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CONDITIONAL FLATNESS, FIBERWISE LOCALIZATIONS, AND ADMISSIBLE REFLECTIONS

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Abstract

We extend the group-theoretic notion of conditional flatness for a localization functor to any pointed category, and investigate it in the context of homological categories and of semi-abelian categories. In the presence of functorial fiberwise localization, analogous results to those obtained in the category of groups hold, and we provide existence theorems for certain localization functors in specific semi-abelian categories. We prove that a Birkhoff subcategory of an ideal determined category yields a conditionally flat localization, and explain how conditional flatness corresponds to the property of admissibility of an adjunction from the point of view of categorical Galois theory. Under the assumption of fiberwise localization, we give a simple criterion to determine when a (normal epi)-reflection is a torsion-free reflection. This is shown to apply, in particular, to nullification functors in any semi-abelian variety of universal algebras. We also relate semi-left-exactness for a localization functor L with what is called right properness for the L-local model structure.

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1. Introduction

In [18] the notion of *conditionally flat functor* was introduced by the second author and Farjoun in order to investigate pullback preservation properties related to homotopical localization functors. This was first done in the category of topological spaces, and then it was interpreted in the context of the category of groups, where short exact sequences replace fibration sequences. Given a fibration sequence

$$F \longrightarrow E \longrightarrow B \tag{1-1}$$

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of topological spaces and a morphism $X \to B$, a natural question is to study the properties of the original fibration (1-1) that are inherited by the pullback fibration along $X \to B$. Given a functor L, a fibration sequence (1-1) over a connected space B is called *L-flat* in analogy with the algebraic notion if the sequence

$$L(F) \longrightarrow L(E) \longrightarrow L(B)$$

is again a homotopy fibration sequence. There are several examples of such preservation in the literature on localizations in homotopy theory (see, for instance, [9, 13]). A functor L is then a *conditionally flat functor* when any pullback of an L-flat fibration is again L-flat. In the topological context, the property of conditional flatness was shown to characterize nullification functors among localization functors: for this, see [18, Theorem 2.1]), which follows from Berrick and Farjoun's work in [1] and relies on the existence of fiberwise localization. The latter is a construction which was already in use before 1980, but which May brought forward in [31], after noticing the key role it played in Sullivan's paper on the Adams conjecture [36].

When moving from the category of topological spaces to the category Grp of groups, fibration sequences were replaced by short exact sequences

$$0 \longrightarrow K \longrightarrow A \longrightarrow B \longrightarrow 0,$$

and the 'flatness property' of such exact sequences was then considered with respect to pullbacks along group homomorphisms $C \to B$. In the case of groups, a major difference with the case of topological spaces is that conditional flatness of a functor no longer characterizes nullification functors. These are conditionally flat, but so are all localization functors associated to a variety of groups in the sense of [33].

The aim of the present paper is then twofold. On the one hand, we widely extend the context from the category of groups to the abstract semi-abelian category (in the sense of Janelidze *et al.* [27]), and thus we include many algebraic examples such as the categories of rings, Lie algebras, crossed modules, compact groups [3] and cocommutative Hopf algebras [22]. We show in Proposition 4.4 that conditional flatness is characterized, in a general context, including any semi-abelian category, by the same properties as in the category of groups, as long as a fiberwise version of the localization functor is available. On the other hand, we establish a more useful criterion that implies conditional flatness, namely, that pullbacks of $\eta_C \colon C \to L(C)$ along any regular epimorphism in $\mathcal{C}$ between L-local objects (that is, objects lying in the reflective subcategory) should be inverted by L. This property is well known in category theory, and it has been used to investigate several adjunctions between algebraic categories. Indeed, the latter is exactly the property of *admissibility* of a reflection from the point of view of categorical Galois theory [25], as shown by Janelidze and Kelly [26] (see also [16] and the references therein). It is always true that a conditionally flat reflector induces an *admissible* reflection in the sense of categorical Galois theory.

Moreover, this condition actually turns out to be *equivalent* to the one of *admissibility* under the assumption of functorial fiberwise localization (see Proposition 5.5).

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We also offer one result which does not depend on the existence of fiberwise localization in the case of a reflection onto a Birkhoff subcategory (that is, when the reflective subcategory is closed under regular quotients and subobjects).

THEOREM 6.1. *When $\mathcal{X}$ is a Birkhoff subcategory of an ideal determined category C, the corresponding pointed endofunctor $L = UF: C \to C$ is conditionally flat.*

This result then applies to many interesting examples, since any subvariety of a semi-abelian variety [7] provides an example of Birkhoff subcategory.

Under the assumption of fiberwise localization, we then characterize torsion-free reflections $F: C \to \mathcal{X}$ in any homological category C among (normal epi)-reflections in terms of the property of stability under extensions of $\mathcal{X}$ in C (Proposition 7.1). This result applies, in particular, to any nullification functor in a semi-abelian variety of universal algebras (see [7] and Corollary 8.2). Note that, unlike in the abelian case, in the semi-abelian context a (normal epi)-reflective subcategory stable under extensions is not necessarily a torsion-free subcategory, as Janelidze and Tholen observed in [29].

In the final section of the paper, we adopt a homotopical viewpoint and revisit the results in terms of model categorical properties. Preservation of L-flatness under pullbacks is related here to semi-left-exactness, also known as right properness. This is already present in the pioneering work by Cassidy *et al.* [12], as explained by Rosický and Tholen in [34]. We mention also the article [37] by Wendt, where right properness of the L-local model structure is explicitly related to the work of Berrick and Farjoun [1]. It is also interesting to remark that the ∞-analogs of semi-left-exact-localizations studied by Gepner and Kock in [19] correspond to so-called locally cartesian localizations. In this setting, fiberwise methods (localization in slice categories) are always at hand and are heavily used.

The article is written for a readership of both category theorists and topologists.

2. Regular, homological and semi-abelian categories

2.1. Regular categories. Recall that a finitely complete category C is a *regular category* if the following two properties are satisfied.

- Any morphism can be factored into a regular epimorphism (that is, a coequalizer of a pair of parallel morphisms) followed by a monomorphism.
- Regular epimorphisms in C are stable under pullbacks: this means that the arrow π_1 in a pullback

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

is a regular epimorphism whenever f is a regular epimorphism.

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Recall that, in a regular category, regular epimorphisms compose and, moreover, if a composite $g \circ f$ is a regular epimorphism, then so is g.

2.2. Homological categories. When a regular category C is pointed (that is, it has a zero object, denoted by 0) one says that it is *homological* [2] if the *split short five lemma* holds in C: given a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\mathsf{ker}(f)"] \arrow[d, "u"'] & A \arrow[r, shift left, "f"] \arrow[d, "v"] & B \arrow[l, shift left, "s"] \arrow[d, "w"] \\
0 \arrow[r] & K' \arrow[r, "\mathsf{ker}(f')"'] & A' \arrow[r, shift left, "f'"] & B' \arrow[l, shift left, "s'"]
\end{tikzcd}$$

in C, where $f \circ s = 1_B$ and $f' \circ s' = 1_{B'}$, and $\mathsf{ker}(f)$ is the kernel of f and $\mathsf{ker}(f')$ is the kernel of f', then v is an isomorphism whenever u and w are isomorphisms. It is well known that this assumption implies, in particular, that any regular epimorphism is a cokernel (so that regular epimorphisms coincide with normal epimorphisms), and then the classical short five lemma holds in a homological category. This implies, in particular, the validity of the following useful proposition.

PROPOSITION 2.1 [2]. *Given a commutative diagram of short exact sequences in a homological category C,*

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\mathsf{ker}(f)"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "v"] & B \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & K' \arrow[r, "\mathsf{ker}(f')"'] & A' \arrow[r, "f'"'] & B' \arrow[r] & 0
\end{tikzcd}$$

u is an isomorphism if and only if the right-hand commutative square is a pullback.

2.3. Semi-abelian categories. A *semi-abelian* category [27] is a finitely cocomplete homological category such that every (internal) equivalence relation in C is a kernel pair. This means that C is an exact category (in the sense of Barr). Among the many examples of semi-abelian categories there are, for example, the categories of groups, Lie algebras, crossed modules [27], compact Hausdorff groups [3], nonunital rings, loops [3], cocommutative Hopf algebras over a field [22], nonunital C^*-algebras [21] and Heyting semi-lattices [30].

3. Localization and factorization systems

In this section, we work with a semi-abelian category C as defined above, even though many of the facts that we recall are valid in a more general setting. The main references here are Bousfield [8] for the homotopy theory viewpoint and the article [12] by Cassidy *et al.* for the categorical side.

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3.1. Factorization systems. A *prefactorization system* in C consists of classes of maps $\mathcal{E}$ and $\mathcal{M}$ determining each other by *unique* lifting properties or orthogonality properties. Thus, a morphism f belongs to $\mathcal{E}$ if and only if there is a unique filler in any commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

where p belongs to $\mathcal{M}$. We write $\mathcal{E} = {}^{\perp}\mathcal{M}$. Dually, $\mathcal{M} = \mathcal{E}^{\perp}$. A *factorization system* is a prefactorization system in which every map can be factored into a morphism in $\mathcal{E}$ followed by a morphism in $\mathcal{M}$.

3.2. Localization and reflectors. A pointed endofunctor $(L\colon C \to C, \eta\colon 1_C \Rightarrow L)$ is called idempotent if $L\eta\colon L \to LL$ is an isomorphism and $L\eta = \eta L$. It is common in algebraic topology to call *localization* an idempotent pointed endofunctor (L, η), and we adopt this terminology. Note, however, that in category theory the meaning of the term 'localization' is quite different; it means a reflective subcategory in which the reflector preserves (finite) limits. In this context, our localization functors are usually called idempotent monads since the inverse of $\eta L = L\eta$ defines a monad multiplication.

A factorization system $(\mathcal{E}, \mathcal{M})$ gives rise to a localization functor $L : C \to C$ by factoring the morphism $X \to 0$, as explained in [8, 2.5]. This is a coaugmented and idempotent functor, and any object X comes with a natural morphism $\eta_X : X \to L(X)$ to an object $L(X)$ having the property that $L(X) \to 0$ belongs to $\mathcal{M}$. Such an object is called *L-local*.

Conversely, when C is finitely well-complete, a localization functor L yields a factorization system with $\mathcal{E}(L)$ consisting of all L-equivalences, that is, morphisms turned into isomorphisms by L, and $\mathcal{M}(L) = \mathcal{E}(L)^{\perp}$. This is due to Cassidy *et al.* in [12, Corollary 3.4] (see also the more recent article by Salch, [35, Theorem 3.4], where the author has already rephrased the original results).

REMARK 3.1. There is a well-known one-to-one correspondence between idempotent monads and full reflective subcategories. However, there is no correspondence between localization functors and factorization systems, as shown in [12]. If one associates to a factorization system its canonical localization functor, and then applies the above construction to get a factorization system back, one does not, in general, recover the original factorization system, but gets its *reflective interior*.

3.3. Birkhoff subcategories. In our work, we are interested in the situation where $\mathcal{X}$ is a *Birkhoff subcategory* of a category C

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift right=2, "U"'] \arrow[r, phantom, "\perp"] & C \arrow[l, shift right=2, "F"'] \end{tikzcd}$$

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where U is the inclusion functor and F is its left adjoint. Being a Birkhoff subcategory means that $\mathcal{X}$ is a full (replete) and (regular epi)-reflective subcategory of C with the additional property that it is closed in C under regular quotients. Accordingly, each component $\eta_A: A \to UF(A)$ of the unit of the adjunction is a regular epimorphism and, moreover, $\mathcal{X}$ is also stable in C under regular quotients: if $A \twoheadrightarrow B$ is a regular epimorphism in C with A in $\mathcal{X}$, then B also belongs to $\mathcal{X}$.

Example 3.2. Any subvariety $\mathcal{X}$ of a variety C of universal algebras is a Birkhoff subcategory by the classical Birkhoff theorem. This applies to many situations: by adding *any* identity to the ones of a given algebraic theory one always determines a Birkhoff subcategory. This includes, of course, the classical examples of abelian groups or, more generally, of nilpotent or of solvable groups of a fixed class $\leq c$ in the category Grp of groups.

4. Conditional flatness

Our aim in this section is to study the notions of flatness and conditional flatness associated with a localization functor in a semi-abelian category. By analogy with the algebraic notion of flatness (tensoring by a flat ring preserves exactness), flatness for homotopy functors was defined by Farjoun and the second author in [18] in terms of preservation of fibration sequences. The same was done in the category of groups in terms of preservation of extensions.

In a pointed category C one can translate this definition as follows.

Definition 4.1. An extension $0 \to K \to E \to Q \to 0$ is called *L-flat* if the functor $L: C \to C$ sends it again to an extension: $0 \to L(K) \to L(E) \to L(Q) \to 0$.

The definition of conditional flatness from [18] still makes sense in any pointed category.

Definition 4.2. A functor $L: C \to C$ in a pointed category C is called *conditionally flat* if any pullback of an L-flat extension is again L-flat.

4.1. Fiberwise localization

Definition 4.3. Given a functor $L: C \to C$ in a pointed category C, we say that an extension $0 \to K \to E \to Q \to 0$ in C admits a *fiberwise localization* if there is a commutative diagram of horizontal extensions

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "\eta_K"] & E \arrow[r] \arrow[d, "e"] & Q \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L(K) \arrow[r] & \overline{E} \arrow[r] & Q \arrow[r] & 0 \end{tikzcd} \tag{4-1}$$

where $e : E \to \overline{E}$ is inverted by L. If the assignment $E \to \overline{E}$ is functorial (in the obvious sense), one says that it forms a *functorial fiberwise localization* for L.

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Any localization in the category Grp of groups enjoys fiberwise localization as shown by Casacuberta and Descheemaeker [10]. Other interesting examples will be considered at the end of this section, but there are also localization functors in certain homological categories that do not admit fiberwise versions (see [32]).

4.2. Pullbacks along reflections. We are now going to show that, in any homological category, the existence of a functorial fiberwise localization has an interesting consequence. The following result refines and generalizes [18, Proposition 4.1] from the category of groups to any homological category. The second, more amenable, condition describes admissible reflections in the sense of Janelidze–Kelly [26], as we discuss in the next section.

PROPOSITION 4.4. *Let $\mathcal{X}$ be a full reflective subcategory of a homological category C*

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] & C \arrow[l, shift right, "F"'] \end{tikzcd}$$

that admits a functorial fiberwise localization. Then the following conditions are equivalent.

(1) *The corresponding localization $(L = UF, \eta)$ is conditionally flat.*
(2) *The pullback of $\eta_C \colon C \to L(C)$ along any regular epimorphism in C between L-local objects is inverted by L.*

PROOF. Condition (1) clearly implies (2), and we prove that (2) implies conditional flatness of L. Let $0 \to K \to E \to Q \to 0$ be an L-flat extension and let $f : X \to Q$ be any arrow. First, we observe that, by applying fiberwise localization, there is no restriction in assuming that K is L-local. In order to see this, consider the right-hand pullback along f and the kernel κ of p_2

$$\begin{tikzcd} & & P \arrow[r, two heads, "p_2"] \arrow[d, "p_1"] & X \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & K \arrow[r, "k"'] \arrow[ur, dotted, "\kappa"] & E \arrow[r, two heads, "p"'] & Q \arrow[r] & 0 \end{tikzcd}$$

By using the functorial fiberwise localization of L, one gets the following commutative diagram of short exact sequences (here we use the same notation as in Definition 4.3).

$$\begin{tikzcd} & & \overline{P} \arrow[r, two heads] \arrow[d] & X \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & L(K) \arrow[r] \arrow[ur] & \overline{E} \arrow[r, two heads] & Q \arrow[r] & 0 \end{tikzcd}$$

In any homological category, the right-hand square in the above diagram is then again a pullback (this follows from Proposition 2.1, since C is assumed to be a homological category). If we write $\pi\colon P \to \overline{P}$ for the L-equivalence in the construction (4-1) of the exact sequence $0 \to L(K) \longrightarrow \overline{P} \longrightarrow X \to 0$ by fiberwise localization, we obviously have that $\eta_P \cong \eta_{\overline{P}} \circ \pi$, and this implies that the latter exact sequence is L-flat if and only if the sequence $0 \to K \longrightarrow P \longrightarrow X \to 0$ is so as well.

The next step is to reduce the proof to the case of an extension of L-local objects. This is done by noticing that, in the L-flat exact sequence $0 \to K \longrightarrow E \longrightarrow Q \to 0$, the kernel K can already be assumed to be L-local, so that the square

$$\begin{tikzcd} E \arrow[r, two heads] \arrow[d] & Q \arrow[d] \\ L(E) \arrow[r, two heads] & L(Q) \end{tikzcd}$$

is a pullback (we again use Proposition 2.1). Finally, we use the universal property of the localization and factor any map $X \to L(Q)$ through $\eta_X\colon X \to L(X)$ to decompose the pullback P of $L(E) \to L(Q)$ and $X \to L(Q)$ as the composite of two pullbacks:

$$\begin{tikzcd} P \arrow[r, two heads] \arrow[d] & X \arrow[d] \\ P' \arrow[r, two heads] \arrow[d] & L(X) \arrow[d] \\ L(E) \arrow[r, two heads] & L(Q) \end{tikzcd}$$

Here P' is a limit of L-local objects, and hence is L-local, and therefore yields, by regularity of C, another regular epimorphism $P' \to L(X)$ of L-local objects. The upper square is of the form required in order to apply assumption (2). $\square$

4.3. Existence of functorial fiberwise localization. In addition to the example of the category Grp of groups, there are many other examples of categories admitting functorial fiberwise localizations in certain circumstances. We focus from now on on localization functors for which the coaugmentation morphisms $\eta_X\colon X \to L(X)$ are always normal epimorphisms. In that case, we write $t_X\colon T(X) \to X$ for the kernel of η_X and identify the latter with the quotient map $X \to X/T(X)$.

PROPOSITION 4.5. *Let C be a homological category and let $L\colon C \to C$ be a localization functor such that any coaugmentation morphism $\eta_X\colon X \to L(X)$ is a normal epimorphism. Then C admits functorial fiberwise localization (with respect to L) if and only if one of the following conditions holds.*

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(1) *For any normal monomorphism $k\colon K \to E$ in C, the pushout of k along η_K exists*

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "\eta_K"'] & E \arrow[d] \\ L(K) \arrow[r, "\overline{k}"'] & \overline{E} \end{tikzcd} \tag{4-2}$$

and the square (4-2) is a pullback.

(2) *For any normal monomorphism $k\colon K \to E$ in C, the pushout (4-2) of k along η_K exists and the morphism $\overline{k}$ is a monomorphism.*

PROOF. First, let us assume that fiberwise localization exists. Given any extension

$$0 \longrightarrow K \xrightarrow{k} E \xrightarrow{f} B \longrightarrow 0$$

a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "k"] \arrow[d, "\eta_K"'] & E \arrow[r, "f"] \arrow[d, "e"] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L(K) \arrow[r, "\overline{k}"'] & \bar{E} \arrow[r, "g"] & B \arrow[r] & 0 \end{tikzcd}$$

exists by the assumption of fiberwise localization. The left-hand square is clearly a pullback, and the middle vertical morphism e is a regular epimorphism since the base category C is homological (see Proposition 8 in [5]). In this context, any pullback of a regular epimorphism along any morphism is a pushout (see [4]), and this proves that (1) holds. It is clear that (1) implies (2) since pullbacks reflect monomorphisms in C [4].

Now assume that (2) holds. In particular, one has the lower left-hand pushout in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "k"] \arrow[d, "\eta_K"'] & E \arrow[r, "f"] \arrow[d, "\pi"] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K/T(K) \arrow[r, "\overline{k}"'] & \overline{E} \arrow[r, dotted, "\overline{f}"'] & B \arrow[r] & 0 \end{tikzcd} \tag{4-3}$$

where $\overline{k}$ is a monomorphism, by assumption. Since $f \circ k \circ t_K = 0$, the universal properties of the cokernel η_K, and then of the left-hand pushout, yield a unique arrow $\overline{f}\colon \overline{E} \to B$ making the right-hand square above commute. The canonical factorization ϕ from $K/T(K)$ to the kernel $\mathsf{Ker}(\overline{f})$ of $\overline{f}$ such that $\mathsf{ker}(\overline{f}) \circ \phi = \overline{k}$ is then a monomorphism (since $\overline{k}$ is a monomorphism). It is also a regular epimorphism, since

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so is $\phi \circ \eta_K$, the latter being the pullback of π along $\mathsf{ker}(\overline{f})$. It follows that ϕ is an isomorphism, and the lower sequence in diagram (4-3) is then exact.

The proof will be then complete if we show that π is inverted by L. For this, consider the commutative diagram

$$\begin{tikzcd}
 & T(K) \arrow[d, "T(k)"'] & & & \\
0 \arrow[r] & T(E) \arrow[r, "t_E"] \arrow[d, "q"'] & E \arrow[r, "\eta_E"] \arrow[d, "\pi"] & L(E) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T(E)/T(K) \arrow[r, dotted, "j"'] & \overline{E} \arrow[r, dotted, "p"'] & L(E) \arrow[r] & 0
\end{tikzcd}$$

where:

- $T(k)$ is a normal monomorphism since so is $t_E \circ T(k) \colon T(K) \to E$ (the assumption that $\overline{k}$ is a monomorphism implies that $T(K)$ is the intersection $T(E) \cap \mathsf{Ker}(\pi)$ of two normal monomorphisms) and t_E is a monomorphism;
- q is the quotient of $T(E)$ by $T(K)$;
- j is the unique morphism such that $j \circ q = \pi \circ t_E$; and
- p is the unique morphism such that $p \circ \pi = \eta_E$.

Now, if $f \colon \overline{E} \to A$ is any morphism with A a local object, then there is a unique morphism $\psi \colon L(E) \to A$ such that $\psi \circ \eta_E = f \circ \pi$. This morphism ψ is also the unique one such that $\psi \circ p = f$ (since π is an epimorphism), which proves that $p = \eta_{\overline{E}}$ (and $L(E) = L(\overline{E})$), so that π is indeed inverted by L. One can easily check that this construction is functorial, and this completes the proof. □

REMARK 4.6. In any homological category, a property equivalent to properties (1) and (2) used in Proposition 4.5 consists of requiring that the monomorphism

$$k \circ t_K \colon T(K) \to K \to E$$

is normal. This was called condition (N) in [15]. The fact that (N) is equivalent to condition (1) easily follows from Proposition 2.1 by choosing the quotient $E/T(K)$ as $\overline{E}$ in diagram (4-2). We used condition (N) in a previous version of the article, and we thank the referee for suggesting to also consider condition (2) in the above proposition. The equivalent property (N) will now be useful in the following examples.

EXAMPLE 4.7. Proposition 4.5 can be applied to any homological category, and hence, in particular, to the category $\mathsf{Grp}(\mathsf{Top})$ of topological groups [3] which has the property that any regular epimorphism is normal. Consider a localization functor $L \colon \mathsf{Grp}(\mathsf{Top}) \to \mathsf{Grp}(\mathsf{Top})$ for which each coaugmentation morphism η_X of the localization is a normal epimorphism (= open surjective group homomorphism).

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Given any short exact sequence

$$0 \longrightarrow K \longrightarrow E \longrightarrow B \longrightarrow 0 \tag{4-4}$$

in $\mathsf{Grp}(\mathsf{Top})$, by taking the kernel $t_K\colon T(K) \to K$ of the unit η_K of any localization, one obtains a *characteristic* subgroup $T(K)$ of K (see, for instance, Example 2.2 in [15]). Accordingly, the subgroup $T(K)$ is also normal in E, and hence condition (N) in Remark 4.6 holds, as desired. The same observation also applies to the category $\mathsf{Grp}(\mathsf{Haus})$ of Hausdorff groups.

EXAMPLE 4.8. Now let $\mathsf{Hopf}_{A,coc}$ be the category of cocommutative Hopf algebras over a field A, which was shown to be semi-abelian in [22]. Given an extension (4-4), by using the same notation as above, the Hopf subalgebra $T(K) \to K \to E$ induced by any localization functor $L\colon \mathsf{Hopf}_{A,coc} \to \mathsf{Hopf}_{A,coc}$ is also a *normal* Hopf subalgebra of E. Indeed, denote by $S\colon K \to K$ the antipode of E and denote by $\phi_e\colon K \to K$ the map defined, for any $e \in E$, by $\phi_e(t) = e_1 t S(e_2)$ for any $t \in K$. Here we use the usual Sweedler convention for Hopf algebras, so that we write $\Delta(e) = e_1 \otimes e_2$, with $\Delta\colon E \to E \otimes E$ being the comultiplication. This map $\phi_e\colon K \to K$ is seen to be a Hopf algebra morphism. By functoriality of the natural transformation $t\colon T \Rightarrow \mathsf{id}_C$, it follows that ϕ_e restricts to $T(K)$, which yields a morphism $T(K) \to T(K)$. This means that, for any $t \in T(K)$, $\phi_e(t) \in T(K)$, and hence $T(K)$ is normal in E, and condition (N) then holds. We conclude that, whenever we have a localization functor $L\colon \mathsf{Hopf}_{A,coc} \to \mathsf{Hopf}_{A,coc}$ with the property that the coaugmentation morphism $\eta_X\colon X \to L(X)$ is a normal epimorphism, the category $\mathsf{Hopf}_{A,coc}$ admits fiberwise localization. For instance, the 'abelianization functor' $\mathsf{ab}\colon \mathsf{Hopf}_{A,coc} \to \mathsf{Hopf}^{comm}_{A,coc}$, as described in Section 4 in [22], necessarily yields a functorial fiberwise localization, with $L = U\,\mathsf{ab}\colon \mathsf{Hopf}_{A,coc} \to \mathsf{Hopf}_{A,coc}$ (here $U\colon \mathsf{Hopf}^{comm}_{A,coc} \to \mathsf{Hopf}_{A,coc}$ is the forgetful functor).

REMARK 4.9. One might hope that similar results hold whenever one is dealing with a category of internal groups in a finitely complete category, since the examples mentioned above, such as groups, topological groups, Hausdorff groups and cocommutative Hopf algebras, are of this type (the category $\mathsf{Hopf}_{A,coc}$ can also be seen as the category of internal groups in the category of cocommutative coalgebras). However, this is not the case, as follows from the results in [32], where some counterexamples are given in the semi-abelian category XMod of crossed modules, which can be seen also (up to a category equivalence) as the category $\mathsf{Grp}(\mathsf{Cat})$ of internal groups in the category Cat of (small) categories (see, for instance, [27] and the references therein).

5. Admissible reflectors with respect to Galois theory

The type of pullbacks that appear in Proposition 4.4 are the ones appearing in categorical Galois theory, in the form presented in the article [26] by Janelidze and Kelly. In the whole section, we work in a homological category C, where we fix a full

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reflective subcategory $\mathcal{X}$ as in

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "F"'] \end{tikzcd}$$

and the corresponding localization functor $L = UF: \mathcal{C} \to \mathcal{C}$.

Definition 5.1 [26]. The reflector $F: \mathcal{C} \to \mathcal{X}$ is *admissible* for the class of regular epimorphisms if it preserves any pullback of the form

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U(E) \arrow[d, "U(x)"] \\ X \arrow[r, "\eta_X"'] & UF(X) \end{tikzcd} \tag{5-1}$$

where $x: E \to FX$ is a regular epimorphism in $\mathcal{X}$.

One could also require F to preserve pullbacks as above for *any* morphism x in $\mathcal{X}$, in which case we are looking at semi-left-exact reflections as introduced by Cassidy *et al.* in [12]. We come back to this in Section 7.

Definition 5.2 [12]. Let $\mathcal{X}$ be a (normal epi)-reflective subcategory of $\mathcal{C}$. Then F is *semi-left-exact*, that is, F preserves all pullbacks of the form

$$\begin{tikzcd} P \arrow[r, two heads, "p_2"] \arrow[d, "p_1"'] & U(E) \arrow[d, "x"] \\ X \arrow[r, two heads, "\eta_C"'] & UF(X) \end{tikzcd}$$

where x is any morphism in $\mathcal{X}$.

In other words, the morphism $P \to U(E)$ coincides with the P-component of the unit $\eta_P: P \to UF(P)$ of the adjunction (up to unique isomorphism). There are several equivalent ways to characterize admissibility, as we recall in the next proposition. They hold, in particular, for all the examples of semi-localizations of semi-abelian categories given in [20].

Proposition 5.3. *Let $\mathcal{C}$ be a homological category, let $F: \mathcal{C} \to \mathcal{X}$ be a reflector and let $(L = UF, \eta)$ be the corresponding localization. The following conditions are then equivalent.*

(1) *The reflector F is admissible for the class of regular epimorphisms.*
(2) *The pullback of $\eta_C: C \to L(C)$ along any regular epimorphism in $\mathcal{C}$ between L-local objects is inverted by L.*

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(3) *The functor $L = UF\colon \mathcal{C} \to \mathcal{C}$ preserves any pullback of the form*

$$\begin{tikzcd} C \times_{L(C)} X \arrow[r, two heads, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, two heads, "g"] \\ C \arrow[r, two heads, "\eta_C"'] & UF(C) \end{tikzcd}$$

where g is a regular epimorphism in $\mathcal{C}$ between objects in $\mathcal{X}$.

PROOF. The equivalence (2) $\Leftrightarrow$ (3) is obvious, while the equivalence between (3) and (1) follows easily from the fact that the functor $U\colon \mathcal{X} \to \mathcal{C}$ reflects limits, since it is a fully faithful right adjoint. $\square$

PROPOSITION 5.4. *If $L = UF\colon \mathcal{C} \to \mathcal{C}$ is a conditionally flat functor, then the reflector $F\colon \mathcal{C} \to \mathcal{X}$ is admissible for the class of regular epimorphisms.*

PROOF. Let K be the object part of the kernel of the vertical morphism $U(x)$ in (5-1). This is the object part of a limit of a diagram lying in $\mathcal{X}$, and hence it lies itself in $\mathcal{X}$. The extension

$$0 \longrightarrow K \longrightarrow U(E) \xrightarrow{Ux} UF(X) \longrightarrow 0$$

is thus L-flat. If L is conditionally flat, then the (induced) pullback extension

$$0 \longrightarrow K \longrightarrow P \longrightarrow X \longrightarrow 0$$

must be L-flat as well. This means that $L = UF$ takes it to an extension

$$0 \longrightarrow K \longrightarrow UF(P) \longrightarrow UF(X) \longrightarrow 0$$

where $K \cong UF(K)$ remains unchanged since it lies in $\mathcal{X}$. This extension comes with a natural transformation to the original extension.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, equal] & UF(P) \arrow[r] \arrow[d, dotted] & UF(X) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K \arrow[r] & U(E) \arrow[r] & UF(X) \arrow[r] & 0 \end{tikzcd}$$

We conclude by the short five lemma (see [2]) that the middle dotted arrow is an isomorphism, and the arrow $P \to U(E)$ in the pullback (5-1) is then isomorphic to the unit $\eta_P\colon P \to UF(P)$. This means that the reflector is admissible with respect to the class of regular epimorphisms, as desired. $\square$

We can also reinterpret Proposition 4.4 as follows.

PROPOSITION 5.5. *Let $\mathcal{C}$ be a homological category and assume that the localization functor L admits a functorial fiberwise localization. Then the functor L is conditionally flat if and only if it is admissible with respect to regular epimorphisms.*

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EXAMPLE 5.6. It is well known that any Birkhoff subcategory of a semi-abelian category induces an admissible reflector with respect to regular epimorphisms [14, 26]. Together with the remarks in Section 4.3, this implies, in particular, that any Birkhoff subcategory of the category $\mathsf{Hopf}_{A,coc}$ of cocommutative Hopf algebras over a field A induces a conditionally flat functor L. In the semi-abelian category $\mathsf{Grp}(\mathsf{Comp})$, its Birkhoff subcategory $\mathsf{Grp}(\mathsf{Prof})$ of profinite groups also induces a conditionally flat functor, since the adjunction is admissible [16].

6. The case of Birkhoff subcategories

We restrict our attention to a Birkhoff subcategory $\mathcal{X}$ of a regular category C, as in Section 3.3. The suitable context in which to obtain the result of this section is the one of *ideal determined* categories, as introduced in [28] by Janelidze *et al.* These are regular categories C with binary coproducts such that:

(1) any regular epimorphism in C is *normal* (that is, a *cokernel*); and
(2) *normal monomorphisms are stable under images*; in any commutative square

$$\begin{tikzcd} A \arrow[r, tail, "a"] \arrow[d, two heads, "f"'] & A' \arrow[d, two heads, "f'"] \\ B \arrow[r, tail, "b"'] & B' \end{tikzcd}$$

in C where f and f' are normal epimorphisms, a is a normal monomorphism and b is a monomorphism, b is also a normal monomorphism.

As explained in [28] any semi-abelian category is ideal determined. In particular, all the examples mentioned before (groups, loops, rings, commutative algebras, associative algebras, cocommutative Hopf algebras, crossed modules, compact groups, $\mathsf{C}^\star$-algebras, and so on) are ideal determined. There are also some examples of ideal determined varieties that are not semi-abelian, for instance, the variety of implication algebras [23].

The following theorem gives a natural condition that guarantees the conditional flatness of the pointed endofunctor L, without the toolkit of fiberwise localization.

THEOREM 6.1. *When $\mathcal{X}$ is a Birkhoff subcategory of an ideal determined category C, the corresponding pointed endofunctor $L = UF : C \to C$ is conditionally flat.*

PROOF. We prove that $L : C \to C$ is conditionally flat. We consider an L-flat extension

$$0 \longrightarrow K \xrightarrow{k} E \xrightarrow{f} X \longrightarrow 0 \tag{6-1}$$

and a morphism $g : A \to X$ in C. We construct the pullback of the original extension along g and need to prove that this extension $0 \to K \longrightarrow P \xrightarrow{p_2} A \to 0$ is again L-flat.

We know that the induced arrow $L(p_2) : L(P) \to L(A)$ is a normal epimorphism, since the arrow $\eta_A \circ p_2 : P \to L(A)$ is a normal epimorphism as it is a composite of two

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normal epimorphisms (see Section 2.1). We now prove that the arrow $L(K) \to L(P)$ is the kernel of $L(p_2)\colon L(P) \to L(A)$. First, observe that the arrow $L(K) \to L(P)$ is a monomorphism, since the arrow

$$L(K) \to L(P) \to L(E) = L(K) \xrightarrow{L(k)} L(E)$$

is a monomorphism (the original extension (6-1) being L-flat). Since the category C is ideal determined and the square

$$\begin{tikzcd} K \arrow[r, two heads, "\eta_K"] \arrow[d] & L(K) \arrow[d] \\ P \arrow[r, two heads, "\eta_P"'] & L(P) \end{tikzcd}$$

is commutative with $K \to P$, which is a normal monomorphism, and $L(K) \to L(P)$, which is a monomorphism, it follows that the arrow $L(K) \to L(P)$ is a normal monomorphism as well. Consequently, $L(K) \to L(P)$ is the kernel of its cokernel $q\colon L(P) \to Q$ in C. However, this latter is isomorphic to $L(p_2)\colon L(P) \to L(A)$. Indeed, this follows from the fact that the functor $F\colon C \to \mathcal{X}$ preserves cokernels (as it is a left adjoint) while $U\colon \mathcal{X} \to C$ preserves them since $\mathcal{X}$ is closed in C under (regular) quotients, by the Birkhoff assumption. □

7. Fiberwise localizations and stability under extensions

In this section, we show that, when C is homological, torsion-free reflections $F\colon C \to \mathcal{X}$ can be characterized among (normal epi)-reflections admitting fiberwise localization in terms of the property of *stability under extensions* of $\mathcal{X}$ in C. We recall that a torsion-free reflection is associated to a torsion theory (see, for example, [20, Definition 1.1]). In particular, the only morphism from a torsion object to a local object is the zero morphism.

Recall that a full (replete) subcategory $\mathcal{X}$ of a pointed category C is stable under extensions (in C) if, given any short exact sequence

$$0 \longrightarrow K \longrightarrow X \longrightarrow Y \longrightarrow 0 \tag{7-1}$$

in C with K and Y in $\mathcal{X}$, we have that X is also in $\mathcal{X}$.

Proposition 7.1. *Let C be a homological category and let $\mathcal{X}$ be a (normal epi)-reflective subcategory of C with the property that the reflector $F\colon C \to \mathcal{X}$ admits fiberwise localization. We write $T(X)$ for the kernel of the X-reflection $\eta_X\colon X \to F(X)$ of any X in C. Then the following conditions are equivalent.*

(1) *$\mathcal{X}$ is stable in C under extensions.*
(2) *$F(T(X)) = 0$ for any object X in C.*
(3) *F is semi-left-exact.*
(4) *$\mathcal{X}$ is a torsion-free subcategory in C.*

PROOF. (1) ⇒ (2) We consider the short exact sequence

$$0 \longrightarrow T(X) \xrightarrow{t_X} X \xrightarrow{\eta_X} F(X) \longrightarrow 0 \tag{7-2}$$

and the associated exact sequence $0 \to F(T(X)) \to \overline{X} \to F(X) \to 0$ that exists by the assumption of fiberwise localization. Since the subcategory $\mathcal{X}$ is closed in C under extensions, $\overline{X}$ is in $\mathcal{X}$ and the fiberwise morphism $X \to \overline{X}$ is, therefore, an F-equivalence to an object in $\mathcal{X}$. Thus, it must be $\eta_X : X \to F(X)$ (up to isomorphism). This implies that the kernel $F(T(X))$ of the morphism $\overline{X} \to F(X)$ is zero.

(2) ⇒ (3) and (3) ⇒ (4) both follow from Theorem 4.12 in [6].

(4) ⇒ (1) We briefly recall the known argument showing that a torsion-free subcategory $\mathcal{X}$ is closed under extensions in C. Given a short exact sequence (7-1) with K and Y in $\mathcal{X}$, consider the canonical short exact sequence (7-2), where $T(X)$ is torsion and $F(X)$ is torsion-free. Clearly, $T(X) \to X \to Y$ is the zero morphism, and hence t_X factors through K. Since $T(X)$ is a subobject of K, $T(X) \in \mathcal{X}$ ($\mathcal{X}$ is closed under subobjects). Since it is also in the torsion subcategory, $T(X) \cong 0$ and $X \cong F(X) \in \mathcal{X}$, as desired. □

Unlike in the abelian case, in homological categories, the property of stability under extensions of a (normal epi)-reflective subcategory $\mathcal{X}$ is not strong enough to guarantee that $F : C \to \mathcal{X}$ is a reflector to a torsion-free subcategory, as observed in [29]. The lemma above shows that, under the assumption of fiberwise localization, this is indeed the case.

REMARK 7.2. From Propositions 4.4 and 7.1 above we deduce that, under the assumption of functorial fiberwise localization, any semi-left-exact reflector $F : C \to \mathcal{X}$ gives rise to a corresponding conditionally flat localization $L = UF : C \to C$. The converse does not hold however, even in the case of a (normal epi)-reflection associated to a Birkhoff subcategory, as illustrated in the following classical example in the category of groups.

EXAMPLE 7.3. We write L_{ab} for the abelianization functor. The dihedral group D_8 of order eight abelianizes to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, an elementary abelian 2-group of rank two. Consider the following pullback in the category of groups.

$$\begin{tikzcd}
\mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\
D_8 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}
\end{tikzcd}$$

The vertical morphism on the right-hand side is a homomorphism of abelian groups and the bottom morphism is the abelianization morphism of D_8. Its pullback, however, is the map $\mathbb{Z}/2\mathbb{Z} \to 0$, which is not the abelianization morphism for $\mathbb{Z}/2\mathbb{Z}$. Since fiberwise localization always exists in the category Grp of groups, Proposition 7.1 applies and tells us that the above problem reflects the fact that the subcategory Ab of

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abelian groups is not closed under extensions in Grp and it is not torsion-free (in the categorical sense).

The fact that the abelianization functor is not semi-left-exact is well known. The ‘relative version’ of the Galois theory, which was developed by Janelidze in [25] and later in collaboration with Kelly in [26], where the class of morphisms to be classified by the Galois theorem is the one of regular epimorphisms, was partly motivated by the possibility of applying their approach to any Birkhoff subcategory of a ‘sufficiently good’ algebraic category. Here ‘sufficiently good’ could mean being a semi-abelian variety of universal algebras [7], for instance, yielding many examples of interest in algebra.

8. The case of nullifications

The results of the previous section apply to nullification functors. Let C be a semi-abelian category, let A be an object in C and define $\mathcal{X} \subset C$ to be the (replete) reflective subcategory of *A-null* objects, that is, of those objects Z such that $\mathrm{Hom}(A, Z) = 0$.

When it exists, the associated localization functor is written P_A and is called *A-nullification* (or *A-periodization*). The construction is due to Bousfield in a homotopical setting and can be found, for example, in Hirschhorn’s [24]. A reference in an algebraic context is Casacuberta *et al.* [11, Theorem 1.4]. In all cases, $P_A(X)$ is constructed as a transfinite filtered colimit of iterated quotients of all morphisms from A. A cardinality argument is invoked to explain when one can stop the iteration.

In the recent preprint [32], Monjon *et al.* gave, in Proposition 2.7, an explicit construction of the nullification functor in the (semi-abelian) category of crossed modules. By looking at the arguments in their proof, one realizes that these still apply to any semi-abelian variety of universal algebras [7]. These are precisely those varieties (= finitary equational classes) whose algebraic theories have a unique constant 0, $n \geq 1$ binary terms $\alpha_i(x, y)$ and one $(n + 1)$ary term β satisfying the identities $\alpha_i(x, x) = 0$ (for $i \in \{1, \dots, n\}$), $\beta(\alpha_1(x, y), \dots, \alpha_n(x, y), y) = x$. For example, in the case of the variety of groups, by using the multiplicative notation for the group operation, one can choose $0 = 1$, $\alpha_1(x, y) = x \cdot y^{-1}$ and $\beta(x, y) = x \cdot y$. Note that, for a variety of universal algebras, being homological or being semi-abelian are equivalent properties, since a variety is always Barr-exact and cocomplete. We work here with *sets* equipped with finitary operations satisfying a *set* of identities, so set-theoretic arguments are available. Moreover, any variety of universal algebras is cocomplete. Hence, the proof of [32, Proposition 2.7] applies.

PROPOSITION 8.1. *Let C be a semi-abelian variety of universal algebras and let A be an object of C. Then the A-nullification functor P_A exists and the coaugmentation morphism $\eta_X : X \to P_A(X)$ is a normal epimorphism, for any object X.*

PROOF. We only need to note that the construction yields a surjective coaugmentation morphism, which is thus a regular epimorphism. The semi-abelian assumption on C then implies that $\eta_X : X \to P_A(X)$ is actually a normal epimorphism. □

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We now show that, in the presence of fiberwise localization, nullification functors are conditonally flat, in fact, even semi-left-exact. We write $\overline{P}_A(X)$ for the kernel of the A-nullification $\eta_X \colon X \to P_A(X)$. The equivalent characterization from Proposition 7.1(2) that $P_A(\overline{P}_A(X)) = 0$ for any object X in C is an algebraic analog of Farjoun's [17, Theorem 1.H.2].

COROLLARY 8.2. *Consider a nullification functor P_A on a semi-abelian variety of universal algebras C and assume that P_A admits a functorial fiberwise localization. Then P_A is semi-left-exact. In particular, P_A is conditionally flat.*

PROOF. In view of Proposition 7.1, it is sufficient to verify one of the equivalent conditions. By definition of A-local objects, it is easy to see that they are closed under extensions. Hence, P_A, which exists by Proposition 8.1, is semi-left-exact, a property which is stronger than admissibility for all regular epimorphisms. We conclude by Proposition 5.5 that P_A is conditionally flat. □

9. A model categorical interpretation

In this article, we study how pullbacks of exact sequences behave and, in the previous sections, we relate this to semi-left-exactness, which is a stronger admissibility property (preservation of pullbacks along *any* morphism between local objects versus preservation of pullbacks along any regular epimorphism between local objects). From a model theoretic perspective, this corresponds to right properness, as we explain next.

Any category with finite limits and colimits admits a *discrete* model structure where weak equivalences are isomorphisms and all morphisms are fibrations and cofibrations. This easy observation has been previously made by Bousfield [8, Examples 2.3], who also constructed new model structures where the class of weak equivalences is $\mathcal{E}(L)$ that is all morphisms inverted by a localization functor L, that is, L-equivalences. Cofibrations do not change and the class of fibrations coincides now with $\mathcal{M}(L)$ (using the notation from Section 3). This is not immediately obvious as we require the lift to be unique in a factorization system, but not in a model category. This is because the model categorical lift is unique up to homotopy in the associated homotopy factorization system and, in the discrete setting, 'unique up to homotopy' means unique.

This process is called left Bousfield localization; we cite Salch [35, Proposition 3.5] for a statement in line with the present work. Our final propositions are just reformulations of the fact that a semi-left-exact reflection is also characterized by the property that, for the induced factorization system $(\mathcal{E}, \mathcal{M})$, the morphisms in $\mathcal{E}$ are stable under pullback along morphisms in $\mathcal{M}$.

PROPOSITION 9.1 [Salch]. *Let C be a finitely cocomplete, finitely well-complete category and let L be a localization functor. There is an L-local model structure on C where weak equivalences are the L-equivalences $\mathcal{E}(L)$, all morphisms are cofibrations and the class of fibrations is $\mathcal{M}(L)$.*

In a model category, it is a direct consequence of the axioms that the pullback of a fibration is a fibration. But weak equivalences need not be preserved by pullbacks, not even by homotopy pullbacks. A model category is *right proper* if the pullback of any weak equivalence $X \to B$ along any fibration $E \twoheadrightarrow B$ is a weak equivalence. The discrete model structure is right proper since the pullback of an isomorphism along any map is an isomorphism.

Now, given a localization functor L on C, it is then a natural question to ask when the left Bousfield localized model structure described in Proposition 9.1 is again right proper. Rosický and Tholen noticed in [34, 3.6] that a result by Cassidy *et al.* [12, Theorem 4.3] allows one to characterize right proper localized model structures as those corresponding to semi-left-exact reflections.

PROPOSITION 9.2 [Rosický–Tholen]. *Let C be a finitely complete category and let L be a localization functor. The L-local model structure is right proper if and only if L is semi-left-exact.*

Therefore, Corollary 8.2 tells us that the P_A-local model structure is right proper, which is an analog of the well-known fact that nullification functors in spaces yield a right proper left Bousfield localized model structure [1] (see also Wendt [37, Corollary 6.1] for simplicial sheaves on a site). However, in an algebraic setting, conditional flatness is different from right properness because pulling back an L-equivalence along a regular epimorphism is not as general as pulling back along an arbitrary fibration, that is, an arbitrary morphism in the localized model structure.

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