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NOTE ON BEST APPROXIMATION OF $|x|$

BY

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In this note the best uniform approximation on $[-1, 1]$ to the function $|x|$ by symmetric complex valued linear fractional transformations is determined. This is a special case of the more general problem studied in [1]. Namely, for any even, real valued function $f(x)$ on $[-1, 1]$ satsifying $0 = f(0) \leq f(x) \leq f(1) = 1$, determine the degree of *symmetric* approximation

$$E_S(f) = \inf\left\{\|U - f\|_\infty \colon U(x) = \frac{ax+b}{cx+d},\ a, b, c \text{ and } d \text{ complex},\ U(x) = U(-x)\right\}$$

and the extremal transformations U whenever they exist. The authors completely solved this problem for two classes of functions f (cf. Theorems C and D, [1]) and in particular solved it for the functions $|x|^\alpha$, provided $\alpha \geq \kappa = 1.4397589\ldots$. Here κ is the unique solution in $(1, \infty)$ of $(2\kappa - 1)^{2\kappa - 1} = \kappa/\kappa - 1$. Furthermore, A. Ruttan [3] has shown that $E_S(|x|^\alpha)$, $\alpha \geq \kappa$, is also the degree of approximation when the symmetry condition $U(x) = \overline{U(-x)}$ is dropped.

The method which was used in [1] to determine best approximations consisted of two basic steps. First the interval $[-1, 1]$ was replaced by the four point set $\{-1, -\omega, \omega, 1\}$, $0 < \omega < 1$, and $U_{(\omega)}(x)$ was chosen so as to minimize

$$\max\{|U_{(\omega)}(x) - f(x)| : x = -1, -\omega, \omega, 1\}.$$

This was achieved by geometric considerations described in [2]. The second step was to show that for certain functions $f(x)$ the best global approximation was attained by $U_{(\omega)}$ for a suitable choice of $\omega \in (0, 1)$. For $f(x) = |x|^\alpha$, $\alpha < \kappa$, this method failed. However we conjectured that the method could be modified so as to successfully handle the case $\alpha < \kappa$ if the four point set were replaced by the five point set $\{-1, -\omega, 0, \omega, 1\}$, $0 < \omega < 1$. In the present note we show that $E_S(|x|)$ and the corresponding extremal transformations can be determined by algebraic means. Moreover, if U^* is extremal then $|U^*(x) - |x||$ does indeed attain its maximum on a certain five point set. Thus our result may be of use in finding a general method for determining best approximations on such five point sets.

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THEOREM. *For any symmetric transformation U (that is $U(x)=\overline{U(-x)}$) we have*

$$\|U(x)-|x|\|_\infty \geq \frac{\sqrt{5}-1}{4},$$

with equality if and only if $U(x)=U^*(x)$ or $\overline{U(x)}=U^*(x)$, where

$$U^*(x)=s_0+r_0\left(\frac{x+it_0}{x-it_0}\right), \qquad r_0=\frac{1}{4},$$

$$s_0=\frac{\sqrt{5}}{4}, \qquad (t_0)^2=\frac{\sqrt{5}-1}{8}.$$

Proof. First we observe that

$$|U^*(x)-|x||^2-\left(\frac{\sqrt{5}-1}{4}\right)^2=|x|\,(|x|-1)\left(|x|-\frac{\sqrt{5}-1}{4}\right)^2\leq 0$$

for $|x|\leq 1$. Hence $\|U^*(x)-|x|\|_\infty=(\sqrt{5}-1)/4$ and the norm is attained precisely when x is in the five point set $Z=\{0, \pm(\sqrt{5}-1)/4, \pm 1\}$. Thus it suffices to show that no other symmetric transformation can attain the degree of approximation $(\sqrt{5}-1)/4$ on Z.

As in [1] we need only consider symmetric transformations $U(x)$ of the form

$$U(x)=s+r\left(\frac{x+it}{x-it}\right),$$

with r, s and t real, $r\neq 0$, $t\neq 0$. Thus we may suppose that

$$(1) \qquad r=r_0+\varepsilon, \qquad s=s_0+\eta, \qquad t^2=(t_0)^2+\xi,$$

and that the corresponding $U(x)$ satisfies

$$(2) \qquad |U(1)-1|\leq\frac{\sqrt{5}-1}{4},$$

$$(3) \qquad \left|U\left(\frac{\sqrt{5}-1}{4}\right)-\frac{\sqrt{5}-1}{4}\right|\leq\frac{\sqrt{5}-1}{4},$$

$$(4) \qquad |U(0)|\leq\frac{\sqrt{5}-1}{4}.$$

From (2) we have

$$(1-r-s)^2+t^2(1+r-s)^2\leq\left(\frac{3-\sqrt{5}}{8}\right)(1+t^2),$$

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and then substituting from (1) we obtain

$$\text{(5)}\qquad \left(\xi+\frac{7+\sqrt{5}}{8}\right)\varepsilon^2+2\left(\frac{9-\sqrt{5}}{8}-\xi\right)\varepsilon\eta + \left(\xi+\frac{7+\sqrt{5}}{8}\right)\eta^2+\left(\frac{-17+7\sqrt{5}}{8} + \frac{5-\sqrt{5}}{2}\xi\right)\varepsilon+\left(\frac{-7+\sqrt{5}}{8}-\frac{5-\sqrt{5}}{2}\xi\right)\eta + \left(\frac{3-\sqrt{5}}{2}\right)\xi\le 0.$$

Similarly, from (3) and (1) we have

$$\text{(6)}\qquad \left(\frac{1}{4}+\xi\right)\varepsilon^2+2\left(\frac{2-\sqrt{5}}{4}-\xi\right)\varepsilon\eta + \left(\frac{1}{4}+\xi\right)\eta^2+\left(\frac{3-\sqrt{5}}{8}\right)\varepsilon+\left(\frac{3-\sqrt{5}}{8}\right)\eta - \left(\frac{3-\sqrt{5}}{8}\right)\xi\le 0.$$

Finally, from (4) and (1) we find that

$$\text{(7)}\qquad \eta-\varepsilon\le 0.$$

Next we multiply (5) by 8, we multiply (6) by 32, and adding the two inequalities that result we obtain

$$\text{(8)}\qquad a\varepsilon^2+2b\varepsilon\eta+a\eta^2\le c(\eta-\varepsilon),$$

where

$$a=15+\sqrt{5}+40\xi,\qquad b=25-9\sqrt{5}-40\xi,$$
$$c=-5+3\sqrt{5}+4(5-\sqrt{5})\xi.$$

It follows from (1) that $a>0$ and $c>0$. The discriminant of the quadratic form on the left of (8) is

$$4a^2-b^2=320(5-\sqrt{5})(\sqrt{5}-1+8\xi)$$

which is positive by (1). Thus the form is positive definite and so (7) and (8) can hold if and only if $\varepsilon=\eta=0$. But then (5) shows that $\xi\le 0$ and (6) implies $\xi\ge 0$. Hence the inequalities (2), (3) and (4) hold if and only if $U(x)=U^*(x)$ or $\overline{U(x)}=U^*(x)$.

References

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