On some results concerning integral equations.

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1. *Description of the paper.*

It is proposed in this paper to show how the well-known Laplace's transformation,

$$y(z)=\int_0^b e^{azt}v(t)dt,$$

which is of great help in finding the solution of linear differential equations, gives also interesting results concerning the theory of integral equations. In §2 we shall study its application to certain differential equations, and find a large class of equations which remain unchanged by this transformation. Then, (§3), taking instead of e^{azt}, a more general function of the product zt, we shall find a solution for some homogeneous integral equations; in §4 we shall describe a method of solving a very general type of integral equation of the first kind, namely,

$$F(z)=\int_0^\infty f(zt)\Phi(t)dt\ ;$$

a further extension to integral equations with the kernel $e^{f(z)f(t)}$ is the object of §5. Then, studying an extension of Euler's transformation, we shall (§6) consider equations such as

$$y(z)=\int_{-\infty}^{+\infty} f(t-z)y(t)dt,$$

which will prove to be singular; and finally, in §7, we shall give other examples of singular integral equations.

2. *Laplace's transformation.*

Let us consider a linear differential equation of the n^{th} order

$$F(y, y', y'', \ldots, y^{(n)}, z)=0 \qquad \ldots\ldots(1)$$

and try to solve it by putting

$$y(z)=\int_a^b e^{azt}v(t)dt \qquad \ldots\ldots(2)$$

If the limits a and b are conveniently chosen, we obtain for v a linear differential equation of the n^{th} order,

$$G(v, v', v'', \dots v^{(n)}, t) = 0 \dots\dots\dots\dots\dots\dots\dots\dots(3)$$

Between a solution y_1 of (1) and a solution v_1 of (3) exists the relation (2), which is an integral equation of the first kind. If in particular we have

$$y_1 = \lambda_1 v_1$$

where λ_1 is a constant, y_1 will be a solution of the homogeneous integral equation

$$y(z) = \lambda \int_a^b e^{azt} y(t) dt$$

for the value λ_1 of λ. This case will always occur if, the equation (1) being of the first order, the equation (3) turns out to be precisely the same as (1).

If the two equations are the same, but of order n, then we shall have a relation such as

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n = \lambda \int_a^b e^{at}[c_1' y_1 + c_2' y_2 + \dots + c_n' y_n] dt,$$

which is an integral equation, generally of the first kind.

If we now consider the most general linear differential equation of order p, in which the coefficients are polynomials of degree p with respect to z, we can enquire what form these coefficients must have in order that they may remain unchanged by Laplace's transformation.

If we write the equation in the following form

$$\left.\begin{aligned}&(a_{p,p} z^p + a_{p,p-1} z^{p-1} + a_{p,p-2} z^{p-2} + \dots + a_{p,0}) \frac{d^p y}{dz^p} \\ &+ (a_{p-1,p} z^p + a_{p-1,p-1} z^{p-1} + \dots + a_{p-1,0}) \frac{d^{p-1} y}{dz^{p-1}} \\ &+ \dots + (a_{0,p} z^p + a_{0,p-1} z^{p-1} + \dots + a_{0,0}) y = 0,\end{aligned}\right\} \dots\dots\dots(4)$$

it can be easily seen, by forming the coefficient of the term $t^n \frac{d^{p-j} v}{dt^{p-j}}$, in the transformed equation (n and j being arbitrary integers), that the equation will remain unchanged if, between the $(p+1)^2$ coefficients of (4) there exist the following $(p+1)^2$ relations, where the symbol ${}_iC_r$ stands for the number of combinations of r numbers i at a time

$$\left.\begin{aligned} a_{p-j,\,n} = \alpha^{n+j-p}[(-1)^p{}_jC_p(n+1)(n+2)\ldots(n+j)a_{n+j,\,p} \\ +(-1)^{p-1}{}_{j-1}C_{p-1}(n+1)(n+2)\ldots(n+j-1)a_{n+j-1,\,p-1} \\ +\ldots+(-1)^{p-j}{}_0C_{p-j}a_{n,\,p-j}] \end{aligned}\right\} \quad (5)$$

where n and j take all the values 0, 1, 2, ...p,

With these formulae, it is possible to form the most general equation such as (4), of any order, which remains unaltered by Laplace's transformation.

Many remarks can be made from the formulae (5).

For instance, making

$$j=0 \quad n=p,$$

we obtain

$$a_{p,\,p} = (-1)^p a_{p,\,p}$$

which shows that, if p is an even number, $a_{p,\,p}$ can be taken arbitrarily. But, if p is an odd number, $a_{p,\,p}=0$.

In the same way, making

$$n=0 \quad j=p$$

we find that $a_{0,\,0}$ is always arbitrary.

These results give for the equation of the second order

$$(a_{2,\,2}z^2 + a_{2,\,0})y'' + 2a_{2,\,2}zy + (\alpha^2 a_{2,\,0}z^2 + a_{0,\,0})y = 0,$$

and for the equation of the third order,

$$\begin{aligned} a_{3,\,1}zy''' + (a_{2,\,2}z^2 + a_{2,0})y'' \\ + (-\alpha^2 a_{3,\,1}z^3 + 2a_{2,\,2}z)y' + [\alpha^2 z^2(-3a_{3,\,1} + a_{2,\,0}) + a_{0,\,0}]y = 0. \end{aligned}$$

The above equation of the second order comprises, as a particular case, the well-known Euler's equation; it shows that our integral equation can be satisfied by the root $y(z) = \dfrac{1}{\sqrt{z}}$; but we do not insist now upon this result, which we shall soon consider in a more general case.

The case of the parabolic cylinder equation is also a particular case of this above equation; such is also the equation

$$y'' + z^2 y = 0,$$

whose solution can be expressed by the aid of Bessel's functions.

We shall now extend Laplace's transformation to integral equations with a more general kernel.

3. *Extension of Laplace's transformation.*

In the differential equation

$$2zy' + y = 0$$

let us make the following change of variable

$$y(z) = \int_a^b f(zt)v(t)dt$$

where f is an arbitrary function of the product zt. As we have

$$y'(z) = \int_a^b f'(zt)tv(t)dt,$$

our equation becomes

$$\int_a^b dt[2zf'(zt)tv(t) + f(zt)v(t)] = 0.$$

But we have, by integrating by parts,

$$2\int_a^b zf'(zt)tv(t)dt = \Big[2f(zt)tv(t)\Big]_a^b - 2\int_a^b f(zt)[tv' + v]dt.$$

Let us take now for a and b the values 0 and $\pm\infty$, if for these values

$$\Big[2f(zt)tv(t)\Big]_a^b = 0.$$

Then the differential equation which we obtain for v is

$$2tv' + v = 0,$$

and so is the same as the original equation for y. This equation having the solution $v = \dfrac{1}{\sqrt{t}}$, we find that the homogeneous integral equation

$$y(z) = \lambda\int_0^{\pm\infty} f(zt)y(t)dt$$

has, for a certain value of λ, the solution $y = \dfrac{1}{\sqrt{z}}$.

This value of λ is obtained by making $z = 1$, which gives

$$\frac{1}{\lambda} = \int_0^{\pm\infty} f(t)\frac{dt}{\sqrt{t}}.$$

By a change of variable, we find also that the integral equation

$$y(z) = \lambda\int_a^{\pm\infty} f[(z-a)(t-a)]y(t)dt$$

where $\dfrac{1}{\lambda} = \displaystyle\int_a^{\pm\infty} f[t-a]\frac{dt}{\sqrt{t-a}}$ has the solution $y = \dfrac{1}{\sqrt{z-a}}$.

4. *Application of the method to integral equations of the first kind.*

Instead of the above differential equation, let us now consider the following

$$zy' - ny = 0$$

whose solution is

$$y = z^n.$$

If we make the same change of variable, namely

$$y(z) = \int_0^\infty f(zt)v(t)dt,$$

we find in the same way that the transformed equation for v is

$$tv' + v(-n-1) = 0$$

which has the solution

$$v = t^{-1-n}.$$

So we find this result, that the homogeneous integral equation of the first kind

$$z^n = \int_0^\infty f(zt)v(t)dt$$

has the solution

$$v = \lambda_n t^{-1-n},$$

where λ_n is a constant determined by

$$\frac{1}{\lambda_n} = \int_0^\infty f(t)t^{-1-n}dt.$$

If we now consider an integral equation of the first kind

$$F(z) = \int_0^\infty f(zt)\Phi(t)dt,$$

where F is known, and Φ unknown, we can find its solution in the following way:

Let us expand the function $F(z)$, for points z within an annulus whose centre is at the origin, in a Laurent series, multiplied by a fractional power of z, if necessary, to allow for multiformity: so

$$F(z) = z^h[a_0 + a_1 z + a_2 z^2 + \ldots + b_1 z^{-1} + b_2 z^{-2} + \ldots].$$

Then the unknown function $\Phi(t)$ will be

$$\Phi(t) = t^{-1-h}[\lambda_0 a_0 + \lambda_1 a_1 t^{-1} + \lambda_2 a_2 t^{-2} + \ldots + \lambda_{-1} b_1 t + \lambda_{-2} b_2 t^2 + \ldots],$$

where the λ's are given by the formula

$$\lambda_i = \frac{1}{\int_0^\infty f(t)\, t^{-1-h-i}\, dt}.$$

As an example of this, we can take the integral equation

$$\frac{1+cz}{z^2}=\int_0^\infty e^{-\frac{1}{2}z^2t^2}\Phi(t)dt,$$

and it can be immediately seen that its solution, obtained by this method, is

$$\Phi(t)=t+c\sqrt{\frac{2}{\pi}}.$$

5. *Further extension.*

Let us consider now the equation

$$f(z)y'-\mathrm{A}f'(z)y=0$$

where f is an arbitrary function of z, and make the transformation

$$y(z)=\int_a^b e^{f(z)f(t)}v(t)dt.$$

It becomes, after dividing by $f'(z)$,

$$\int_a^b dt\, e^{f(z)f(t)}[f(z)f(t)v(t)-av(t)].$$

On the other hand, we have

$$\int_a^b f(z)e^{f(z)f(t)}f(t)v(t)dt=\left[e^{f(z)f(t)}\frac{f(t)v(t)}{f'(t)}\right]_a^b-\int_a^b e^{f(z)f(t)}dt\frac{f'^2v+ff'v'-ff''v}{f'^2}$$

So that if we take for a and b two roots of $f(t)=0$, the differential equation for v will be

$$f'(t)f(t)v'+v[(1+\mathrm{A})f'^2(t)-f(t)f''(t)]=0.$$

This equation is not the same as the original one; and, if we choose the function f in order to make it the same, we shall find nothing but

$$f(z)=(z+m)^n,$$

which gives only a particular case of the kernel studied in § 2. However, we can go on with these differential equations. The solution of the equation for y is

$$y(z)=[f(z)]^{\mathrm{A}},$$

and for v,

$$v(t)=f'(t)[f(t)]^{-1-\mathrm{A}}.$$

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So that we have the result that, a and b being two roots of $f(t)$, (or also values making $e^{f(t)}=0$, *i.e.* $f(t)=-\infty$), the homogeneous integral equation of the first kind

$$[f(z)]^{A}=\int_a^b e^{f(z)f(t)}v(t)dt$$

has the solution

$$v(t)=\lambda_A f'(t)[f(t)]^{-1-A}$$

where λ_A is a constant determined by

$$\frac{1}{\lambda_A}=\int_a^b e^{f(t)}f'(t)[f(t)]^{-1-A}dt.$$

We can extend this result, in the same way as in the preceding section, to an integral equation

$$F[f(z)]=\int_a^b e^{f(z)f(t)}\Phi(t)dt$$

where F can be developed in powers of $f(z)$:

$$F[f(z)]=a_0+a_1f(z)+a_2f^2(z)+\ldots.$$

We shall have then

$$\Phi(t)=f'(t)\left[\frac{a_0\lambda_0}{f(t)}+\frac{a_1\lambda_1}{f^2(t)}+\ldots\right].$$

the λ's being determined by the above formula.

6. *Other transformations.*

Euler has used the transformation

$$y(z)=\int_a^b (t-z)^n v(t)dt$$

to solve linear differential equations; we can extend it, as we did Laplace's, to the theory of integral equations.

Let $f(u)$ be an even function of u, which is zero for $u=\pm\infty$; put

$$y(z)=\int_{-\infty}^{+\infty} f(t-z)v(t)dt.$$

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The kernel is, as required in the ordinary theory, symmetrical, f being even. We have then

$$y'(z) = -\int_{-\infty}^{+\infty} f'(t-z)v(t)dt$$
$$= -\Big[f(t-z)v(t)\Big]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} f(t-z)v'(t)dt$$
$$= \int_{-\infty}^{+\infty} f(t-z)v'(t)dt.$$

so that, if we consider the equation

$$y' - ay = 0,$$

it becomes, by this change of variable,

$$v' - av = 0.$$

The solution of this equation being $v = e^{at}$, we find in that way that the homogeneous integral equation of the second kind

$$y(z) = \int_{-\infty}^{+\infty} f(t-z)y(t)dt$$

has, for a certain value of λ, the solution $y = e^{az}$.

The question is now, what is this value of λ.

We obtain it as we did before, and find

$$\frac{1}{\lambda} = \int_{-\infty}^{+\infty} f(t)e^{at}dt.$$

But a is an arbitrary constant, to which we can give an infinity of values; to each of these values corresponds a value of λ, and inversely, to a given value of λ corresponds a quantity a, and therefore a solution e^{az} for the integral equation. This equation appears then to be a singular integral equation, not having, as in the general case, a limited number of *auto-values* for the parameter λ, but having an infinity of them; in fact, any given number is an auto-value. The relation between the given λ and the corresponding a is the above. It can also be pointed out that to a chosen λ correspond two values of a, namely a and $-a$, for

$$\int_{-\infty}^{+\infty} e^{at}f(t)dt = \int_{-\infty}^{+\infty} e^{-at}f(t)dt.$$

But these are not the only singularities of that equation. The special value of λ corresponding to $a = 0$, *i.e.* λ_0, where

$$\frac{1}{\lambda_0} = \int_{-\infty}^{+\infty} f(t)dt = 2\int_0^{\infty} f(t)dt$$

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is remarkable in another way. For if we consider the integral

$$I = \int_{-\infty}^{+\infty} f(t-z)t\,dt,$$

we can, to find its value, make the change of variable $t = u + z$, so that

$$I = \int_{-\infty}^{+\infty} f(u)u\,du + z\int_{-\infty}^{+\infty} f(u)du = \frac{z}{\lambda_0},$$

for, owing to the fact of f being even, the first of these two integrals is equal to nothing. So, for the particular value $\lambda = \lambda_0$, our equation is satisfied not only by $y(z) = \text{const.}$, but also by $y(z) = z$. It can be shown in a similar way that if the integral

$$\int_{-\infty}^{+\infty} f(u)u^2 du$$

is equal to nothing, the equation possesses also the root $y(z) = z^2$, for the same value of λ. And then, in this case, as

$$\int_{-\infty}^{+\infty} f(u)u^3 du = 0,$$

it has also the root $y(z) = z^3$, and so on.

Remarks of the same kind can be made on the subject of the integral equation

$$y(z) = \lambda\int_{-\infty}^{+\infty} f(z+t)y(t)dt,$$

which can be connected with the differential equation

$$y'' + ay = 0,$$

and is a singular integral equation, any value of λ being an auto-value; and the two values

$$\frac{1}{\lambda} = \pm\int_{-\infty}^{+\infty} f(u)du$$

having special properties.

We can also extend our method to integral equations of the type

$$y(z) = \int F[f(z) - f(t)]v(t)dt,$$

with convenient limits; they can be connected to the equation

$$y' - af'(z)y = 0,$$

and we obtain the integral equation of the first kind

$$e^{af(z)} = \lambda \int F[f(z) - f(t)]f'(t)e^{af(t)}dt.$$

As we did in some of the preceding sections, we can apply this result to the case of a function such as

$$y(z) = a_0 + a_1 e^{f(z)} + a_2 e^{2f(z)} + \dots .$$

But this transformation, as well as the similar one,

$$y(z) = \int F[f(z) + f(t)]v(t)dt,$$

is of no use to obtain roots for new types of integral equations of the second kind.

7. *Singular integral equations.*

This brings us to say some few words about homogeneous singular integral equations. It is well known that the equation

$$y(z) = \lambda \int_0^\infty \cos zt \, . \, y(t)dt$$

has, for the special value $\lambda = \sqrt{\frac{2}{\pi}}$, an infinite number of solutions, owing to the fact that the integral

$$f(z) = \int_0^\infty \cos zt \phi(t)dt$$

possesses an inversion formula, which is Fourier's formula,

$$\phi(z) = \frac{2}{\pi}\int_0^\infty \cos zt \, f(t)dt.$$

We can extend this as follows:

Let us consider the integral equation

$$f(z) = \int_a^b K(z, t)g(t)\phi(t)dt,$$

where $K(z, t)$ is a symmetrical kernel, $g(t)$ a known function, and $\phi(t)$ an unknown one, and suppose that an inversion formula exists for this integral, and is of the type

$$\phi(z) = A\int_a^b K(z, t)g(t)f(t)dt,$$

where A is a constant. (Fourier's formula is a particular case of this inversion).

Then let us consider the function

$$\psi(z) = f(z) + \frac{1}{\sqrt{A}}\phi(z).$$

Replacing f and ϕ by their expressions as integrals, we have

$$\psi(z) = \int_a^b dt K(z, t) g(t)\left[\phi(t) + \sqrt{A} f(t)\right] = \sqrt{A}\int_a^b dt K(z, t) g(t)\psi(t).$$

We obtain thus an integral equation of the first kind. Let us make the change of variable

$$t = \chi(\theta)$$

where θ is defined by

$$\theta = \int g(t)dt.$$

Let us put also $z = \chi(u)$, and denote $\psi[\chi(u)]$ by $\Psi(u)$. Our integral equation becomes an integral equation of the second kind

$$\Psi(u) = \sqrt{A}\int_a^b K[\chi(u), \chi(\theta)]\Psi(\theta)d\theta,$$

and, for this special value $\sqrt{A}$ of the parameter, it has an infinite number of roots, namely,

$$\Psi(u) = f[\chi(u)] + \frac{1}{\sqrt{A}}\phi[\chi(u)].$$

A remarkable example of this happening is given by using Hankel's formula

$$f(z) = \int_0^\infty J_n(zt)t\phi(t)dt, \quad \phi(z) = \int_0^\infty J_n(zt)tf(t)dt\,;$$

it shows that the integral equation

$$\psi(z) = \int_0^\infty J_n(2\sqrt{zt})\psi(t)dt$$

has an infinite number of solutions.

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