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Finally, replacing in J_l, one obtains $J_l = I_{l+1}$, together with the absolute convergence of I_{l+1}. Notice that when $l = m-1$, the last bit of the proof becomes

$$\begin{aligned} J_m &= \int_{P^{\sigma}_{n-1}(k)U^{\sigma}_n(\mathbb{A})\backslash H_n(\mathbb{A})} F(h)\Big(\int_{U_n(k)\backslash U_n(\mathbb{A})} \phi(nh)\theta^{-1}(n)dn\Big)dh \\ &= \int_{P^{\sigma}_{n}(k)U^{\sigma}_n(\mathbb{A})\backslash H_n(\mathbb{A})} F(h) \sum_{\gamma\in w'_n(P_{n-1}(k)\backslash G_{n-1}(k))}\Big(\int_{U_n(k)\backslash U_n(\mathbb{A})} \phi(n\gamma h)\theta^{-1}(n)dn\Big)dh \\ &= \int_{P^{\sigma}_{n}(k)U^{\sigma}_n(\mathbb{A})\backslash H_n(\mathbb{A})} F(h)\Big(\int_{U^{\sigma}_n(k)\backslash U^{\sigma}_n(\mathbb{A})} \phi(nh)\theta^{-1}(n)dn\Big)dh \\ &= \int_{P^{\sigma}_{n}(k)(\mathbb{A})\backslash H_n(\mathbb{A})} F(h)\phi(h)dh = I_m. \end{aligned}$$

∎

As a corollary, we see that $\Psi(s, W_\phi, \chi, \Phi)$ extends to a meromorphic function (namely $I(s,\phi,\chi,\Phi)$). Writing π as the restricted tensor product $\otimes_v \pi_v$, for any $W = \prod_v W_v$ in $W(\pi,\theta)$, any decomposable $\Phi = \prod_v \Phi_v$ in $\mathcal{S}(\mathbb{A}^n)$, one has

$$\Psi(s, W, \chi, \Phi) = \prod_v \Psi(s, W_v, \chi_v, \Phi_v).$$

4.3 The Partial L-function

Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $G_n(\mathbb{A})$, and let S be the finite set of places of k, such that for v in S, π_v is archimedean or ramified. Let $\alpha = \otimes_v \alpha_v$ be a character of $\mathbb{A}^*/k^*$ (with α_v unramified for v outside S), and let χ be the character $h(h_1,h_2) \mapsto \alpha(\det(h_1)/\det(h_2))$ of $H_n(\mathbb{A})$. We define the partial L-function $L^{\mathrm{lin},S}(s,\pi,\chi)$ to be the product

$$L^{\mathrm{lin},S}(s,\pi,\chi) = \prod_{v\notin S} L(s,\alpha_v\otimes\pi_v)L(2s,\Lambda^2,\pi_v),$$

where $L(s,\alpha_v\otimes\pi_v)$ and $L(2s,\Lambda^2,\pi_v)$ are the corresponding L-functions of the Galois parameter of π_v. Hence, if θ_v has conductor $\mathfrak{O}_v$ at every unramified place v, the function $L^{\mathrm{lin},S}(s,\pi,\chi)$ is the product $\prod_{v\notin S}\Psi(s, W_v^0, \chi_v, \Phi_v^0)$ thanks to the unramified computation. Because of this, we see that it is meromorphic (it is equal to $\Psi(s,W,\Phi)/\prod_{v\in S}\Psi(s,W_v,\Phi_v)$ for a well chosen Φ an W). We now show that when $\mathrm{Re}(\alpha)\geq 0$, the partial L function $L^{\mathrm{lin},S}(s,\pi,\chi)$ has at most a simple pole at $1/2$, and that this happens if and only if π admits a χ^{-1}-period.

Theorem 4.5 *Suppose that* $\mathrm{Re}(\alpha)\geq 0$ *and that the partial* L*-function* $L^{\mathrm{lin},S}(s,\pi,\chi)$ *is holomorphic for* $\mathrm{Re}(s) > 1/2$ *and has a pole at* $1/2$ *if and only if* π *has an* $(H_n(\mathbb{A}),\chi^{-1})$*-period. If it is the case, this pole is simple.*

Proof Let S_∞ be the archimedean places of k, and let S_f be the set of finite places in S. First, for v in S_f, thanks to Corollary 3.3, we take W_v and Φ_v such that $\Psi(s,W_v,\chi_v,\Phi_v) = 1$ for all s (and $\widehat{\Phi_v}(0) > 0$ because Φ_v is positive). For any $s_0 \geq 1/2$, if v belongs to S_∞, it is possible to take W_{v,s_0} and Φ_{v,s_0} with $\widehat{\Phi_{v,s_0}}(0) > 0$ such that

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$\Psi(s, W_{v,s_0}, \chi_v, \Phi_{v,s_0})$ is convergent for $s \geq 1/2 - \epsilon$, and $\neq 0$ for $s = s_0$ according to Propositions 3.4 and 3.5. We write

$$W_{s_0} = \prod_{v \in S_\infty} W_{v,s_0} \prod_{v \in S_f} W_v \prod_{v \notin S} W_v^0,$$

notice that W_{s_0} is equal to W_ϕ for some cusp ϕ in the space of π, thanks to the restricted tensor product decomposition of π, which by multiplicity 1 for local Whittaker functionals, implies the same decomposition of the global Whittaker model $W(\pi, \theta)$. If we now write

$$\Phi_{s_0} = \prod_{v \in S_\infty} \Phi_{v,s_0} \prod_{v \in S_f} \Phi_v \prod_{v \notin S} \Phi_v^0,$$

the theorem follows from the equality

$$\prod_{v \in S} \Psi(s, W_{v,s_0}, \chi_v, \Phi_{v,s_0}) L^{\mathrm{lin},S}(s, \pi) = \Psi(s, W_{s_0}, \chi, \Phi_{s_0}),$$

and Theorems 4.4 and 4.2, as $\widehat{\Phi}(0) \neq 0$ (it is positive). ∎

We then recall the following lemma.

Lemma 4.6 *If the partial exterior square L-function $L^S(\pi, \Lambda^2, \pi)$ has a pole at* 1*, then π is self-dual.*

Proof It is a consequence of the main theorem of [16] that if $L^S(\pi, \Lambda^2, \pi)$ has a pole at 1, then π has a non-vanishing Shalika period, hence all its non-archimedean components admit a Shalika model. Then, by [13], all the non archimedean components must be self-dual, hence π as well by strong multiplicity one. ∎

We will use it in the following proof, to remove the assumption $\mathrm{Re}(\alpha) \geq 0$. We reobtain a theorem of Friedberg and Jacquet ([10]), using the Bump–Friedberg L-function directly.

Theorem 4.7 *The cuspidal automorphic representation π of $G_n(\mathbb{A})$ admits a global χ^{-1}-period if and only if $L^S(s, \Lambda^2, \pi)$ has a pole at* 1 *and $L(1/2, \alpha \otimes \pi) \neq 0$.*

Proof We first give the proof for α satisfying $\mathrm{Re}(\alpha) \geq 0$. We will deduce the general case form this particular one. It is well known (see [7]) that $L(s, \alpha \otimes \pi)$ is entire, hence $L^S(s, \alpha \otimes \pi)$. Moreover, $L(1/2, \alpha \otimes \pi) = 0$ if and only if $L^S(1/2, \alpha \otimes \pi) = 0$. Indeed $L^S(s, \alpha \otimes \pi)$ is an entire multiple of $L(s, \alpha \otimes \pi)$, hence one implication. Using the Rankin–Selberg convolution for $G_n(\mathbb{A}) \times \mathbb{A}^*$, then for any W in $W(\pi, \theta)$, denoting $\int_{\mathbb{A}^*} W(a, 1, \ldots, 1)\alpha(a)|a|^{(n-1)/2} d^*a$ by $\Psi(s, W, \alpha)$,

$$\prod_{v \in S} \int_{k_v^*} W(t_v, 1, \ldots, 1)\alpha_v(t_v)|t_v|^{(n-1)/2} d^*t_v$$

by $\Psi(s, W_S, \alpha_S)$, and $\prod_{v \in S} L(s, \alpha_v \otimes \pi_v)$ by $L_S(s, \alpha \otimes \pi)$, one has

$$\Psi(s, W_S, \alpha_S) L^S(s, \alpha \otimes \pi) = \left[\Psi(s, W_S, \alpha_S)/L_S(s, \alpha \otimes \pi)\right] L(s, \alpha \otimes \pi).$$

But there is $\epsilon > 0$ such that $\Psi(s, W_S, \alpha_S)$ converges for $\mathrm{Re}(s) > 1/2 - \epsilon$ according to the estimates for the W_v's retriction to A_{n-1} given in [16, Proposition 3]. Hence, if

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$L^S(1/2, \alpha \otimes \pi) = 0$, then $[\Psi(1/2, W_S, \alpha_S)/L_S(1/2, \alpha \otimes \pi)]L(1/2, \alpha \otimes \pi) = 0$, but one can always choose W such that

$$\left[\Psi(1/2, W_S, \alpha_S)/L_S(1/2, \alpha \otimes \pi)\right] \neq 0 \quad \text{and} \quad L_S(1/2, \alpha \otimes \pi) = 0.$$

It is also proved in [16] that the partial exterior square L-function $L^S(s, \Lambda^2, \pi)$ can have a pole at 1 that is at most simple. Now the theorem follows from the equality $L^{\mathrm{lin},S}(s, \pi, \chi) = L^S(s, \alpha \otimes \pi)L^S(2s, \Lambda^2, \pi)$ and Theorem 4.5.

Now assume $\mathrm{Re}(\alpha) < 0$. If π admits a global χ^{-1}-period, then its dual representation $\widetilde{\pi}$ admits obviously a χ-period. By the previous case, we know that $L^S(s, \Lambda^2, \widetilde{\pi})$ has a pole at 1, and $L(1/2, \alpha^{-1} \otimes \widetilde{\pi}) \neq 0$. But then, by Lemma 4.6, we have $\widetilde{\pi} = \pi$, hence $L^S(s, \Lambda^2, \pi)$ has a pole at 1, and we obtain that $L(1/2, \alpha \otimes \pi) \neq 0$ thanks to the functional equation of the Godement–Jacquet L-function. Conversely, if $L^S(s, \Lambda^2, \pi)$ has a pole at 1, and $L(1/2, \alpha \otimes \pi) \neq 0$, by Lemma 4.6 again, we know that π is equal to $\widetilde{\pi}$, hence $L^S(s, \Lambda^2, \widetilde{\pi})$ has a pole at 1. Using the functional equation again, we also obtain $L(1/2, \alpha^{-1} \otimes \widetilde{\pi}) \neq 0$, hence, as $\mathrm{Re}(\alpha^{-1}) \geq 0$, we deduce that $\widetilde{\pi}$ has a χ-period, *i.e.*, that π has a χ^{-1}-period. ∎

5 The Odd Case

In this section, we just state the results for the odd case, which is very similar to the even case.

In the local non-archimedean case, for π a generic representation of G_n, the integrals we consider are the following for W in $W(\pi, \theta)$, and Φ in $\mathcal{C}_c^\infty(F^n)$:

$$\Psi(s, W, \chi, \Phi) = \int_{N_n \cap H_n \backslash H_n} W(h)\Phi\big(L_{m+1}(h_1)\big)|h|^s \chi(h)dh.$$

We have the following theorem.

Theorem 5.1 *There is a real number $r_{\pi,\chi}$, such that each integral $\Psi(s, W, \chi, \Phi)$ converges for $\mathrm{Re}(s) > r_\pi$. Moreover, when W and Φ vary in $W(\pi, \theta)$ and $C_c^\infty(F^n)$ respectively, they span a fractional ideal of $\mathbb{C}[q^s, q^{-s}]$ in $\mathbb{C}(q^{-s})$, generated by an Euler factor which we denote $L^{\mathrm{lin}}(s, \pi, \chi)$.*

Let π^0 be a generic unramified representation of G_n. Thanks to the relation $\delta_{B_n}(a) = \delta_{B_n}(a')\delta_{B_n}(a'')$, with $a' = (a_1, a_3, \dots, a_n)$ and $a'' = (a_2, a_4, \dots, a_n)$, the unramified computation gives again the equality

$$\Psi(s, W^0, \Phi^0) = \sum_{\lambda \in \Lambda^{++}} \alpha(\omega)^{c_\lambda} s_\lambda(q^{-s}z) = L(s, \alpha \circ \pi^0)L(s, \Lambda^2, \pi^0)$$

In the archimedean case, for π a generic representation of G_n, and $\mathrm{Re}(\alpha) \geq 0$, the integrals

$$\Psi(s, W, \chi, \Phi) = \int_{N_n \cap H_n \backslash H_n} W(h)\Phi\big(L_{m+1}(h_1)\big)\chi(h)|h|^s dh$$

converge again for $\mathrm{Re}(s) \geq 1/2 - \epsilon$ for a positive number ϵ depending on π. For any such s, one can chose W and Φ such that they do not vanish.

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In the global situation, for Φ in $\mathcal{S}(\mathbb{A}^n)$, we define

$$f_{\chi,\Phi}(s,h) = |h|^s\chi(h)\int_{\mathbb{A}^*}\Phi\big(aL_{m+1}(h_1)\big)|a|^{(n)s}d^*a$$

for h in H_n. Associated with this is the Eisenstein series

$$E(s,h,\chi,\Phi) = \sum_{\gamma\in Z_n(k)P_n^\sigma(k)\backslash H_n(k)} f_{\chi,\Phi}(s,\gamma h),$$

which converges absolutely for $\mathrm{Re}(s) > 1/2$, extends meromorphically to $\mathbb{C}$, with possible poles simple and located at 0 and $1/2$. It satisfies the functional equation

$$E(1/2-s,{}^th^{-1},\chi^{-1}\delta'^{1/2},\widehat{\Phi}) = E(s,h,\chi,\Phi).$$

Then if π is a cuspidal automorphic representation of $G_n(\mathbb{A})$ and ϕ is a cusp form in the space of π, we define for $\mathrm{Re}(s) > 1/2$ the integral

$$I(s,\phi,\chi,\Phi) = \int_{Z_n(\mathbb{A})H_n(k)\backslash H_n(\mathbb{A})} E(s,h,\chi,\Phi)\phi(h)dh,$$

which satisfies the statement of the following theorem.

Theorem 5.2 *The integral $I(s,\phi,\chi,\Phi)$ extends to an entire function on $\mathbb{C}$. The integral $I(s,\phi,\chi,\Phi)$ also admits the functional equation*

$$I(1/2-s,\widetilde{\phi},\chi^{-1}\delta^{1/2},\widehat{\Phi}) = I(s,\phi,\chi,\Phi),$$

where $\widetilde{\phi}\colon g\mapsto\phi({}^tg^{-1})$.

The proof is the same up to the following extra argument. It is clear that the residue of $I(s,\phi,\chi,\Phi)$ at $1/2$ is

$$c\widehat{\Phi}(0)\int_{Z_n(\mathbb{A})H_n(k)\backslash H_n(\mathbb{A})}\chi(h)\phi(h)dh,$$

but these integrals are known to vanish according to [5, Proposition 2.1], hence there is actually no pole at $1/2$.

Again, we define

$$\Psi(s,W_\phi,\chi,\Phi) = \int_{N_n(\mathbb{A})\cap H_n(\mathbb{A})\backslash H_n(\mathbb{A})} W_\phi(h)\Phi\big(L_{m+1}(h_1)\big)|h|^s\chi(h)dh,$$

and this integral converges for $\mathrm{Re}(s)$ large, and is in fact equal to $I(s,\phi,\chi,\Phi)$.

From this we deduce that the partial L-function $L^{\mathrm{lin},S}(s,\pi,\chi)$ is meromorphic, and, when $\mathrm{Re}(\alpha)\geq 0$, it is holomorphic for $\mathrm{Re}(s)\geq 1/2-\epsilon$, for some positive ϵ (corresponding to the ϵ of the archimedean case).

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