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ON TATE-DRINFELD MODULES

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ABSTRACT. The Tate-Drinfeld module is defined by Gekeler. We define the Tate-Drinfeld map and show the analogous properties concerning Tate elliptic curves and Tate map.

0. **Introduction.** The analogy between number fields and function fields is an important subject in number theory. In 1973, Drinfeld introduced the notion of elliptic modules, now called Drinfeld modules, in function fields to replace elliptic curves in the classical case. Then Goss, Gekeler and others used Drinfeld modules to investigate arithmetic properties of function fields, which are analogous to those of number fields.

In [4], Gekeler defined the Tate-Drinfeld module using t-expansions to see the congruence properties of Drinfeld modular forms. In this article we define the Tate-Drinfeld map analogous to the Tate map in the classical case. In the classical case, the Tate map gives an isomorphism between the Tate elliptic curve and the multiplicative group L^* of a complete field modulo a free group generated by $q \in L$ with $|q| < 1$. In our case we change the A-module structure on L using the Carlitz module. Then the Tate-Drinfeld map is an A-module homomorphism from L with this A-module structure to L with the A-module structure given by the Tate-Drinfeld module, and the kernel is a free A-submodule of L generated by t^{-1} when $|t|$ is small enough.

1. **Preliminiaries.** Let $A = \mathbf{F}_q[T]$, $K = \mathbf{F}_q(T)$, $K_\infty = \mathbf{F}_q\big((T^{-1})\big)$, and C be the completion of algebraic closure of K_∞. Let L be an A-field, that is, there is a structure map

$$\gamma : A \longrightarrow L.$$

Let $\mathbf{G}_a$ be the additive group scheme, and $\mathrm{End}_L(\mathbf{G}_a)$ the ring of endomorphisms of $\mathbf{G}_a$ which are defined over L. Then $\mathrm{End}_L(\mathbf{G}_a)$ is isomorphic to $L\{\tau_p\}$, the noncommutative polynomial ring, where τ_p is the Frobenius map

$$\tau_p(x) = x^p.$$

It is known that any injective ring homomorphism

$$\phi : A \longrightarrow \mathrm{End}_L(\mathbf{G}_a) = L\{\tau_p\}$$

has its values in $L\{\tau\}$ where $\tau = \tau_p^s$ with $q = p^s$.

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By a Drinfeld module of rank r we mean an injective ring homomorphism

$$\phi : A \longrightarrow \mathrm{End}_L(\mathbf{G}_a)$$
$$a \longrightarrow \phi_a$$

such that for all $a \in A$, we have

(i) $\deg \phi_a = r \cdot \deg a$

(ii) $D(\phi_a) = \gamma(a)$

where $\deg \phi_a$ is the degree of ϕ_a in τ and $D(\phi_a)$ is the coefficient of the constant term of ϕ_a.

Because ϕ is a ring homomorphism, a Drinfeld module ϕ is completely determined by

$$\phi_T = \gamma(T) + a_1\tau + a_2\tau^2 + \cdots + a_r\tau^r$$

where $a_i \in L$ and $a_r \neq 0$. We can write this additively by

$$\phi_T(X) = \gamma(T)X + a_1X^q + a_2X^{q^2} + \cdots + a_rX^{q^r}.$$

By a *homomorphism* u from a Drinfeld module ϕ to another Drinfeld module ϕ' we mean an element $u \in \mathrm{End}_L(\mathbf{G}_a)$ with the property

$$u \circ \phi_a = \phi_a \circ u$$

for all $a \in A$. A homomorphism u is called an *isomorphism* if u is an automorphism of $\mathbf{G}_a$, that is, a constant different from 0. A nonzero homomorphism can exist only between Drinfeld modules of the same rank, and is called an *isogeny*.

We now review some results in the theory in Drinfeld modules over C. Let Λ be an A-lattice in C, which is defined to be a projective A-submodule of finite rank in C and discrete in the topology of C. Define the exponential function

$$e_\Lambda : C \longrightarrow C$$

given by

$$e_\Lambda(z) = z \prod_{\lambda \in \Lambda - \{0\}} \left(1 - \frac{z}{\lambda}\right).$$

Then it is known that e_Λ is entire, $\mathbf{F}_q$-linear and surjective. To each A-lattice Λ in C, we associate a Drinfeld module ϕ by the following commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & C & \xrightarrow{e_\Lambda} & C & \longrightarrow & 0 \\ & & a\big\downarrow & & a\big\downarrow & & \big\downarrow\phi_a & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & C & \xrightarrow{e_\Lambda} & C & \longrightarrow & 0. \end{array}$$

The association gives a 1-1 correspondence between the set of Drinfeld modules of rank r and the set of rank r A-lattices in C. Then the set of isomorphism classes of Drinfeld modules of rank r over C is isomorphic to the set of similarity classes of rank r A-lattices in C.

In this article, we are mainly interested in the Drinfeld modules of rank 2. A Drinfeld module ϕ of rank 2 is determined by

$$\phi_T(X) = \gamma(T)X + gX^q + \Delta X^{q^2}.$$

The quantity $j = g^{q+1}/\Delta$ is called the *j-invariant* of ϕ and j identifies the isomorphism class of Drinfeld modules of rank 2 with the affine j-line. By the above analytic description of Drinfeld modules by lattices, the set of isomorphism classes of Drinfeld modules of rank 2 over C is parameterised by $\mathrm{GL}_2(A)\backslash\Omega$, where $\Omega = C - K_\infty$, the algebraists upper half plane. Then we can speak of modular forms and modular functions (For precise definitions of these, see [4]). In fact, g and Δ are modular forms of weight $q-1$ and q^2-1, respectively, and j is a modular function, viewing them as functions on Ω.

Let ρ be the rank 1 Drinfeld module, which is often called the *Carlitz module*, given by

$$\rho_T(X) = TX + X^q.$$

Then ρ corresponds to the lattice $L = \bar{\pi}A$ for $\bar{\pi} \in C$. Let $t = e_L^{-1}(\bar{\pi}z)$. Then each modular form f has t-expansion

$$f = \sum a_i t^i$$

with a positive radius of convergence. In fact, only powers of t^{q-1} can appear.

Let $g_{\text{new}} = \bar{\pi}^{1-q}g$ and $\Delta_{\text{new}} = \bar{\pi}^{1-q^2}\Delta$. From now on, we write g and Δ for g_{new} and Δ_{new}, respectively, for simplicity. It is shown in [4] that the t-expansions of g and Δ lie in $A[[t]]$.

Define the *Tate-Drinfeld module* TD over a complete field L by

$$\mathrm{TD}_T(X) = TX + g(t)X^q + \Delta(t)X^{q^2}.$$

This is well-defined if $|t| < \delta'$ where δ' is the minimum of the radii of convergences of g and Δ.

2. **Main results.** In this section, we will show that the Tate-Drinfeld module TD is a nice analog of Tate-elliptic curve in the classical case. Let

$$\delta = \min\{\delta', 1\}.$$

For each rank 2-lattice $\Lambda_z = Az + A$, the exponential function

$$e_{\Lambda_z}(\omega) = \omega \prod_{\substack{a,b\in A \\ (a,b)\neq(0,0)}} \left(1 - \frac{\omega}{az+b}\right)$$

has the following form ([3])

$$e_{\Lambda_z}(\omega) = \bar{\pi}^{-1} e_L(\bar{\pi}\omega) \prod_{a\in A-\{0\}} \frac{e_L(a\bar{\pi}z) - e_L(\bar{\pi}\omega)}{e_L(a\bar{\pi}z)}.$$

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Let $u = e_L^{-1}(\bar{\pi}\omega)$. Then

$$e_{\Lambda_z}(\omega) = \bar{\pi}^{-1}u^{-1} \prod_{a\in A-\{0\}} \frac{\rho_a(t^{-1}) - u^{-1}}{\rho_a(t^{-1})}.$$

Define

$$\check{e}_{\mathrm{TD}}(u) = u^{-1} \prod_{a\in A-\{0\}} \frac{\rho_a(t^{-1}) - u^{-1}}{\rho_a(t^{-1})}. \tag{2.1}$$

REMARK. $\bar{\pi}$ disappears in the definition of $\check{e}_{\mathrm{TD}}$, because we used g_{new} and Δ_{new} in the definition of TD.

We can see easily that the zeros of $\check{e}_{\mathrm{TD}}$ are

$$\frac{1}{\rho_a(t^{-1})}, \quad a \in A.$$

Once we define $\rho_0(x) = 0$. Let

$$\check{L} = (L - \{0\}) \cup \{\infty\}.$$

Define a binary operation $\oplus$ on $\check{L}$ as follows:

$$x \oplus y = \frac{xy}{x+y} \text{ for } x, y \in L - \{0\}, \quad x \neq -y$$
$$x \oplus \infty = x$$
$$x \oplus -x = \infty.$$

Also define the A-module structure on $\check{L}$ by, for $a \in A$,

$$a * x = \frac{1}{\rho_a(x^{-1})} \text{ for } x \neq \infty$$
$$a * \infty = \infty.$$

Then $\check{L}$ is an A-module. In fact $\check{L}$ is isomorphic with L via

$$i: L \longrightarrow \check{L}$$

where $i(x) = \frac{1}{x}$ and $i(0) = \infty$, where L has the A-module structure via ρ.

From our construction of TD and $\check{e}_{\mathrm{TD}}$, we have

$$\mathrm{TD}_a\big(\check{e}_{\mathrm{TD}}(u)\big) = \check{e}_{\mathrm{TD}}(a * u). \tag{2.2}$$

PROPOSITION 2.3. $\check{e}_{\mathrm{TD}}(u \oplus v) = \check{e}_{\mathrm{TD}}(u) + \check{e}_{\mathrm{TD}}(v)$.

PROOF. We follow the method in [2]. Let H be a finite subgroup of A. Write

$$P_H(u) = \prod_{h\in H} \big(u^{-1} - \rho_h(t^{-1})\big).$$

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Define

$$Q_v(u) = P_H(u \oplus v) - P_H(v).$$

Since $(u \oplus v)^{-1} = \frac{1}{u} + \frac{1}{v}$ for $v \neq -u$, we have

$$P_H(u \oplus v) = \prod_{h \in H} \Big(\frac{1}{u} + \frac{1}{v} - \rho_h(t^{-1}) \Big).$$

Then, for each $h' \in H$,

$$\begin{aligned} Q_v\Big(\frac{1}{\rho_{h'}(t^{-1})}\Big) &= \prod_{h \in H} \Big(\frac{1}{v} + \rho_{h'}(t^{-1}) - \rho_h(t^{-1}) \Big) - \prod_{h \in H} \Big(\frac{1}{v} - \rho_h(t^{-1}) \Big) \\ &= \prod_{h \in H} \Big(\frac{1}{v} + \rho_{h'-h}(t^{-1}) \Big) - \prod_{h \in H} \Big(\frac{1}{v} - \rho_h(t^{-1}) \Big) \\ &= 0, \end{aligned}$$

since H is a subgroup. Since Q_v and P_H have same degree and same zeros, $Q_v(u) = P_H(u)$. Now take limit on H; one gets the result.

Define

$$e_{\mathrm{TD}} \colon L \longrightarrow L$$

by $e_{\mathrm{TD}} = \check{e}_{\mathrm{TD}} \circ i$. We call e_{TD} the *Tate-Drinfeld map* and $\check{e}_{\mathrm{TD}}$ the *inverse Tate-Drinfeld* map. Then the set of zeros of e_{TD} is

$$\{ \rho_a(t^{-1}) : a \in A \}.$$

THEOREM 2.4. *(i)* $e_{\mathrm{TD}}(u + v) = e_{\mathrm{TD}}(u) + e_{\mathrm{TD}}(v)$.
(ii) $\mathrm{TD}_a\big(e_{\mathrm{TD}}(u)\big) = e_{\mathrm{TD}}\big(\rho_a(u)\big)$.

PROOF. (i) $e_{\mathrm{TD}}(u + v) = \check{e}_{\mathrm{TD}}(\frac{1}{u} * \frac{1}{v}) = \check{e}_{\mathrm{TD}}(\frac{1}{u}) + \check{e}_{\mathrm{TD}}(\frac{1}{v}) = e_{\mathrm{TD}}(u) + e_{\mathrm{TD}}(v)$.
(ii)

$$\begin{aligned} \mathrm{TD}_a\big(e_{\mathrm{TD}}(u)\big) &= \mathrm{TD}_a\Big(\check{e}_{\mathrm{TD}}\Big(\frac{1}{u}\Big)\Big) \\ &= \check{e}_{\mathrm{TD}}\Big(\frac{1}{\rho_a(u)}\Big) = e_{\mathrm{TD}}\big(\rho_a(u)\big). \end{aligned}$$

REMARK. We have two A-module structures on L. One is given by

$$a \cdot x = \rho_a(x),$$

the other is given by

$$a \cdot x = \mathrm{TD}_a(x).$$

Then e_{TD} is an A-module homomorphism from the former A-module struture to the latter with kernel the A-submodule generated by t^{-1}.

Let $n \in A$ and Δ_n^* be the subset of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A)$ such that $\det \gamma = \mu \cdot n$ for some $\mu \in \mathbf{F}_q^*$ and the ideal generated by a, b, c and d is the unit ideal. Then $\mathrm{GL}_2(A)$ acts on Δ_n^*. Let $\{ \alpha_i \}$ be the representatives of right cosets of Δ_n^* for $\mathrm{GL}_2(A)$. In [1], the modular polynomial Φ_n is defined by

$$\Phi_n(X, j) = \prod_{i=1}^{\psi(n)} (X - j \circ \alpha_i)$$

and the following proposition is proved.

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PROPOSITION 2.5. *Let ϕ, ϕ' be two Drinfeld modules of* rank 2 *over C. There exists an isogeny*

$$u\colon \phi' \longrightarrow \phi$$

with cyclic kernel of degree n iff $j_{\phi'}$ is a root of the equation

$$\Phi_n(X, j_\phi) = 0.$$

For $t \in L$, we write $\mathrm{TD}^{\langle t \rangle}$ the Tate-Drinfeld module associated to t. Using Proposition 2.5 and following the method in [5], one gets

THEOREM 2.6. *Assume that L is without characteristic. Let $t \in L$ satisfy $|t| < \delta$. Then for any $n \in A$, the Tate-Drinfeld modules $\mathrm{TD}^{\langle t \rangle}$ and $\mathrm{TD}^{\langle t_n \rangle}$ are isogenous, where $t_n = \frac{1}{\rho_n(t^{-1})}$.*

3. **Drinfeld modules over a complete local ring.**

Let R be a complete local domain over A with maximal ideal $\mathfrak{m}$ and quotient field L. We assume that R is Noetherian and integrally closed. From the t-expansion of j (see [4]), one can get

$$t^{q-1} = \frac{1}{j} + f\Big(\frac{1}{j}\Big)$$

where f is a power series with coefficients in A. Hence for $j \in L$ such that $|j| > (\frac{1}{\delta})^{q-1}$, we can find $t \in L$ with $|t| < \delta$ and get a Tate-Drinfeld module having invariant j. Let

$$D_t = \{\rho_a(t^{-1}) : a \in A\}.$$
$$D_t^{\frac{1}{n}} = \{u \in L : \rho_n(u) \in D_t\}.$$

View L as an A-module via $a \cdot x = \rho_a(x)$. Then $D_t^{\frac{1}{n}}$ is the submodule generated by the n-th roots of ρ and any n-th root of t^{-1}, say $t^{-\frac{1}{n}}$. If L contains all the n-th roots of ρ and $t^{-\frac{1}{n}}$ and characteristic of L is prime to n, then $D_t^{\frac{1}{n}} / D_t$ is isomorphic to a direct product of cyclic A-modules of degree n, generated by a primitive n-th root λ_n of ρ and $t^{-\frac{1}{n}}$ (mod t^{-1}).

THEOREM 3.1. *Let $\mathrm{TD}^{\langle t \rangle}$ have invariant j with $|j| > (\frac{1}{\delta})^{q-1}$. Let R_n be the integral closure of R in $K_\infty = K(\lambda_n, t^{-\frac{1}{n}})$. Then e_{TD} induces a homomorphism of $D_q^{\frac{1}{n}}$ into $\mathrm{Ker}\,\mathrm{TD}_n^{\langle t \rangle}$. Furthermore, if characteristic of L is prime to n, then it induces a Galois isomorphism of $D_t^{\frac{1}{n}} / D_t$ onto $\mathrm{Ker}(\mathrm{TD}_n^{\langle t \rangle})$, and*

$$L\big(\mathrm{Ker}(\mathrm{TD}_n^{\langle t \rangle})\big) = L(\lambda_n, t^{-\frac{1}{n}}).$$

PROOF. Let $\omega = \omega_{a,b} = \rho_a(\lambda_n) + \rho_b(t^{-\frac{1}{n}})$, $a, b \in A/n$. The only nontrivial part is about Galois isomorphism. Let $\sigma \in \mathrm{Gal}\big(L\big(\mathrm{Ker}(\mathrm{TD}_n^{\langle t \rangle})\big)/L\big)$. Then

$$e_{\mathrm{TD}}(\omega^\sigma) = \omega^\sigma \prod_{a \in A}{}' \frac{\rho_a(t^{-1}) - \omega^\sigma}{\rho_a(t^{-1})}$$
$$= e_{\mathrm{TD}}(\omega)^\sigma$$

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since $\rho_a(X) \in A[X]$. But $\mathrm{Ker}(\mathrm{TD}_n^{\langle t \rangle})$ is just

$$\{ e_{\mathrm{TD}}(\omega_{a,b}) : a, b \in A/n \}.$$

Hence the result follows.

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