

RESTRICTED DIOPHANTINE APPROXIMATION

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Abstract

The problem considered is that of approximating irrationals α by rationals p/q where p and q avoid certain congruence classes mod 2^k for certain integers k . Results are obtained which give close bounds on a number c such that $|\alpha - p/q| < c/q^2$ has infinitely many solutions where p and q can be expressed as the sum of three squares.

1. Introduction

Scott (1940) considered the problem of approximating real irrational numbers α by rationals p/q where p and q are of given parity. Others (see section 47.6 of Lekkerkerker (1969)) have generalized this to the case $p \equiv r \pmod{m}$, $q \equiv s \pmod{m}$ where $(r, s, m) = 1$. The problem to be discussed here is the complementary problem, namely the restriction $p \not\equiv r \pmod{m}$, $q \not\equiv s \pmod{m}$, in the cases $m = 4, 8$ and 16 . We define $k(r, s, m)$ to be the infimum of the values of k for which, for every real irrational α , the inequality

$$(1) \quad q |q\alpha - p| < k$$

has infinitely many solutions in integers p, q with $q > 0$ satisfying

$$(2) \quad p \not\equiv r \pmod{m}, \quad q \not\equiv s \pmod{m}.$$

In Section 2 we prove

THEOREM 1. *For $m = 4$ or 8 and r and s both odd*

$$k(r, s, m) = 2/\sqrt{5}$$

and this is an attained minimum.

THEOREM 2. $k(0, 1, 8) = 7\sqrt{2}/12$ is not attained.

It can be seen from the proofs that the constants $2/\sqrt{5}$ and $7\sqrt{2}/12$ are isolated, but in each case the irrationals α requiring the critical value of k do not form a full equivalence class. The contrast between the two results is a little surprising. The method of proof used here enables $k(r, s, m)$ to be evaluated for most triples (r, s, m) , but it appears that a general proof giving $k(r, s, m)$ as a specific function of r, s, m will not exist. Indeed the actual value of $k(r, s, m)$ seems almost to occur by accident. We shall also prove the following result.

THEOREM 3. For every irrational $\alpha > 0$ there are infinitely many solutions of (1) with $k = 27/2\sqrt{145}$ and p and q sums of three squares.

It should be noted that this value of k is not shown to be best possible. However by combining this with the result of Theorem 1, remembering that numbers congruent to $7 \pmod{8}$ cannot be written as the sum of three squares, we have the inequality:

COROLLARY. The infimum κ of the values of k for which (1) has infinitely many solutions with p and q being sums of three squares satisfies

$$2/\sqrt{5} \leq \kappa \leq 27/2\sqrt{145}.$$

It will be noted that the proof of Theorem 1 also holds for the case $m = 16$, $rs \equiv \pm 1 \pmod{16}$ with little modification. The proof of Theorem 3 uses the well-known result that a positive integer can be written as the sum of three squares if and only if it is not of the form $4^a(8b + 7)$ for integers a and b . At one stage much of the strength of this result is used, though for the rest of the proof the simple fact that numbers $4^a(8b + 7)$ are congruent to 0, 4 or 7 mod 8 is sufficient.

The proofs of Theorems 1, 2 and 3 seek α for which the convergents (to the simple continued fraction of α) violate the congruence condition. The convergents investigated include not only the primary convergents but also the secondary, and even "tertiary" convergents.

2.

In the proof of Theorems 1 and 2, we assume $\alpha > 0$, for if $\alpha < 0$ we just consider $-\alpha$ and replace r by $-r \pmod{m}$. The proofs depend on the theory of simple continued fractions and we list the results needed from the standard theory in the following lemmas, proofs of which are essentially to be found in chapter 1 of Lang (1966).

LEMMA 1. Let α have continued fraction $[a_0, a_1, a_2, \dots]$, and let p_n and q_n be defined inductively for $n \geq 0$ by the equations

$$p_0 = a_0, \quad p_1 = a_1 a_0 + 1, \quad p_n = a_n p_{n-1} + p_{n-2},$$

$$q_0 = 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

Furthermore, for $n \geq 0$, define

$$P_n = p_{n+1} + p_n, \quad P'_n = (a_{n+2} - 1)p_{n+1} + p_n,$$

$$Q_n = q_{n+1} + q_n, \quad Q'_n = (a_{n+2} - 1)q_{n+1} + q_n.$$

Then if $(p, q) = 1$ and $q | q\alpha - p | < 1$ either

- (i) $(p, q) = (p_n, q_n)$ for some $n \geq 0$, or
- (ii) $(p, q) = (P_n, Q_n)$ or (P'_n, Q'_n) for some $n \geq 0$.

Furthermore (i) must hold if $q | q\alpha - p | < 1/2$.

LEMMA 2. Let $\alpha_n = [a_{n+1}, a_{n+2}, \dots]$ and $\alpha'_n = [0, a_n, a_{n-1}, \dots, a_1]$. Then with the notation of Lemma 1

- (i) $q_n | q_n \alpha - p_n | = 1/(\alpha_n + \alpha'_n) = k_n$,
- (ii) $Q_n | Q_n \alpha - P_n | = (\alpha_{n+1} - 1)(\alpha'_{n+1} + 1)/(\alpha_{n+1} + \alpha'_{n+1}) = k'_n$, and
- (iii) $Q'_n | Q'_n \alpha - P'_n | = (\alpha_{n+1} + 1 - a_{n+2})(\alpha'_{n+1} + a_{n+2} - 1)/(\alpha_{n+1} + \alpha'_{n+1}) = k''_n$.

From the formula

$$k'_n = 1 - (2 - \alpha_{n+1} + 1/\alpha'_{n+1})/(1 + \alpha_{n+1}/\alpha'_{n+1})$$

it is clear that $k'_n > 1$ unless

$$[a_{n+2}, a_{n+3}, \dots] < 2 + [a_{n+1}, a_n, \dots, a_1].$$

Similarly from the formula

$$k''_n = 1 - (2 + \alpha_{n+2} - 1/\alpha'_{n+2})/(1 + \alpha_{n+2}/\alpha'_{n+2})$$

it is clear that $k''_n > 1$ unless

$$[a_{n+2}, a_{n+1}, \dots, a_1] < 2 + [a_{n+3}, a_{n+4}, \dots].$$

From these two formulae we also have $k'_n + k''_{n-1} < 2$.

LEMMA 3. (i) Let $\alpha = [a_0, \dots, a_r, a_{r+1}, \dots]$, and $\beta = [a_0, \dots, a_r, b_{r+1}, \dots]$. Then $\alpha < \beta$ if $a_{r+1} < b_{r+1}$ and r is odd or if $a_{r+1} > b_{r+1}$ and r is even.

(ii) $[a_0, a_1, \dots, a_r] < [a_0, a_1, \dots, a_r, \dots] < [a_0, a_1, \dots, a_r + 1]$ if r is even, otherwise the reverse inequalities hold.

We can use these results to show that, for rs odd, $k(r, s, 8) \geq 2/\sqrt{5}$ as follows.

LEMMA 4. Let rs be odd, and let $\alpha = [a_0, a_1, \dots]$ be such that for some integer l the following conditions hold.

- (i) $a_n = 4$ for $n \geq l + 2$,
- (ii) $p_l \equiv r \pmod{8}$ and $q_{l+1} \equiv s \pmod{8}$, and
- (iii) both q_l and p_{l+1} are even.

Then there are infinitely many pairs (p, q) of integers with $q > 0$ satisfying (1) and (2) with $m = 8$ and $k = 2/\sqrt{5}$, but there are only finitely many pairs when $k < 2/\sqrt{5}$.

PROOF. Let p and q satisfy (1) with $k \leq 2/\sqrt{5}$. Then by Lemma 1 one of the following must be true.

- (i) $(p, q) = (p_n, q_n)$ and $q | q\alpha - p | = k_n$,
- (ii) $(p, q) = (P_n, Q_n)$ and $q | q\alpha - p | = k'_n$,
- (iii) $(p, q) = (P'_n, Q'_n)$ and $q | q\alpha - p | = k''_n$,
- (iv) $(p, q) = (tp_n, tq_n)$ and $q | q\alpha - p | = t^2 k_n \geq 4k_n$.

Now $[4, \dot{4}] = 2 + \sqrt{5}$, so since

$$\lim_{n \rightarrow \infty} k'_n = \lim_{n \rightarrow \infty} k''_n = 4 \lim_{n \rightarrow \infty} k_n = 2/\sqrt{5}$$

it is clear that with $k < 2/\sqrt{5}$ equations (1) and (2) can have at most finitely many solutions arising from cases (ii), (iii) and (iv). In addition since for $n \geq l$ we have either $p_n \equiv r \pmod{8}$ or $q_n \equiv s \pmod{8}$ only finitely many solutions of (1) and (2) can arise from case (i) when $k < 2/\sqrt{5}$. To complete the proof of the lemma it remains to show that $(p, q) = (2p_n, 2q_n)$ satisfies (1) and (2) with $k = 2/\sqrt{5}$ infinitely often. This is a trivial consequence of the fact that for $n \geq l$, $4k_n < 2/\sqrt{5}$ if $\alpha'_n > \sqrt{5} - 2$ and $4k_{n+1} < 2/\sqrt{5}$ if $\alpha'_n < \sqrt{5} - 2$.

It must of course be noted that it is possible to construct, for each choice of odd r and s , an irrational satisfying these conditions: for example for $(r, s) = (1, 1)$ we can take $\alpha = [8, \dot{4}]$, for $(r, s) = (1, 3)$ we can take $\alpha = [0, 1, 1, 1, \dot{4}]$. Essentially all that is required is to choose that one of $rs + 1$, $rs - 1$ that is divisible by 4 and let ab be a factorization of that number into even factors. Then $r/a, b/s$ are consecutive convergents $p_i/q_i, p_{i+1}/q_{i+1}$ of some finite continued fraction $[a_0, a_1, \dots, a_{i+1}]$ which can be found by reversing the continued fraction algorithm. Setting $\alpha = [a_0, a_1, \dots, a_{i+1}, \dot{4}]$ satisfies the requirements.

It can also be verified that if $rs \equiv \pm 1 \pmod{16}$ and we choose a and b both divisible by 4 such that $ab = rs \pm 1$ then the above procedure yields an α that satisfies conditions (i), (ii) and (iii) of the lemma where the congruences in (ii) are now modulo 16, and in (iii) both q_i and p_{i+1} are divisible by 4. Following the proof of the lemma through yields $k(r, s, 16) \geq 2/\sqrt{5}$ for $rs \equiv \pm 1 \pmod{16}$.

To show $k(r, s, 8) = 2/\sqrt{5}$ for rs odd it remains to show the following result.

LEMMA 5. *Let rs be odd, $m = 4$ and $k = 2/\sqrt{5}$. Then equations (1) and (2) have infinitely many solutions.*

PROOF. If $\alpha = [a_0, a_1, a_2, \dots]$ then one of the following possibilities must occur.

- (i) $a_n \geq 5$ for infinitely many n ,
- (ii) $a_n \leq 3$ for all sufficiently large n ,
- (iii) $a_n = 4$ for all sufficiently large n ,
- (iv) $a_n \leq 4$ for all sufficiently large n and either
 - (a) $a_{n-1} = 1$ or $a_{n+1} = 1$ for infinitely many n for which $a_n = 4$, or
 - (b) $a_{n-1} \geq 2$ and $a_{n+1} \geq 2$ for all sufficiently large n for which $a_n = 4$ and $a_{n+1} \leq 3$ for infinitely many of these n .

The result for case (iii) follows by taking $(p, q) = (2p_n, 2q_n)$ as in the proof of Lemma 4, while in cases (i), (iv)(a) and (iv)(b) we only need to take $(p, q) = (2p_n, 2q_n)$ for infinitely many n for which $k_n < 1/2\sqrt{5}$. In case (i) we have $k_n < 1/5$ for the infinitely many n for which $a_{n+1} \geq 5$, in case (iv)(a) we have

$$k_n^{-1} > 4 + [0, 1, 1] + [0, 4, 1] = 4.7$$

infinitely often, and in case (iv)(b) we have

$$k_n^{-1} > [4, 3, 1, 5] + [0, 4, 2, 5] > 4.48$$

infinitely often. We are therefore left with case (ii).

Let l be chosen so that $a_n \leq 3$ for $n \geq l$. For $n > l$

$$\alpha_n + \alpha'_n > [1, 3, 1] + [0, 3, 1] = 1.5$$

and setting $(p, q) = (p_n, q_n)$ gives infinitely many solutions to (1) and (2) with $k = 2/\sqrt{5}$ unless (p_n, q_n) does not satisfy (2) for all n greater than some l_0 . We therefore assume from now on that l' ($\geq l$) is such that for all $n > l'$ either $p_n \equiv r \pmod{4}$ or $q_n \equiv s \pmod{4}$.

For the present we shall also assume that $r = s = 1$ and denote by h an integer greater than $l' + 6$. We let T_h denote the matrix

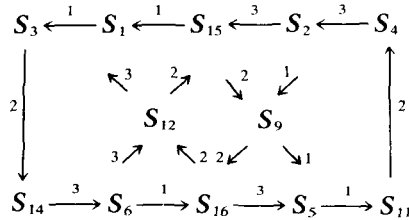
$$\begin{pmatrix} p_h & q_h \\ p_{h+1} & q_{h+1} \end{pmatrix}$$

where each entry has been reduced mod 4 to lie in $\{0, 1, 2, 3\}$. Using the conditions (a) $p_h \equiv 1 \pmod{4}$ or $q_h \equiv 1 \pmod{4}$ and (b) $|p_h q_{h+1} - p_{h+1} q_h| = 1$, this latter being a standard property of continued fractions, either $S_i = T_h$ or

$S_h^* = T_h$ where S_h^* denotes S_i with transposed columns and S_i denotes one of the following matrices:

$$\begin{aligned}
 S_1 &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & S_2 &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} & S_3 &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} & S_4 &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \\
 S_5 &= \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} & S_6 &= \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} & S_7 &= \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix} & S_8 &= \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} \\
 S_9 &= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} & S_{10} &= \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} & S_{11} &= \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} & S_{12} &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\
 S_{13} &= \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} & S_{14} &= \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} & S_{15} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & S_{16} &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.
 \end{aligned}$$

Since $T_{h+1} = \begin{pmatrix} 0 & 1 \\ 1 & a_{h+2} \end{pmatrix} T_h \pmod{4}$ and $a_{h+2} \leq 3$ we can form the following graph, where $S_i \xrightarrow{j} S_j$ denotes that if T_h is S_i or S_i^* then $T_{h+1} = S_j$ or S_j^* if $a_{h+2} = j$.



It will be noted that S_7, S_8, S_{10} and S_{13} disappear from consideration. Using Lemma 3 we find that $\alpha_{h+1} \leq [2, 3, 3, 2, 2, \dots]$ and $\alpha'_{h+1} \leq [0, 1, 3, 2, 2, 1, \dots]$ when $T_h = S_3$ or S_3^* , and $\alpha_{h+1} \leq [2, 3, 3, 1, 1, \dots]$ and $\alpha'_{h+1} \leq [0, 1, 3, 1, 3, 2, \dots]$ when $T_h = S_{11}$ or S_{11}^* . Thus since k'_h is a maximum when both α_{h+1} and α'_{h+1} are a maximum we have that $k'_h < .755$ whenever $T_h \in \mathcal{S} = \{S_3, S_3^*, S_{11}, S_{11}^*\}$. Now $(p, q) = (P_h, Q_h)$ satisfies (2) whenever $T_h \in \mathcal{S}$, and so if $T_h \in \mathcal{S}$ for infinitely many h we get infinitely many solutions of (1) and (2) with $k = 2/\sqrt{5}$. We therefore assume that $T_h \in \mathcal{S}$ for only finitely many h , that is for $h \geq h_0, T_h$ or T_h^* cycles around the subgraph

$$\xrightarrow{2} S_{12}^* \xrightarrow{2} S_{15} \xrightarrow{2} S_9^* \xrightarrow{2} S_{16} \xrightarrow{2} .$$

Since $[2] = 1 + \sqrt{2}$ we have $\lim_{h \rightarrow \infty} k'_h = 2^{-\frac{1}{2}} < 0.71$. This means that, since $T_h = S_{16}$ or S_{16}^* implies that $(p, q) = (P_h, Q_h)$ satisfies (2), we again have infinitely many solutions of (1) and (2) with $k = 2/\sqrt{5}$.

This completes the discussion of case (ii) for $r = s = 1$. The same proof applies for the other three cases on letting T_h denote, respectively, the matrix

$$\begin{pmatrix} p_h & -q_h \\ p_{h+1} & -q_{h+1} \end{pmatrix}, \begin{pmatrix} -p_h & -q_h \\ -p_{h+1} & -q_{h+1} \end{pmatrix}, \begin{pmatrix} -p_h & q_h \\ -p_{h+1} & q_{h+1} \end{pmatrix}$$

reduced mod 4 in the cases $(r, s) = (1, 3), (3, 3), (3, 1)$.

Theorem 1 now follows as a consequence of Lemma 4 and Lemma 5. The proof of Theorem 2 is basically similar, though it is more complicated since $(2p, 2q)$ will not always satisfy (2). To save repetition we shall from now on take $r = 0, s = 1, m = 8$ in (2).

LEMMA 6. *If, for some $l > 0, a_n = 1$ for $n \geq l$ then $q \mid q\alpha - p \mid < 0.82$ has infinitely many solutions satisfying (2).*

PROOF. If $n \geq l$ then one of the pairs $(p_h, q_h), h = n, n + 1, n + 2$ satisfies (2), for it is easily verified that the equations

- (a) $p_{n+1} + p_n \equiv 0 \pmod{8}$ or $q_{n+1} + q_n \equiv 1 \pmod{8}$,
- (b) $p_{n+1} \equiv 0 \pmod{8}$ or $q_{n+1} \equiv 1 \pmod{8}$,
- (c) $p_n \equiv 0 \pmod{8}$ or $q_n \equiv 1 \pmod{8}$,
- (d) $|p_n q_{n+1} - p_{n+1} q_n| = 1$

have no solution. Now $[\dot{1}] = \frac{1}{2}(1 + \sqrt{5})$ and $k_n \rightarrow 1/\sqrt{5}$, so clearly $q \mid q\alpha - p \mid < 0.82$ must have infinitely many solutions $(p, q) = (p_n, q_n)$ satisfying (2).

In the following Lemmas 7–10 we mean by assumptions (A) the following: $q \mid q\alpha - p \mid < .82$ has finitely many solutions satisfying (2), and l is chosen such that none of the solutions is of the form (dp_n, dq_n) with $n \geq l$. In the proof of the lemmas we make frequent use of the standard result $|p_{n+1}q_n - p_nq_{n+1}| = 1$.

LEMMA 7. *Under assumptions (A) let $h > l$ be such that $a_{h+1} \geq 2, q_h \equiv q_{h+1} \equiv 1 \pmod{8}$, and $p_{h+1} \not\equiv 0 \pmod{8}$. Then*

- (i) $a_{h+2} \leq 4$ and $p_{h+2} \equiv 0 \pmod{8}$ if $p_{h+1} \not\equiv 0 \pmod{4}$, or
- (ii) $a_{h+2} = 8$ if $p_{h+1} \equiv 4 \pmod{8}$.

PROOF. (i) Suppose $p_{h+1} \not\equiv 0 \pmod{4}$. We have $a_{h+2} \leq 4$ immediately, for $(2p_{h+1}, 2q_{h+1})$ satisfies (2), implying $4k_{h+1} \geq .82$. Now suppose that $p_{h+2} \not\equiv 0 \pmod{8}$. Since $q_{h+2} \equiv a_{h+2} + 1$ it is clear that (p_{h+2}, q_{h+2}) satisfies (2), so $k_{h+2} \geq .82$. This is impossible unless $a_{h+2} = 4, a_{h+3} = 1$ and $a_{h+4} \geq 50$. But now (p_{h+3}, q_{h+3}) satisfies (2) and $k_{h+3} < 1/50$, contrary to assumptions (A). This proves that $p_{h+2} \equiv 0 \pmod{8}$.

(ii) Suppose $p_{h+1} \equiv 4 \pmod{8}$. Then $a_{h+2} \leq 10$, for $9k_{h+1} \geq .82$ since $(3p_{h+1}, 3q_{h+1})$ satisfies (2). Now suppose $a_{h+2} \neq 8$. Then (p_{h+2}, q_{h+2}) satisfies (2), so $k_{h+2} \geq .82$, from which it follows, since $\alpha'_{h+2} > 1/11$, that $a_{h+3} = 1$ and

$a_{h+4} \geq 7$. But now $(2p_{h+3}, 2q_{h+3})$ satisfies (2) and $4k_{h+3} < 4/7$ contradicting (A). This proves that $a_{h+2} = 8$, as desired.

COROLLARY. *Under the assumptions of Lemma 7, if $p_{h+1} \equiv 4 \pmod 8$, then $a_{h+3} \leq 4$ and $p_{h+3} \equiv 0 \pmod 8$.*

PROOF. Apply the lemma with h replaced by $h + 1$, since not both p_{h+1} and p_{h+2} can be even.

LEMMA 8. *Under assumptions (A) let $h \geq l$ be such that $a_{h+2} \geq 2$. Then there exists $j \geq h$ such that either $p_j \equiv 0 \pmod 8$ and $q_{j+1} \equiv 1 \pmod 8$ or $q_j \equiv 1 \pmod 8$ and $p_{j+1} \equiv 0 \pmod 8$.*

PROOF. Since $k_h < 1/2$ we either have $p_h \equiv 0 \pmod 8$ or $q_h \equiv 1 \pmod 8$. Suppose firstly that $p_h \equiv 0 \pmod 8$. We show that we may take $j = h$, for suppose to the contrary that $q_{h+1} \not\equiv 1 \pmod 8$. Then (p_{h+1}, q_{h+1}) satisfies (2), so $k_{h+1} \geq .82$, $a_{h+2} = 1$ and $a_{h+2} \geq 4$. Now $(2p_{h+2}, 2q_{h+2})$ satisfies (2), so $4k_{h+2} \geq .82$ and $a_{h+3} \leq 4$, i.e. $a_{h+3} = 4$. Furthermore as $k_{h+2} < .82$ we have $q_{h+2} \equiv 1 \pmod 8$ since p_{h+2} is odd. This implies q_{h+1} is even, so (p_{h+3}, q_{h+3}) satisfies (2). For $k_{h+3} \geq .82$ we require $a_{h+4} = 1$ and $a_{h+5} \geq 50$. However $2p_{h+4} \equiv 4 \pmod 8$ since $p_{h+4} \equiv 6p_{h+1} \pmod 8$, so $(2p_{h+4}, 2q_{h+4})$ satisfies (2) and $4k_{h+4} < .82$. This contradiction shows that we may take $j = h$, as claimed.

We now suppose that $q_h \equiv 1 \pmod 8$. Since we could set $j = h$ if $p_{h+1} \equiv 0 \pmod 8$, and $j = h + 1$ or $j = h + 2$ if $q_{h+1} \equiv 1 \pmod 8$ (by Lemma 7 and its corollary) we assume $p_{h+1} \not\equiv 0 \pmod 8$ and $q_{h+1} \not\equiv 1 \pmod 8$. Now (p_{h+1}, q_{h+1}) satisfy (2), so $k_{h+1} \geq .82$, $a_{h+2} = 1$, $a_{h+1} \geq 4$ and $a_{h+3} \geq 4$. Since $k_{h+2} < .82$ it is clear that (p_{h+2}, q_{h+2}) cannot satisfy (2). We assume $q_{h+2} \equiv 1 \pmod 8$, for $p_{h+2} \equiv 0 \pmod 8$ would imply we could take $j = h + 2$ by the first part of this proof. Repeating the above argument with h replaced by $h + 2$ we can take $a_{h+4} = 1$, $a_{h+5} \geq 4$ and $q_{h+4} \equiv 1 \pmod 8$. Now $q_{h+1} \equiv q_{h+2} - q_h \equiv 0 \pmod 8$ and $q_{h+3} \equiv 0 \pmod 8$ similarly, so $a_{h+3} \equiv 0 \pmod 8$. In particular $a_{h+3} \geq 8$ and $4k_{h+2} < .82$. Thus $(2p_{h+2}, 2q_{h+2})$ cannot satisfy (2), so $p_{h+2} \equiv 0 \pmod 4$. Similarly we can show $p_{h+4} \equiv 0 \pmod 4$, so $p_{h+3} \equiv p_{h+4} - p_{h+2}$ is also even. However this contradicts $|p_{h+2}q_{h+3} - p_{h+3}q_{h+2}| = 1$, and this is sufficient to prove the lemma.

LEMMA 9. *Under assumptions (A) let $h > l$ be such that $q_h \equiv 1 \pmod 8$ and $p_{h+1} \equiv 0 \pmod 8$. Then $a_{h+2} \equiv 0 \pmod 8$ and $q_{h+2} \equiv 1 \pmod 8$.*

PROOF. Suppose $a_{h+2} \not\equiv 0 \pmod 8$. Then (p_{h+2}, q_{h+2}) satisfy (2) since $p_h q_{h+1}$ is odd, so $k_{h+2} \geq .82$, $a_{h+3} = 1$, $a_{h+4} \geq 4$ and $a_{h+2} \geq 4$. Now p_{h+3} is odd, but (p_{h+3}, q_{h+3}) doesn't satisfy (2) since $k_{h+3} < .82$, so $q_{h+3} \equiv 1 \pmod 8$. This implies that $(2p_{h+3}, 2q_{h+3})$ satisfies (2). We therefore have $a_{h+4} = 4$ and $a_{h+5} \geq 2$ in order to have $4k_{h+4} \geq .82$. Since q_{h+4} is even we have (p_{h+4}, q_{h+4}) satisfying (2)

and $k_{h+4} < 1/2$. This contradiction implies that $a_{h+2} \equiv 0 \pmod 8$ as desired.

LEMMA 10. *Under assumptions (A) let $h > l$ be such that $p_h \equiv 0 \pmod 8$ and $q_{h+1} \equiv 1 \pmod 8$. Then $a_{h+3} \geq 2$ and $a_{h+2} = 4$.*

PROOF. Since both p_{h+1} and q_h are odd, $(2p_{h+1}, 2q_{h+1})$ satisfies (2), so $a_{h+2} \leq 4$ to ensure $4k_{h+1} \geq .82$. Now if $a_{h+2} \neq 4$ we have $k_{h+3} < .82$, so $q_{h+2} \equiv 1$ since $p_{h+2} \not\equiv 0 \pmod 8$. By Lemma 7 we therefore have, since a_{h+2} must be even, that $p_{h+3} \equiv 0 \pmod 8$. However since $a_{h+2} = 2$ is even p_{h+3} is odd, so $p_{h+3} \equiv 0 \pmod 8$ is impossible. This leaves only the possibility that $a_{h+2} = 4$. Now if $a_{h+3} = 1$ we would have p_{h+3} odd, q_{h+3} even, so (p_{h+3}, q_{h+3}) satisfies (2) yet $k_{h+3} < .82$. This contradiction shows $a_{h+3} \geq 2$ as desired.

LEMMA 11. *Under the same assumptions as Lemma 10 $a_{h+3} = 8$, $q_h \equiv 5 \pmod 8$, $q_{h+2} \equiv q_{h+3} \equiv 1 \pmod 8$, $a_{h+4} = 4$ and $p_{h+4} \equiv 0 \pmod 8$.*

PROOF. From Lemma 10 we have $k_{h+2} < 1/2$, so either $p_{h+2} \equiv 0 \pmod 8$ or $q_{h+2} \equiv 1 \pmod 8$. The first alternative cannot hold, since p_{h+2} is four times an odd number, so the second alternative holds. Hence $q_h \equiv 1 - 4 \equiv 5 \pmod 8$, as desired. Furthermore, by Lemma 7, $a_{h+3} = 8$ and again by Lemma 7, since p_{h+3} is odd, $p_{h+4} \equiv 0 \pmod 8$ and $a_{h+4} \leq 4$. However $p_{h+4} \equiv 4 + a_{h+4}p_{h+3}$ where p_{h+3} is odd certainly cannot hold for $a_{h+4} < 4$ and $p_{h+4} \equiv 0 \pmod 8$, so we have $a_{h+4} = 4$.

COROLLARY. *Assumptions (A) imply that for some $j > l$, $p_{j+4n} \equiv 0 \pmod 8$, $q_{j+4n} \equiv 5 \pmod 8$, $q_{j+1+4n} \equiv q_{j+2+4n} \equiv q_{j+3+4n} \equiv 1 \pmod 8$, $p_{j+4n+2} \equiv 4 \pmod 8$, $p_{j+1+2n} \equiv p_{j+1}$, $a_{j+2n} = 4$ and $a_{j+2n+1} = 8$ for all $n \geq 0$.*

PROOF. This is just a straightforward application of Lemmas 6, 8 and 9 followed by repeated application of Lemmas 11 and 9.

Now that we have isolated the critical case all that remains is to prove that this case is critical.

LEMMA 12. *Let $\alpha = [a_0, a_1, \dots]$ be such that for some integer j , $q_i \equiv 5 \pmod 8$, $p_j \equiv 0 \pmod 8$, $q_{j+1} \equiv 1 \pmod 8$, $a_{j+2n} = 4$ for $n \geq 0$ and $a_{j+2n+1} = 8$ for $n \geq 0$. Then (1) and (2) have infinitely many solutions for $k > 7\sqrt{2}/12$. However if $[8, a_j, \dots, a_1] < [8, 4]$ and $k = 7\sqrt{2}/12$ there are only finitely many solutions.*

PROOF. By Lemmas 1 and 2 we only need to investigate as possibilities for (p, q) the pairs (p_h, q_h) , (P_h, Q_h) , (P'_h, Q'_h) and (dp_h, dq_h) . We only consider $h \geq j$ since we are looking for infinitely many solutions. There are a number of cases to be considered.

- (i) The pairs (p_h, q_h) . These never satisfy (2).
- (ii) The pairs $(2p_h, 2q_h)$. These satisfy (2) only for $h = j + n$ with n odd.

In this case

$$4k_h = 4/(\overline{[4, 8]} + [0, 8, \dots, a_1]) > 16/17.$$

- (iii) The pairs (dp_h, dq_h) for $d \geq 3$. In this case $9k_h > 1$.
- (iv) The pairs (P_h, Q_h) for $h = j + n$ with n odd. In this case $k'_h > 1$.
- (v) The pairs (P'_h, Q'_h) for $h = j + n$ with n even. In this case $k''_h > 1$.
- (vi) The pairs (P_h, Q_h) for $h = j + n$ with n even. In this case

$$k'_h = 1 - (2 - \overline{(4, 8)} + [8, \dots, a_1]) / (1 + \overline{[4, 8]} [8, \dots, a_1])$$

tends to $7\sqrt{2}/12$ as $h \rightarrow \infty$. Furthermore $k'_h < 7\sqrt{2}/12$ if and only if $[8, \dots, a_1] > \overline{[8, 4]}$, that is $[8, a_i, \dots, a_1] > \overline{[8, 4]}$.

(vii) The pairs (P'_h, Q'_h) for $h = j + n$ with n odd. In this case $k''_h \rightarrow 7\sqrt{2}/12$ as $h \rightarrow \infty$ and $k''_h < 7\sqrt{2}/12$ if and only if $[4, 8, \dots, a_1] < \overline{[4, 8]}$, that is $[8, a_i, \dots, a_1] > \overline{[8, 4]}$. The lemma follows immediately from the results of these cases.

Theorem 2 now follows immediately from the above lemma and corollary on observing that the hypotheses of Lemma 12 are satisfied by $\alpha = [4, 1, \overline{4, 8}]$.

3.

The proof of Theorem 3 depends on the result that numbers not of the form $4^a(8k + 7)$ can be written as the sum of three squares. The first point to note is that $27/2\sqrt{145} = 1.121\dots > 1$ and we therefore need a modified form of Lemma 1.

LEMMA 13. *Let α be as in Lemma 1 and let p and q be integers such that $(p, q) = 1$ and $1 < q | q\alpha - p | < 2$. Then either (ii) of Lemma 1 holds or for some integer n*

- (iv) $(p, q) = (ip_n + p_{n-1}, iq_n + q_{n-1})$ with $0 < i < a_{n+1}$, or
- (v) $(p, q) = ((2i + 1)p_n + 2p_{n-1}, (2i + 1)q_n + 2q_{n-1})$ with $0 \leq i < a_{n+1}$.

PROOF. Since $(p, q) \neq (p_n, q_n)$ for any n (Lemma 2) there either exists n such that

$$p_{n-1}/q_{n-1} < p/q < p_{n+1}/q_{n+1} < \alpha < p_n/q_n$$

or there exists n such that the above holds with all inequalities reversed. We consider the first possibility only, as the second requires a similar argument. From the definition of p_{n+1}, q_{n+1} there exists $i, 0 \leq i < a_{n+1}$, such that

$$\frac{{}^{(i)}P_n}{{}^{(i)}Q_n} < \frac{p}{q} \leq \frac{{}^{(i+1)}P_n}{{}^{(i+1)}Q_n} < \alpha$$

where ${}^{(i)}P_n, {}^{(i)}Q_n$ denote $ip_n + p_{n-1}, iq_n + q_{n-1}$ respectively. Since ${}^{(i+1)}P_n {}^{(i)}Q_n - {}^{(i)}P_n {}^{(i+1)}Q_n = 1$ we have

$$(3) \quad \frac{1}{{}^{(i)}Q_n {}^{(i+1)}Q_n} = \frac{{}^{(i+1)}P_n}{{}^{(i+1)}Q_n} - \frac{{}^{(i)}P_n}{{}^{(i)}Q_n} \cong \frac{s}{q {}^{(i)}Q_n}$$

where $s = {}^{(i)}Q_n p - {}^{(i)}P_n q > 0$. In addition

$$(4) \quad 2q^{-2} > |\alpha - p/q| > r/q {}^{(i+1)}Q_n$$

where $r = {}^{(i+1)}P_n q - {}^{(i+1)}Q_n p \geq 0$. Combining (3) and (4) gives

$$rs {}^{(i+1)}Q_n < rq < 2 {}^{(i+1)}Q_n$$

and so $rs < 2$. The proof of the lemma is now complete on observing that (iv) holds if $r = 0$ and (v) holds if $r = s = 1$.

We note in passing that if (iv) holds then

$$q |q\alpha - p| = (\alpha_n - i)(\alpha'_n + i)/(\alpha_n + \alpha'_n), = {}^{(i)}k_n \text{ say,}$$

and if (v) holds then

$$q |q\alpha - p| = (2\alpha_n - (2i + 1))(2\alpha'_n + (2i + 1))/(\alpha_n + \alpha'_n), = {}^{(i)}k'_n \text{ say.}$$

The following lemma is required to show that certain irrational α can be approximated by a quotient of sums of three squares sufficiently well.

LEMMA 14. *Let $\alpha = [a_0, a_1, \dots]$ be such that for some integer l the following conditions hold.*

- (i) $p_{l-1} \equiv 0 \pmod{8}, q_{l-1} \equiv 1 \pmod{8}, p_l \equiv q_l \equiv 7 \pmod{8},$
- (ii) $a_{l+3m+1} = 2, a_{l+3m+2} = 7$ and $a_{l+3m} = 1$ for all integers $m \geq 0$.

Then there exist infinitely many pairs (p, q) of integers with $q > 0$ satisfying (1) with $k = 27/2\sqrt{145} (= k^, \text{ say})$ and neither p nor q congruent to 0, 4 or 7 mod 8, but there are finitely many such pairs if $k < k^*$.*

PROOF. For convenience we shall let (c) denote the condition: neither p nor q is congruent to 0, 4 or 7. Let (p, q) satisfy (1) with $k \leq k^*$ and condition (c). Then by Lemmas 1 and 14 either

- (i) $(p, q) = (tp_n, tq_n)$ for some integer t ,
- (ii) $(p, q) = ({}^{(i)}P_n, {}^{(i)}Q_n)$ with $0 < i < a_{n+1}$, or

(iii) $(p, q) = ({}^{(i)}P_n + {}^{(i+1)}P_n, {}^{(i)}Q_n + {}^{(i+1)}Q_n)$ with $0 \leq i < a_{n+1}$.

We therefore need to consider $t^2k_n, {}^{(i)}k_n$ and ${}^{(i)}k'_n$. For convenience we denote by $\lim_3 k_l$ the limit

$$\lim_{m \rightarrow \infty} k_{l+3m}$$

with similar notation for ${}^{(i)}k_l, {}^{(i)}k'_l$. We look at three cases.

(a) Consider $n \equiv l + 2 \pmod 3$. Then possibility (ii) is vacuous and t must be at least 2 in case (i).

Since $\lim_3 k_{l+2} = 15/2\sqrt{145}$ and $\lim_3 {}^{(0)}k'_{l+2} = 37/2\sqrt{145}$ there are clearly only finitely many possibilities for (p, q) in this case.

(b) Consider $n \equiv l \pmod 3$. Then t must be at least 2 in case (i) and case (ii) is excluded since $P_n \equiv 0 \pmod 8$. Since $\lim_3 k_l > \frac{1}{3}k^*$, $\lim_3 {}^{(0)}k'_l > k^*$ and $\lim_3 {}^{(1)}k_l > k^*$ there are again only finitely many possibilities (p, q) .

(c) Consider $n \equiv l + 1 \pmod 3$. Then t must be at least 3 in case (i). Furthermore $\lim_3 k_{l+1} = \frac{1}{9}k^*$, $\lim_3 {}^{(1)}k_{l+1} = k^*$, and $\lim_3 {}^{(0)}k_{l+1}$ and $\lim_3 {}^{(0)}k'_{l+1}$ are greater than k^* in all other cases. Thus we clearly have only finitely many possibilities for (p, q) in this case when $k < k^*$. However if $k = k^*$ there are infinitely many possibilities for (p, q) since $k_{3m+l+4} < k^*$ whenever $k_{3m+l+1} > k^*$.

To complete the proof of Theorem 3 it remains to prove

LEMMA 15. Let $\alpha = [a_0, a_1, \dots] > 0$ be such that with $k = 1.12$ equation (1) has only finitely many solutions (p, q) such that neither p nor q is of the form $4^a(8k + 7)$. Then α satisfies the conditions of Lemma 14.

PROOF. Before proceeding further it should be mentioned that no α satisfying the conditions of Lemma 15 actually exists, due to the fact that numbers congruent to 0 or 4 mod 8 need not be of the form $4^a(8k + 7)$. It is precisely this fact that makes the reduction of $k = 1.12$ to say $k = 1$ extremely difficult in this approach to the problem. We first observe that if one of p_n, q_n is of the form $4^a(8k + 7)$ then unless the other is of the form $2 \cdot 4^a(8k + 7)$ the choice $(p, q) = (2p_n, 2q_n)$ has neither p nor q of the form $4^a(8k + 7)$. Secondly, for at least one of $t = 1, t = 2, t = 3$ the choice $(p, q) = (tp_n, tq_n)$ has neither p nor q of the form $4^a(8k + 7)$. Consequently, if α is as in the statement of the lemma, then for all sufficiently large n we must have

- (i) At least one of p_n, q_n is congruent to 0, 4 or 7 mod 8,
- (ii) $\alpha_n + \alpha'_n \leq 9/1.12$, and hence $a_{n+1} \leq 7$,
- (iii) $\alpha_n + \alpha'_n \leq 4/1.12$, and hence $a_{n+1} \leq 3$, unless one of p_n, q_n is congruent to 7 mod 8 and the other is congruent to 0 or 6 mod 8. Considering (P'_n, Q'_n) and (P_n, Q_n) we also have for sufficiently large n :

(iv) At least one of $P_n, Q_n, P'_{n-1} = p_{n+1} - p_n, Q'_{n-1} = q_{n+1} - q_n$ must be congruent to 0, 4 or 7 mod 8 (since $k'_n + k''_{n-1} < 2$).

(v) If neither $p_{n+1} + p_n$ nor $q_{n+1} + q_n$ is congruent to 0, 4 or 7 mod 8 then (p_{n+1}, q_{n+1}) must be one of (0, 7), (7, 0), (6, 7) or (7, 6) mod 8. For otherwise we would have $a_{n+2} \leq 3$, and $k'_n < 1.12$ unless $a_{n+2} = 3, a_{n+1} = a_{n+3} = 1$ and $a_n \geq 5$. But then $4k_{n+1} < 1$ and $(2p_{n+1}, 2q_{n+1})$ is valid.

(vi) Similarly if neither $p_{n+1} - p_n$ nor $q_{n+1} - q_n$ is congruent to 0, 4 or 7 mod 8 then (p_n, q_n) must be one of (0, 7), (7, 0), (6, 7) or (7, 6) mod 8.

The remainder of the proof relies heavily on the above conditions. If n is sufficiently large there are 236 possibilities for the 4-tuple $F_n = (p_n, q_n, p_{n+1}, q_{n+1}) \pmod 8$ satisfying (i) and the condition $p_n q_{n+1} - p_{n+1} q_n \equiv \pm 1 \pmod 8$. Since the roles of p and q can be interchanged without affecting (i) to (vi) we really only have 118 possibilities to consider. These can be eliminated one by one till we are left only with the possibility that α satisfies the conditions of Lemma 14.

The elimination proceeds as follows.

(a) Ten possibilities for F_n can be eliminated immediately because they do not transform by valid a_{n+2} to valid F_{n+1} . These are (0, 1, 1, 0), (0, 1, 1, 4), (0, 3, 3, 0), (0, 5, 3, 0), (0, 5, 3, 4), (1, 4, 4, 7), (3, 4, 0, 5), (4, 5, 5, 0), (4, 5, 5, 4) and (4, 7, 7, 2). We can now eliminate (1, 7, 0, 1), (1, 7, 4, 3), (1, 7, 4, 5), (2, 7, 1, 4), (3, 7, 0, 3), (3, 7, 0, 5) and (4, 7, 7, 4), followed by (3, 4, 1, 7), (4, 5, 7, 3), (5, 7, 7, 2) and (6, 7, 7, 4), since these transform by valid a_{n+2} only into valid F_{n+1} that have already been eliminated.

(b) Condition (iv) enables (0, 3, 5, 0), (0, 5, 5, 0), (1, 4, 2, 7), (1, 4, 4, 1), (2, 7, 7, 4) and (4, 7, 1, 4) to be eliminated, and consequently (0, 7, 1, 4), (1, 7, 2, 7), (5, 7, 0, 3), (5, 7, 4, 7), (0, 1, 7, 1), (0, 7, 7, 5) and (3, 4, 5, 7) may be eliminated as in the second part of (a).

(c) Condition (v) enables (0, 3, 3, 7), (0, 3, 5, 7), (0, 5, 5, 4), (0, 7, 1, 7), (1, 4, 0, 1), (1, 7, 5, 4), (2, 7, 1, 7), (2, 7, 3, 7), (3, 4, 3, 7), (3, 4, 7, 1), (3, 4, 7, 7), (3, 7, 2, 7), (3, 7, 3, 4), (4, 5, 7, 1), (4, 5, 7, 5), (4, 7, 5, 7), (5, 7, 5, 4), (6, 7, 5, 7), (6, 7, 7, 3), (6, 7, 7, 7) and (7, 7, 3, 4) to be eliminated. In addition condition (vi) enables (0, 1, 1, 7), (0, 3, 3, 4), (0, 3, 5, 4), (0, 5, 3, 7), (0, 5, 5, 7), (1, 4, 7, 5), (1, 7, 3, 4), (1, 7, 7, 2), (2, 7, 7, 1), (2, 7, 7, 5), (3, 4, 4, 5), (3, 7, 1, 0), (3, 7, 5, 4), (4, 5, 1, 7), (4, 5, 7, 7), (4, 7, 1, 0), (5, 7, 0, 5), (5, 7, 3, 4), (7, 7, 0, 1) and (7, 7, 4, 5) to be eliminated. Consequently (0, 1, 7, 0), (0, 1, 7, 2), (0, 1, 7, 4), (0, 7, 7, 2), (0, 7, 7, 3), (0, 7, 7, 6), (3, 4, 0, 3), (3, 4, 4, 3), (4, 5, 3, 0), (4, 5, 3, 4) and (5, 7, 4, 1) may be eliminated.

(d) The following table lists in the first column as H_1, \dots, H_{32} the remaining thirty two possibilities for F_n . In the second column opposite H_i the entry $H_j(k)$ indicates that if $F_n = H_i$ and $a_{n+2} = k$ then $F_{n+1} = H_j$. All such

possibilities are listed. In the third column opposite H_i the entry $H_j(k)$ indicates that if $F_{n-1} = H_i$ and $a_{n+1} = k$ then $F_n = H_j$.

Table 1

F_n	Transforms to $F_{n+1}(a_{n+2})$	Transforms from $F_{n-1}(a_{n+1})$
$H_1 = (0, 7, 1, 0)$	$H_{23}(3)$	$H_6(4), H_{10}(7)$
$H_2 = (0, 7, 7, 0)$	$H_5(1), H_3(4), H_{12}(7)$	$H_7(4), H_{11}(7), H_{13}(2), H_{15}(6), H_{16}(1), H_{17}(3), H_{18}(5)$
$H_3 = (0, 7, 7, 4)$	$H_9(1), H_{14}(2)$	$H_2(4), H_{11}(3), H_{13}(6), H_{15}(2), H_{16}(5), H_{17}(7), H_{18}(1)$
$H_4 = (0, 1, 7, 7)$	$H_{11}(1), H_{20}(2)$	$H_{29}(2), H_{30}(1)$
$H_5 = (0, 7, 7, 7)$	$H_{20}(1)$	$H_2(1), H_7(5), H_{13}(3), H_{15}(7), H_{16}(2), H_{17}(4), H_{18}(6)$
$H_6 = (1, 4, 0, 7)$	$H_1(4)$	$H_{21}(1), H_{31}(2)$
$H_7 = (4, 7, 7, 0)$	$H_{12}(3), H_2(4), H_5(5)$	
$H_8 = (1, 4, 7, 3)$	$H_{17}(1)$	$H_{31}(1)$
$H_9 = (4, 7, 3, 7)$	$H_{19}(1)$	$H_3(1)$
$H_{10} = (1, 7, 0, 7)$	$H_1(7)$	$H_{12}(1)$
$H_{11} = (7, 7, 0, 7)$	$H_3(3), H_{12}(6), H_2(7)$	$H_4(1)$
$H_{12} = (0, 7, 7, 1)$	$H_{10}(1)$	$H_2(7), H_7(3), H_{11}(6), H_{13}(1), H_{15}(5), H_{17}(2), H_{18}(4)$
$H_{13} = (2, 7, 7, 0)$	$H_{12}(1), H_2(2), H_5(3), H_3(6)$	
$H_{14} = (4, 7, 7, 6)$	$H_{15}(4)$	$H_3(2), H_{22}(1)$
$H_{15} = (6, 7, 7, 0)$	$H_3(2), H_{12}(5), H_2(6), H_5(7)$	$H_{14}(4), H_{32}(7)$
$H_{16} = (1, 7, 7, 0)$	$H_2(1), H_5(2), H_3(5)$	
$H_{17} = (3, 7, 7, 0)$	$H_{12}(2), H_2(3), H_5(4), H_5(7)$	$H_8(1)$
$H_{18} = (5, 7, 7, 0)$	$H_3(1), H_{12}(4), H_2(5), H_5(6)$	
$H_{19} = (3, 7, 7, 6)$	$H_{30}(3), H_{31}(7)$	$H_9(1), H_{23}(2)$
$H_{20} = (7, 7, 6, 7)$	$H_{31}(3), H_{30}(7)$	$H_4(2), H_5(1)$
$H_{21} = (3, 7, 4, 1)$	$H_6(1)$	
$H_{22} = (3, 7, 4, 7)$	$H_{14}(1)$	$H_{23}(1)$
$H_{23} = (0, 1, 7, 3)$	$H_{22}(1), H_{19}(2)$	$H_1(3), H_{22}(2), H_{28}(1)$
$H_{24} = (0, 1, 7, 5)$	$H_{32}(1)$	$H_{28}(1)$
$H_{25} = (0, 1, 7, 6)$	$H_{31}(4)$	$H_{29}(1)$
$H_{26} = (1, 4, 6, 7)$	$H_{30}(4)$	
$H_{27} = (1, 7, 1, 0)$	$H_{23}(2)$	
$H_{28} = (2, 7, 1, 0)$	$H_{23}(1), H_{24}(3)$	
$H_{29} = (5, 7, 1, 0)$	$H_{25}(1), H_4(2)$	
$H_{30} = (6, 7, 1, 0)$	$H_4(1)$	$H_{19}(3), H_{20}(7), H_{26}(4)$
$H_{31} = (6, 7, 1, 4)$	$H_8(1), H_6(2)$	$H_{19}(7), H_{20}(3), H_{25}(4)$
$H_{32} = (5, 7, 6, 7)$	$H_{15}(7)$	$H_{24}(1)$

Clearly by assuming n sufficiently large the transient states $H_7, H_{13}, H_{16}, H_{18}, H_{21}, H_{26}, H_{27}, H_{28}, H_{29}, H_{24}, H_{32}$ and H_{25} may be eliminated.

(e) Consider the possibility $F_n = H_{10}$. Then $F_{n+1} = H_1$ and $F_{n+2} = H_{23}$. If $F_{n+3} = H_{22}$ then $k_{n+2}^{-1} > [3, 2] + [0, 8]$ contradicting (iii), so $F_{n+3} = H_{19}$. But now

$$k_{n+1}^{-1} > [0, 1, 2, 1, 1, 8] + [7, 3, 2, 7, 1, 1, 2] > 9/1.12$$

contradicting (ii). Hence H_{10} , and consequently H_{12} , may be eliminated.

(f) Consider the possibility $F_n = H_1$. We must have

$$k_{n+1}^{-1} > [3, 2, 3, 2] + [0, 4, 2, 8] > 4/1.12$$

contradicting (iii). Thus H_1 may be eliminated, followed by H_6, H_{23} and H_{22} .

(g) Consider the possibility $F_n = H_{19}$. We cannot have $F_{n+1} = H_{31}$ for then

$$k_{n+1}^{-1} = [7, 1, 1, \dots] + [0, 1, 1, \dots] > 9/1.12$$

contrary to (ii). Hence $F_{n+1} = H_{30}$ and $(p, q) = (P'_n, Q'_n)$ is valid. Since k''_n is an increasing function of α_{n+1} and α'_{n+1} we have

$$k''_n < [1, 1, 2, 3, 2][2, 1, 1, 2]/([3, 1, 2, 3, 2] + [0, 1, 1, 2]) < 1.1.$$

Thus, for sufficiently large n we may eliminate H_{19} and consequently H_9 .

(h) Consider the possibility $F_n = H_{14}$. Then $(p, q) = (P_n, Q_n)$ is valid and since k'_n is an increasing function of α_{n+1} and α'_{n+1} we have

$$k'_n < [3, 2, 2, 4][1, 2, 8]/([4, 2, 2, 4] + [0, 2, 8]) < 1.1.$$

Hence for sufficiently large n we may eliminate H_{14} and consequently H_3 and the transient state H_{15} . But now $k_{n+1}^{-1} = [7, 1, \dots] + [0, 1, \dots] > 9/1.12$ if $F_n = H_{11}$ so we may also eliminate H_{11} .

(j) Consider the possibility that $F_n = H_{20}$ and suppose that $F_{n+1} = H_{31}$. Then $(p, q) = (P'_n, Q'_n)$ is valid and

$$k''_n < [1, 1, 1, 3, 2][2, 1, 5]/([3, 1, 1, 3, 2] + [0, 1, 5]) < 1.1.$$

This is impossible for sufficiently large n , so for large n we have that if $F_n = H_{20}$ then $F_{n+1} = H_{30}$. But then we may eliminate H_{31}, H_8, H_{17}, H_2 and H_5 as being transient. We are now left only with the possibility that the conditions of Lemma 14 are satisfied, for H_4, H_{20} and H_{30} are the only remaining possibilities.

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