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RESEARCH ARTICLE

Polypositroids

Thomas Lam[1] and Alexander Postnikov[2]

[1]Department of Mathematics, University of Michigan, 530 Church St., Ann Arbor, MI 48109, USA; E-mail: tfylam@umich.edu.
[2]Department of Mathematics, M.I.T., 77 Massachussetts Ave., Cambridge, MA 02139, USA; E-mail: apost@math.mit.edu.

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Abstract

We initiate the study of a class of polytopes, which we coin *polypositroids*, defined to be those polytopes that are simultaneously generalized permutohedra (or polymatroids) and alcoved polytopes. Whereas positroids are the matroids arising from the totally nonnegative Grassmannian, polypositroids are "positive" polymatroids. We parametrize polypositroids using Coxeter necklaces and balanced graphs, and describe the cone of polypositroids by extremal rays and facet inequalities. We introduce a notion of (W, c)-*polypositroid* for a finite Weyl group W and a choice of Coxeter element c. We connect the theory of (W, c)-polypositroids to cluster algebras of finite type and to generalized associahedra. We discuss *membranes*, which are certain triangulated 2-dimensional surfaces inside polypositroids. Membranes extend the notion of plabic graphs from positroids to polypositroids.

Contents

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1. Introduction

1.1. Polypositroids

The aim of this work is to study a new class of polytopes, which we call *polypositroids*. These are polytopes which are simultaneously generalized permutohedra (or polymatroids) and alcoved polytopes.

A permutohedron is the convex hull of the orbit of a point in $\mathbb{R}^n$ under the action of the symmetric group S_n. A *generalized permutohedron* is obtained from a permutohedron by parallel translation of some of the facets. The class of generalized permutohedra include many classical polytopes: the usual permutohedron, the associahedron, hypersimplices, matroid polytopes, and many others. Generalized permutohedra are polytopal analogues of matroids, and are essentially equivalent to the *polymatroids* of Edmonds and the closely related *submodular functions* [Edm, Po09, CL].

An *alcoved polytope* is a polytope whose facets are normal to roots in the type A_{n-1} root system. In [LP07], we studied the class of integer alcoved polytopes, which are (convex) unions of alcoves in the affine Coxeter arrangement of type A. In the present paper, we work with alcoved polytopes that may not be integral.

Our initial motivation for studying polypositroids as a subclass of polymatroids comes from the theory of total positivity. Lusztig [Lus] and Postnikov [Po06] have defined the totally nonnegative Grassmannian $\mathrm{Gr}(k,n)_{\geq 0}$, a subspace of the real Grassmannian $\mathrm{Gr}(k,n)$ of k-planes in $\mathbb{R}^n$. Any point $X \in \mathrm{Gr}(k,n)$ gives rise to a (realizable) matroid M_X. When $X \in \mathrm{Gr}(k,n)_{\geq 0}$ is totally nonnegative, the matroid M_X is called a *positroid*, short for "positive matroid." Positroids were classified in [Po06] and [Oh], and the geometry of *positroid varieties* in the Grassmannian was studied by Knutson-Lam-Speyer [KLS].

To a matroid M on the set $\{1,2,\ldots,n\}$, one has an associated matroid polytope $P_M \subset \mathbb{R}^n$. Our investigations began with the observation that a matroid M is a positroid if and only if P_M is an alcoved polytope (Theorem 2.1). Polypositroids are thus "positive polymatroids." In the present work, we study polypositroids from a discrete geometer's perspective, leaving aside potential connections to Grassmannians, and so on.

Whereas matroids are notoriously difficult to parametrize, the subclass of positroids were parametrized in [Po06, Oh]. In Theorem 6.12, we give a parametrization of polypositroids, showing that they are in bijection with *Coxeter necklaces* and with *balanced graphs*. Coxeter necklaces and balanced graphs are generalizations of the *Grassmann necklaces* and *decorated permutations* of [Po06]. The set of generalized permutohedra attains a cone structure under the Minkowski sum, and the corresponding cone is the *cone of submodular functions* $\mathcal{C}_{\mathrm{sub}}$ [Edm]. Alcoved polytopes and polypositroids also form cones $\mathcal{C}_{\mathrm{alc}}$ and $\mathcal{C}_{\mathrm{pol}}$. There is a (projection) map of cones $\mathcal{C}_{\mathrm{sub}} \to \mathcal{C}_{\mathrm{alc}}$, and we show in Theorem 4.8 that the image of this map is $\mathcal{C}_{\mathrm{pol}}$. In Corollary 6.14 and Theorem 4.9, we describe the extremal rays and give defining inequalities for the cone $\mathcal{C}_{\mathrm{pol}}$. For example, the extremal rays of $\mathcal{C}_{\mathrm{pol}}$ are indexed by directed cycles on $\{1,2,\ldots,n\}$.

The normal fan of the permutohedron is the *braid fan* associated to the arrangement of hyperplanes $\{x_i - x_j = 0\}$. Generalized permutohedra are exactly the polytopes with a normal fan, a coarsening of the braid fan. In Section 8, we study the possible normal fans $\mathcal{N}(P)$ for a generic simple polypositroid P. In contrast to generalized permutohedra, there is more than one such normal fan, and each such $\mathcal{N}(P)$ is a coarsening of the braid fan.

We show in Lemma 8.2 that each maximal cone C_T of $\mathcal{N}(P)$ is labeled by an *alternating noncrossing* tree. The maximal cones C_w, $w \in S_n$ satisfying $C_w \subset C_T$ are described in Proposition 8.22 in terms of a dual *noncrossing circular-alternating tree*. In Theorem 8.4, we show that the normal fan $\mathcal{N}(P)$ is characterized by a *matching ensemble*, a collection of perfect matchings, one for each bipartite subgraph of the complete graph, satisfying certain axioms. Our matching ensembles are a variant of the matching ensembles of Oh and Yoo [OY], and the matching fields of Sturmfels and Zelevinsky [SZ]. In Theorem 8.1, we show that all generic simple polypositroids have the same f-vector, identical to that of the *cyclohedron*.

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1.2. Coxeter polypositroids

In the second part of this work, we generalize the theory of polypositroids to the root-system theoretic setting. Let V be a real vector space, $R \subset V$ be a crystallographic root system with Weyl group W, and $R^+ \subset R$ be a choice of positive roots. A *generalized W-permutohedron* is a polytope $P \subset V$ whose edges are in the directions of R.

Now fix the choice of a Coxeter element c. Define the *twisted root system* $\tilde{R} := (I - c)^{-1}R$. We define a (W, c)-*twisted alcoved polytope* to be a polytope $P \subset V$ whose facet normals belong to the twisted roots $\tilde{R}$. These polytopes are c-twisted variants of the alcoved polytopes we studied in [LP18]. A (W, c)-*polypositroid* is a polytope that is simultaneously a generalized W-permutohedron and a (W, c)-twisted alcoved polytope. When R is of type A_{n-1} and c is the long cycle $(12\cdots n)$, this definition reduces to the earlier one.

In general, the class of (W, c)-polypositroids cannot be parametrized in the same manner that we did for polypositroids. We introduce a larger class of (W, c)-*prepolypositroids* as a compromise. The set of (W, c)-prepolypositroids is defined as a cone, by giving the facet inequalities (16.1) satisfied by the support function. We define (W, R^+, c)-Coxeter necklaces and (W, c)-balanced arrays and show in Theorem 16.5 that these objects are in bijection with (W, c)-prepolypositroids. We also show that there is a projection map from the cone $\mathcal{C}^W_{\text{sub}}$ of W-submodular functions (see [ACEP]) to the cone $\mathcal{C}^{W,c}_{\text{pre}}$ of (W, c)-prepolypositroids, and we conjecture (Conjecture 17.2) that this map is surjective.

In Section 18, we connect the theory of (W, c)-prepolypositroids to cluster algebras. Let $\mathcal{A}(W, R^+, c)$ be the cluster algebra of finite type associated to the Coxeter element c, as studied by Yang and Zelevinsky [YZ]. We show, using the work of Padrol et al. [P^4], in Theorem 18.5 that each exchange relation of $\mathcal{A}(W, R^+, c)$ gives rise to an inequality satisfied by the cone $\mathcal{C}^{W,c}_{\text{pre}}$ of (W, c)-prepolypositroids. We show that (W, c)-prepolypositroids are closely related to generalized associahedra, one of our main examples of (W, c)-polypositroids.

In Section 19, with the aim of studying the normal fans of (W, c)-prepolypositroids, we develop notions of *alternating* and *c-noncrossing* for root systems, and a notion of *c-noncrossing tree*. These notions are related to the theory of finite type cluster algebras, and to the theory of reflection factorizations of Coxeter elements.

1.3. Membranes

The third part of the paper is devoted to *membranes*. In Section 20, we define R-membranes, for a root system $R \subset V$, which are essentially triangulated 2-dimensional surfaces in V homeomorphic to wedges of disks, such that every edge of every triangle in them is a parallel translation of a root from R. An R-membrane is *minimal* if it has minimal possible surface area among all membranes with the same *boundary loop*. We view the problem of describing minimal membranes as a discrete version of *Plateau's problem* from geometric measure theory concerning the existence of a minimal surface with a given boundary. In the rest of the paper, we discuss membranes of type A. In Section 21, we show that membranes of type A are closely related to Postnikov's *plabic graphs*, introduced in the study of the totally nonnegative Grassmannian $\text{Gr}(k, n)_{\geq 0}$ [Po06]. For each positroid, there is a class of *reduced* plabic graphs connected with each other by local moves. Each reduced plabic graph gives a parametrization of the associated positroid cell in $\text{Gr}(k, n)_{\geq 0}$.

Membranes (of type A) are in bijection with plabic graphs with faces labelled by integer vectors. In Section 22, we show that local moves of plabic graphs correspond to octahedron and tetrahedron moves of membranes. In Section 23, we show that minimal membranes correspond to reduced plabic graphs.

In Sections 24, 25, and 26, we discuss special classes of membranes associated with positroids and polypositroids. In Section 27, we define *semisimple* membranes as membranes that project bijectively onto the Coxeter plane. We show that notions of semisimple membranes and minimal membranes are equivalent to each other (for a particular class of boundary loops).

The structures that we study in this paper are related to the theory of *cluster algebras* in several different ways. While in Section 18, we connect (W, c)-prepolypositroids to cluster algebras of finite

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type, in Section 28, we connect membranes (of type A) to a certain class of cluster algebras, which, in general, are not of finite type. We discuss the *higher octahedron recurrence* as a certain rational recurrence relation on variables x_λ labelled by integer vectors λ. This relation naturally extends the octahedron recurrence on $\mathbb{Z}^3$ [Spe] to a higher dimensional integer lattice. With each polypositroid P, we associate a cluster algebra $\mathcal{A}_P$ generated by some finite subset of variables x_λ. Minimal membranes for the polypositroid P correspond to a class of clusters of this cluster algebra, and local moves of membranes correspond to cluster *mutations*. Remarkably, the class of cluster algebras $\mathcal{A}_P$ is the same as the subclass associated to positroids, which Galashin and Lam have shown [GL] to be isomorphic to the coordinate rings of open positroid varieties [KLS].

Finally, in Section 29, we propose an area of study, which we dub the "Asymptotic Cluster Algebra." We pose a problem related to asymptotics of membranes under dilations. The setup of membranes extends several models from statistical physics.

Part I Polypositroids

Throughout the paper, we use the following notation. Let $[n] := \{1, 2, \ldots, n\}$. Let $\binom{[n]}{k}$ denote the set of k-element subsets of $[n]$. Also, let $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{R}^n$.

2. Positroid polytopes

The material of this section serves as motivation, and is largely independent of the rest of the work.

Let $\mathrm{Gr}(k, n)$ denote the Grassmannian of k-planes in $\mathbb{R}^n$. We may represent a point $X \in \mathrm{Gr}(k, n)$ as a $k \times n$ matrix. For a k-element subset $I \in \binom{[n]}{k}$, the *Plücker coordinate* $\Delta_I(X)$ is defined to be the $k \times k$ minor indexed by the columns I. The *matroid* $\mathcal{M}_X$ is given by $\mathcal{M}_X := \{I \in \binom{[n]}{k} \mid \Delta_I(X) \neq 0\}$. The *totally nonnegative Grassmannian* [Po06] $\mathrm{Gr}(k, n)_{\geq 0}$ is the subspace of $\mathrm{Gr}(k, n)$ represented by matrices, all of whose Plücker coordinates are nonnegative. The matroid of a totally nonnegative point $X \in \mathrm{Gr}(k, n)_{\geq 0}$ is called a *positroid*.

The matroid polytope $P_{\mathcal{M}}$ of a matroid $\mathcal{M}$ is the convex hull of the vectors e_I, $I \in \mathcal{M}$, where $e_I := e_{i_1} + \cdots + e_{i_k} \in \mathbb{R}^n$ if $I = \{i_1, \ldots, i_k\}$. By [G^{2}MS], matroid polytopes are exactly those polytopes whose vertices belong to $\{e_I \mid I \in \binom{[n]}{k}\}$ and whose edges are parallel to vectors of the form $e_i - e_j$. We have the following characterization of the matroid polytopes of positroids.

Theorem 2.1. *A matroid $\mathcal{M}$ is a positroid if and only if the matroid polytope $P_{\mathcal{M}}$ is an alcoved polytope, that is, it is given by inequalities of the form $c_{ij} \leq x_i + x_{i+1} + \cdots + x_j \leq b_{ij}$, for $1 \leq i < j \leq n$, where $c_{ij}, b_{ij} \in \mathbb{R}$.*

We discovered Theorem 2.1 over a decade ago, and it has since found a number of applications, for example, to *positively oriented matroids* [ARW17] (see [ALS, Ear, LPW] for other appearances of positroid polytopes).

To prove Theorem 2.1, we use a classification of positroids due to Oh [Oh] (see also [Lam, Section 8.2]). The *Bruhat partial order* on $\binom{[n]}{k}$ is defined as follows. For two subsets $I, J \in \binom{[n]}{k}$, we write $I \leq J$ if $I = \{i_1 < i_2 < \cdots < i_k\}$, $J = \{j_1 < j_2 < \cdots < j_k\}$, and we have $i_r \leq j_r$ for $r = 1, 2, \ldots, k$. For $I \in \binom{[n]}{k}$, the *Schubert matroid* $\mathcal{S}_I$ is defined as

$$\mathcal{S}_I := \{J \in \binom{[n]}{k} \mid I \leq J\}$$

and has minimal element I in the Bruhat order. For $a \in [n]$, let $<_a$ denote the cyclically rotated order on $[n]$ with minimum a, that is, $a <_a (a+1) <_a \cdots <_a n <_a 1 <_a \cdots <_a (a-1)$, which induces a partial order $\leq_a$ on $\binom{[n]}{k}$. Let $\mathcal{S}_{I,a} := \{J \in \binom{[n]}{k} \mid I \leq_a J\}$ denote the cyclically rotated Schubert matroid. Equivalently, $\mathcal{S}_{I,a} := c^{a-1}(\mathcal{S}_{c^{-a+1}(I)})$, where c is the long cycle $(1, 2, \ldots, n)$ in the symmetric group S_n naturally acting on $[n]$ and $\binom{[n]}{k}$.

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A (k,n)-*Grassmann necklace* $\mathcal{I} = (I_1, I_2, \ldots, I_n)$ is an n-tuple of k-element subsets of $[n]$ satisfying the following condition: for each $a \in [n]$, we have

1. if $a \notin I_a$, then $I_{a+1} = I_a$,
2. otherwise, $a \in I_a$ and $I_{a+1} = (I_a \setminus \{a\}) \cup \{a'\}$ for some $a' \in [n]$,

with indices taken modulo n.

Theorem 2.2 [Oh, Po06]. *Let $\mathcal{I} = (I_1, I_2, \ldots, I_n)$ be a (k,n)-Grassmann necklace. Then the intersection of cyclically rotated Schubert matroids*

$$\mathcal{M}_{\mathcal{I}} = \mathcal{S}_{I_1,1} \cap \mathcal{S}_{I_2,2} \cap \cdots \cap \mathcal{S}_{I_n,n} \tag{2.1}$$

is a positroid, and the map $\mathcal{I} \mapsto \mathcal{M}_{\mathcal{I}}$ gives a bijection between (k,n)-Grassmann necklaces and positroids of rank k on $[n]$.

Proof of Theorem 2.1. Let $\Delta(k,n)$ denote the hypersimplex, the convex hull of all points e_I, for $I \in \binom{[n]}{k}$. The matroid polytope $P_{\mathcal{S}_{I,a}}$ is the intersection of the hypersimplex $\Delta(k,n)$ with the inequalities

$$x_a + x_{a+1} + \cdots + x_b \geq \#(I \cap [a,b])$$

for $i = 1, 2, \ldots, n$. In particular, $P_{\mathcal{S}_{I,a}}$ is an alcoved polytope.

Let $\mathcal{M}$ be an arbitrary matroid. Recall, that any matroid has a unique minimal base in the Bruhat partial order $\leq$ on $\binom{[n]}{k}$, and thus in any cyclically rotated partial order $\leq_a$. Denote by $I_a(\mathcal{M})$ the minimal base of $\mathcal{M}$ with respect to $\leq_a$. Let $Q = \mathrm{env}(P_{\mathcal{M}})$ be the alcoved envelope of $P_{\mathcal{M}}$, that is, the smallest alcoved polytope that contains $P_{\mathcal{M}}$. Then Q is given by the intersection of the rotated Schubert matroid polytopes $P_{\mathcal{S}_{I_a(\mathcal{M}),a}}$ for $a = 1, 2, \ldots, n$ (see Lemma 5.1). It is known [Po06, Lemma 16.3] that for any matroid $\mathcal{M}$, the n-tuple $\mathcal{I}(\mathcal{M}) = (I_1(\mathcal{M}), I_2(\mathcal{M}), \ldots, I_n(\mathcal{M}))$ is a (k,n)-Grassmann necklace, and $\mathcal{M}_{\mathcal{I}}$ is called the *positroid envelope* of $\mathcal{M}$ [KLS]. Thus, Q is the matroid polytope of the positroid envelope of $\mathcal{M}$. In particular, $P_{\mathcal{M}}$ is alcoved if and only if $Q = P_{\mathcal{M}}$ if and only if $\mathcal{M}$ is a positroid. □

A *decorated permutation* on $[n]$ is a pair $\pi^{:} = (\pi, \mathrm{col})$, where π is a permutation of $[n]$ and col is an assignment of one of two colors "black" and "white" to each of the fixed points $\{i \in [n] \mid \pi(i) = i\}$. We say that $i \in [n]$ is an *antiexceedance* of $\pi^{:}$ if $\pi^{-1}(i) > i$ or $\pi(i) = i$ and i is colored white. Given a (k,n)-Grassmann necklace $\mathcal{I} = (I_1, I_2, \ldots, I_n)$, we define a decorated permutation $\pi^{:}(\mathcal{I})$ by

1. if $I_{a+1} = I_a - \{a\} \cup \{a'\}, a' \neq a$, then $\pi(a) = a'$;
2. if $I_{a+1} = I_a$ and $a \notin I_a$, then $\pi(i) = i$ and i is colored black;
3. if $I_{a+1} = I_a$ and $a \in I_a$, then $\pi(i) = i$ and i is colored white.

[Po06, Lemma 16.2] states that the map $\mathcal{I} \to \pi^{:}(\mathcal{I})$ is a bijection between (k,n)-Grassmann necklaces and decorated permutations on $[n]$ with k antiexceedances.

3. Polypositroids

In this paper, we consider several classes of convex polytopes in $\mathbb{R}^n$. All polytopes lie in an affine hyperplane $H = H_k := \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = k\}$, for some constant k. For the majority of this work, the reader may assume that the hyperplane H has been fixed.

3.1. Generalized permutohedra

Definition 3.1 [Po09]. A polytope $P \subset \mathbb{R}^n$ is called a *generalized permutohedron* if all edges of P are parallel to a vector of the form $e_i - e_j$.

The class of generalized permutohedra include many classical polytopes: the usual permutohedron, the associahedron, hypersimplices, and many others (see [Po09]). Let us give several alternative ways

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to describe the class of generalized permutohedra. Let $(\mathbb{R}^n)^*$ denote the vector space of linear functions on $\mathbb{R}^n$. For a face F of a polytope P, the normal cone $C_F \subset (\mathbb{R}^n)^*$ to F is given by

$$C_F = \{h \in (\mathbb{R}^n)^* \mid h(x) = \max\{h(y) \mid y \in P\} \text{ for } x \in F\}.$$

The normal fan $\mathcal{F}_P$ is the complete fan in $(\mathbb{R}^n)^*$ consisting of all cones C_F as F varies over all the faces of P. We say that a complete fan $\mathcal{F}'$ is a coarsening of a complete fan $\mathcal{F}$ if the maximal cones of $\mathcal{F}'$ are unions of the maximal cones of $\mathcal{F}$.

For a permutation $w \in S_n$, let $v_w = -(w^{-1}(1), \ldots, w^{-1}(n)) \in \mathbb{R}^n$, and let $P_n := \text{conv}(v_w \mid w \in S_n)$ be the standard permutohedron in $\mathbb{R}^n$. For the following result, see [PRW, Theorem 15.3] and [CL].

Theorem 3.2. *The following are equivalent for a polytope P in $\mathbb{R}^n$:*

1. *The polytope P is a generalized permutohedron.*
2. *The normal fan of P is a coarsening of the normal fan of the standard permutohedron P_n.*
3. *The vertices of P can be (possibly redundantly) labelled v'_w, $w \in S_n$, such that for any edge (v_u, v_w) in P_n, there is a nonnegative real t, such that $(v'_u - v'_w) = t(v_u - v_w)$.*

For a polytope P, we define the *support function* $f_P : (\mathbb{R}^n)^* \to \mathbb{R}$ given by

$$f_P(h) = \max_{v \in P} h(v).$$

The function f_P is a piecewise linear function on $(\mathbb{R}^n)^*$ whose maximal domains of linearity are exactly the top-dimensional cones of the normal fan of P.

The normal fan of the standard permutohedron P_n is the braid fan (see Section 8.5), and by Theorem 3.2(2), a generalized permutohedron P is uniquely determined by the values $f_P(S) := f_P(h_S)$, where $h_S(x_1, \ldots, x_n) = \sum_{i \in S} x_i$, and S varies over proper nonempty subsets of $[n]$. The polytope P is then given by

$$x_1 + \cdots + x_n = k = f_P([n]) \quad \text{and} \quad \sum_{i \in S} x_i \leq f_P(S). \tag{3.1}$$

We write $f_P|_{2^{[n]}}$ for the function sending a subset $S \subset [n]$ to $f_P(S)$, and, more generally, we will use notation, such as $f_P|_{\mathcal{S}}$ for a collection $\mathcal{S} \subseteq 2^{[n]}$ of subsets.

3.2. Alcoved polytopes

Let $h_1, \ldots, h_n$ be the basis of $(\mathbb{R}^n)^*$, such that $h_i(x) = x_1 + x_2 + \cdots + x_i$. Thus, $h_i := h_{[1,i]}$.

Definition 3.3 [LP07]. A polytope $P \subset \mathbb{R}^n$ is called an *alcoved polytope* if it is given by inequalities of the form

$$(h_i - h_j)(x) \leq a_{ij}, \qquad \text{for } i \neq j \text{ belonging to } [n] \tag{3.2}$$

and the equation $x_1 + \cdots + x_n = k$, for some real[1] numbers a_{ij} and k.

Alcoved polytopes are also called *polytropes* in [JK]. We will always assume in (3.2) that the a_{ij} have been chosen to be minimal, that is, they are values of the support function.

Lemma 3.4. *The set of alcoved polytopes in the affine hyperplane H_k and the set of alcoved polytopes in H_ℓ are in natural bijection via the isomorphism $+(\ell-k)e_1 : H_k \to H_\ell$ adding $(\ell - k)$ to the first coordinate. This bijection preserves the values of a_{ij}.*

[1] In [LP07], we required that a_{ij} and k were integer numbers, while, in this paper, we allow any real coefficients. We will call the polytopes from [LP07] *integer* alcoved polytopes. Their vertices are always integer lattice points.

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An alcoved polytope P is called *generic* if it is full-dimensional in H, and each equality $(h_i - h_j)(x) = a_{ij}$ defines a facet of P, for all $i, j \in [n]$, $i \neq j$.

Definition 3.5. Let $Q \subset H$ be a bounded subset. The *alcoved envelope* $\mathrm{env}(Q) \subset H$ is defined to be the smallest alcoved polytope containing Q. Equivalently, $\mathrm{env}(Q)$ is given by inequalities (3.2), where $a_{ij} = \mathrm{supremum}_{x \in Q}(h_i - h_j)(x)$.

Example 3.6. For $n = 3$, the class of generalized permutohedra coincides with the class of alcoved polytopes. However, for $n \geq 4$, neither of these two classes of polytopes contains the other. For example, the standard permutohedron $P_4 = \mathrm{conv}((w(1), w(2), w(3), w(4)) \mid w \in S_4) \subset \mathbb{R}^4$ is a generalized permutohedron, but it is not an alcoved polytope, because one of its facets is given by $x_1 + x_3 \leq 7$. On the other hand, the simplex $\mathrm{conv}(e_1 + e_3,\ e_2 + e_3,\ e_2 + e_4,\ e_3 + e_4) \subset \mathbb{R}^4$ is an alcoved polytope, but it is not a generalized permutohedron, because it has the edge $(e_1 + e_3,\ e_2 + e_4)$. Here, "conv" means the convex hull of points.

Define the *cyclic interval* $[r, s]$ in $[n]$ as

$$[r,s] := \left\{ \begin{array}{ll} \{r, r+1, \ldots, s\} & \text{if } 1 \leq r \leq s \leq n, \\ \{r, r+1, \ldots, n, 1, 2, \ldots, s\} & \text{if } 1 \leq s < r \leq n. \end{array} \right\},$$

and set $h_{[r,s]} := \sum_{i \in [r,s]} x_i \in (\mathbb{R}^n)^*$. Alcoved polytopes P are exactly all polytopes of the form

$$x_1 + \cdots + x_n = k \quad \text{and} \quad \sum_{i \in [r,s]} x_i \leq f_{[r,s]} \tag{3.3}$$

for *cyclic intervals* $[r, s] \subset [n]$, where $f_{[r,s]} = f_P([r, s]) := f_P(h_{[r,s]})$. Note that we have

$$a_{ij} = \begin{cases} f_{[j+1,i]} & \text{if } i > j \\ f_{[j+1,i]} - f_{[n]} = f_{[j+1,i]} - k & \text{if } i < j, \end{cases} \tag{3.4}$$

and it will be convenient to use both sets of parameters $f_{[r,s]}$ and a_{ij}.

If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we shall use the shorthands $x_{[r,s]} := h_{[r,s]}(x) = \sum_{i \in [r,s]} x_i$ and $x_S := h_S(x) = \sum_{i \in S} x_i$.

Remark 3.7. The class of generalized permutohedra in $\mathbb{R}^n$ is closed under the operation of taking the Minkowski sum polytopes, but is not closed under the operation of taking the intersection of polytopes. On the other hand, the class of alcoved polytopes in $\mathbb{R}^n$ is closed under the operation of taking the intersection of polytopes (if it is nonempty), but is not closed under the operation of taking the Minkowski sum of polytopes.

3.3. Polypositroids

Definition 3.8. A *polypositroid* is a polytope $P \subset H_k$ which is both a generalized permutohedron and an alcoved polytope.

As for alcoved polytopes, polypositroids in H_k and in H_ℓ are naturally in bijection. One of the main results of this paper is an explicit parametrization of all polypositroids.

We give some examples of polypositroids. For $a \in \mathbb{R}$ and P a polytope, the notation aP denotes the polytope $\{ax \mid x \in P\}$. For a cyclic interval $[r, s]$, let

$$\Delta_{[r,s]} := \mathrm{conv}(e_r, e_{r+1}, \ldots, e_s)$$

denote the corresponding coordinate simplex.

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The *cyclohedron* is the Minkowski sum of simplices

$$P = \sum_{[r,s]\neq[n]} c_{r,s}\Delta_{[r,s]} + b\Delta[n], \tag{3.5}$$

where each $c_{r,s} > 0$ and $b > 0$. By [Po09, Theorem 7.4], the cyclohedron P is both an alcoved polytope and a generalized permutohedron, and in addition, P is simple. The $f_{[r,s]}$ from (3.3) are given by

$$f_{[r,s]} = b + \sum_{[t,u]\cap[r,s]\neq\emptyset} c_{t,u}, \tag{3.6}$$

and we also set $k := b + \sum c_{r,s}$ so that $P \subset H_k$.

If we set $c_{r,s} = 0$ whenever $r > s$ (i.e., whenever $[r, s]$ is not an honest interval), then we obtain the *associahedron*, which is also a polypositroid. Another possibility is to consider the polytope $P(b, c_{r,s})$ defined by (3.5), where we allow $b \in \mathbb{R}$ and $c_{r,s} \geq 0$. When $b < 0$, we use the *Minkowski difference*:

$$P - Q := \{x \in \mathbb{R}^n \mid x + Q \subset P\}.$$

It is not hard to see that the deformed cyclohedron $P(b, c_{r,s})$ is either empty, or a polypositroid. When $b \geq 0$, the polytope $P(b, c_{r,s})$ is a deformation of a cyclohedron: the normal fan of $P(b, c_{r,s})$ is a coarsening of that of the cyclohedron. However, when $b < 0$, this may no longer be the case.

4. The cone of polypositroids

The numbers $f_P(S)$ in (3.1) are not arbitrary. It is well-known that support functions of generalized permutohedra are exactly the *submodular functions*, for example, see [MPS²W, Proposition 12], [A², Section 12], [CL, Theorem 3.11].

Theorem 4.1. *The polytope P given by (3.1) is a generalized permutohedron if and only if f_P is submodular, that is*

$$f_P(S) + f_P(T) \geq f_P(S \cap T) + f_P(S \cup T) \tag{4.1}$$

for any subsets $S, T \subset [n]$, where we assume that $f_P(\emptyset) = 0$.

Definition 4.2. The *submodular cone*, or the cone of generalized permutohedra, is the cone $\mathcal{C}_{\text{sub}}$ of all functions $f : 2^{[n]} \to \mathbb{R}$ satisfying (4.1).

The cone structure of $\mathcal{C}_{\text{sub}}$ corresponds to taking Minkowski sums of generalized permutohedra. Let us now turn to alcoved polytopes.

Theorem 4.3. *Suppose $n \geq 3$. Let P be an alcoved polytope given by (3.2) (with a_{ij} minimal). Then the a_{ij} satisfy the triangle inequality*

$$a_{ij} + a_{jk} \geq a_{ik} \tag{4.2}$$

for distinct $i, j, k \in [n]$. Conversely, any a_{ij} satisfying (4.2) define a (nonempty) alcoved polytope in H_k.

For example, for $n = 3$, we have six inequalities (4.2): $a_{12}+a_{23} \geq a_{13}$, $a_{13}+a_{32} \geq a_{12}$, $a_{21}+a_{13} \geq a_{23}$, $a_{23} + a_{31} \geq a_{21}$, $a_{31} + a_{12} \geq a_{32}$, and $a_{32} + a_{21} \geq a_{31}$.

We thank Michael Joswig for pointing out that Theorem 4.3 is known to the optimization community (see [JL, Jos] and the references therein).

Proposition 4.4. *Let $n \geq 3$. Let P be an alcoved polytope given by the inequalities $(h_i - h_j)(x) \leq a_{ij}$. Then P is generic if and only if, for any three pairwise distinct indices $i, j, k \in [n]$, we have the strict triangle inequality $a_{ij} + a_{jk} > a_{ik}$.*

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Proof. The "only if" direction is trivial. We prove the "if" direction. Let P denote the "alcoved polyhedron" inside $\mathbb{R}^n$ given by the inequalities $(h_i - h_j)(x) \le a_{ij}$. The intersections $P \cap H_k$ for varying k are linearly isomorphic via Lemma 3.4. Thus, it suffices to show that P itself is generic.

Since $n \ge 3$, summing $a_{cd} + a_{de} > a_{ce}$ and $a_{dc} + a_{ce} > a_{de}$ for distinct c, d, e, we obtain $a_{cd} + a_{dc} > 0$. It follows that P is not strictly contained in any hyperplane $(h_c - h_d)(x) = a_{cd}$, and therefore P is full-dimensional.

Fix $c \ne d$, and let $A = \{x \in \mathbb{R}^n \mid (h_c - h_d)(x) = a_{cd}\}$. To show that $P \cap A$ is a facet of P, it suffices to show that $P \cap A \ne \emptyset$ because, by the same reasoning as above, $P \cap A$ would be full-dimensional in A.

For $x \in \mathbb{R}^n$, define

$$d_P(x) = \max_{i,j}(\max(0, (h_i - h_j)(x) - a_{ij})).$$

Thus, $x \in P$ if and only if $d_P(x) = 0$. Since the function d_P is continuous, it is straightforward to see that it achieves a minimum on A. Let $v \in A$ be such a minimum, and assume $d_P(v) > 0$. We also assume that v is chosen so that

$$R(v) = \#\{(i, j) \mid (h_i - h_j)(v) = a_{ij} + d_P(v)\}$$

is minimal. Note that since $v \in A$, we have $(c, d) \notin R(v)$. It follows from $a_{cd}+a_{dc} > 0$ that $(d, c) \notin R(v)$.

Suppose $(e, f) \in R(v)$. Assume that $e \notin \{c, d\}$; the case $f \notin \{c, d\}$ is similar. If there does not exist $(g, e) \in R(v)$, then we can modify v slightly so that $h_e(v)$ decreases but the other $h_{e'}(v)$ remain unchanged, and reducing the size of $R(v)$. This would contradict the construction of v. But if $(g, e) \in R(v)$, then using the condition of the proposition, we have $(h_g - h_f)(v) = (h_g - h_e)(v) + (h_e - h_f)(v) = a_{ge} + a_{ef} + 2d_P(v) > a_{gf} + d_P(v)$, contradicting the definition of $d_P(v)$. Thus, we conclude that $d_P(v) = 0$, as desired. □

Proof of Theorem 4.3. By our assumption that the a_{ij}-s are taken minimal, it is clear that (4.2) holds for any alcoved polytope. By Proposition 4.4, the inequalities $a_{ij} + a_{jk} > a_{ik}$ define an open cone $C \subset \mathbb{R}^{n(n-1)}$, each point of which represents a generic alcoved polytope. It follows from (3.6) that C is nonempty: the cyclohedron is a generic alcoved polytope. The closure of C is thus the closed cone cut out by (4.2). The corresponding limits of generic alcoved polytopes are nonempty alcoved polytopes, finishing the proof of the theorem. □

Definition 4.5. The *triangle inequality cone*, or the cone of alcoved polytopes, is the cone $\mathcal{C}_{\text{alc}} \subset \mathbb{R}^{n(n-1)}$ of all a_{ij} satisfying (4.2).

Remark 4.6. We caution the reader that the cone $\mathcal{C}_{\text{sub}}$ contains the information $f_P([n]) = k$ and thus parametrizes generalized permutohedra in various affine hyperplanes H_k. In contrast, an alcoved polytope is determined by a point in $\mathcal{C}_{\text{alc}}$ together with the value of k. Equivalently, $\mathcal{C}_{\text{alc}}$ is the cone of alcoved polytopes inside H_0.

The cone structure of $\mathcal{C}_{\text{alc}}$ corresponds to the composition of the following two operations: first take the Minkowski sum $P_1 + P_2$ of two alcoved polytopes, and then take the alcoved envelope $\text{env}(P_1 + P_2)$ (see Definition 3.5).

Definition 4.7. The *polypositroid cone* $\mathcal{C}_{\text{pol}} \subset \mathcal{C}_{\text{alc}}$ is the subset of $\mathcal{C}_{\text{alc}}$ representing alcoved polytopes that are polypositroids.

The fact that $\mathcal{C}_{\text{pol}}$ is closed under the addition and positive scalar multiplication is a consequence of Theorems 4.8 or 4.9 below. Let $\mathcal{I} \subset 2^{[n]}$ denote the collection of all nonempty cyclic intervals, including $[n]$ itself. We define a map $\pi_{\mathcal{I}} : \mathbb{R}^{2^{[n]}} \to \mathbb{R}^{\mathcal{I}} \simeq \mathbb{R}^{n(n-1)}$ by first projecting to cyclic intervals, and then applying the transformation (3.4). If f_P is the support function of a generalized permutohedron P, considered as an element of $\mathbb{R}^{2^{[n]}}$, then $\pi_{\mathcal{I}}(f_P)$ represents the alcoved envelope $\text{env}(P)$ of P (Definition 3.5).

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Theorem 4.8. *We have* $\pi_{\mathcal{I}}(\mathcal{C}_{\mathrm{sub}}) = \mathcal{C}_{\mathrm{pol}}$.

Theorem 4.8 will be proved in Section 6.6. Note that $\pi_{\mathcal{I}} : \mathcal{C}_{\mathrm{sub}} \to \mathcal{C}_{\mathrm{pol}}$ is a homomorphism of cones: it commutes with addition and with scalar multiplication, and sends 0 to 0.

Theorem 4.9. *The cone* $\mathcal{C}_{\mathrm{pol}}$ *is the subcone of* $\mathcal{C}_{\mathrm{alc}}$ *satisfying*

$$a_{ik} + a_{jl} \geq a_{il} + a_{jk} \tag{4.3}$$

for any four indices i, j, k, l *in cyclic order.*

Theorem 4.9 will be proved in Section 8.3.

Example 4.10. Let P be the cyclohedron of (3.5). Then (4.3) is immediate from (3.6).

Remark 4.11. We have described the three cones $\mathcal{C}_{\mathrm{sub}}, \mathcal{C}_{\mathrm{alc}}, \mathcal{C}_{\mathrm{pol}}$ in terms of defining (possibly redundant) inequalities. The extremal rays of $\mathcal{C}_{\mathrm{sub}}$ (modulo translation) are hard to describe explicitly (see, e.g. [Ng, MPS^{2}W, ACEP] for discussions of the rays). Among the rays of $\mathcal{C}_{\mathrm{sub}}$ are all connected matroids [Ng].

However, we will give in Corollary 6.14 an explicit description of the rays of $\mathcal{C}_{\mathrm{pol}}$ (modulo translation).

5. Alcoved envelopes

Let $C \subset H_0 \subset \mathbb{R}^n$ denote the following polyhedral $(n-1)$-dimensional pointed cone

$$C := \{x \in \mathbb{R}^n \mid x_1 \leq 0,\ x_1 + x_2 \leq 0, \ldots, x_1 + \cdots + x_{n-1} \leq 0,\ x_1 + \cdots + x_n = 0\}. \tag{5.1}$$

Define the *dominance order* on the hyperplane $H = H_k$ as the partial order $x \preceq y$ if and only if $x_1 \leq y_1$, $x_1 + x_2 \leq y_1 + y_2$, …, or equivalently, $x - y \in C$. For any bounded subset $Q \subset H$, there is a unique maximal in the dominance order point $v = v(Q) \in H$, such that $Q \subset v + C$. Informally, $v + C$ is the cone containing Q, such that every facet of $v + C$ touches Q. Note that the cone C is an alcoved polyhedron: its facets have normals given by $h_i - h_n$.

Let $c = (12\cdots n) \in S_n$ be the long cycle, with $c = 23\cdots n1$ in one-line notation. The permutations $w \in S_n$ act on $\mathbb{R}^n$ by the formula

$$w \cdot (x_1, x_2, \ldots, x_n) = (x_{w^{-1}(1)}, x_{w^{-1}(2)}, \ldots, x_{w^{-1}(n)}).$$

Then c acts on $\mathbb{R}^n$ as the *cyclic shift* linear operator given by $c(e_i) = e_{i+1}$, for $i = 1, \ldots, n$ (here and below, we assume that indices i are taken modulo n). For $i \in \mathbb{Z}/n\mathbb{Z}$, define the cyclically shifted cone $C_i := c^{i-1}(C)$. Thus, $C_1 = C$.

If $\mathbf{v} = (v^{(1)}, \ldots, v^{(n)})$ is a sequence of points in H, we denote by $Q(\mathbf{v})$ the intersection

$$Q(\mathbf{v}) := \bigcap_{i \in \mathbb{Z}/n\mathbb{Z}} (v^{(i)} + C_i). \tag{5.2}$$

It is clear that $Q(\mathbf{v})$ is an alcoved polytope whenever it is nonempty.

For a bounded subset $Q \subset H$, define

$$v^{(i)} = v^{(i)}(Q) := c^{i-1}(v(c^{1-i}(Q))). \tag{5.3}$$

Again, the cone $v^{(i)} + C_i$ contains Q, and every facet of $v^{(i)} + C_i$ touches Q. Recall the alcoved envelope $\mathrm{env}(Q)$ defined in Definition 3.5.

Lemma 5.1. *For the cones* C_i *and points* $v^{(i)}$, $i \in \mathbb{Z}/n\mathbb{Z}$, *as above, the alcoved envelope of* Q *is the intersection of the following cones*

$$\mathrm{env}(Q) = \bigcap_{i \in \mathbb{Z}/n\mathbb{Z}} (v^{(i)} + C_i).$$

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Proof. Let $f_{[r,s]}$ be the minimal number, such that Q belongs to the halfspace $S_{[r,s]} = \{x \in H \mid \sum_{i\in[r,s]} x_i \le f_{[r,s]}\}$, that is the (unique) facet of $S_{[r,s]}$ touches Q. Then the alcoved envelope of Q is the alcoved polytope $\mathrm{env}(Q) = \bigcap_{r,s} S_{[r,s]}$. For fixed r, the intersection $\bigcap_s S_{[r,s]}$ is the affine translation $v^{(r)} + C_r$ of the cone C_r that satisfies the conditions of the lemma. So $\mathrm{env}(Q) = \bigcap_{r\in\mathbb{Z}/n\mathbb{Z}}(v^{(r)} + C_r)$, as needed. □

6. Parametrization of polypositroids

6.1. Coxeter necklaces

As before, we fix an affine plane $H = H_k := \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = k\}$.

Definition 6.1. Let $\mathbf{v} = (v^{(1)}, v^{(2)}, \ldots, v^{(n)})$ be a sequence of points in H. We say that $\mathbf{v}$ is a *Coxeter necklace* if, for each i, we have

$$v^{(i+1)} - v^{(i)} \text{ is nonnegative in all coordinates except the } i\text{-th coordinate.} \tag{6.1}$$

Here, the superscript i is taken modulo n. Since $\mathbf{v} \subset H$, (6.1) implies that the i-th coordinate of $v^{(i+1)} - v^{(i)}$ is nonpositive.

Remark 6.2. The set of (k, n)-Grassmann necklaces (see Section 2) is exactly the set of Coxeter necklaces $\mathbf{v} = (v^{(1)}, v^{(2)}, \ldots, v^{(n)})$ in H_k, such that each $v^{(i)}$ is a 01-vector.

The sum of two Coxeter necklaces $\mathbf{v} \in H_k$ and $\mathbf{v}' \in H_{k'}$ is a Coxeter necklace $\mathbf{v}'' \in H_{k+k'}$. Recall that the polytope $Q(\mathbf{v})$ was defined in (5.2).

Lemma 6.3. *The space of Coxeter necklaces for varying $H = H_k$ form a cone. The map $\mathbf{v} \to Q(\mathbf{v})$ induces a homomorphism of cones from Coxeter necklaces to $\mathcal{C}_{\mathrm{alc}}$.*

It will follow from Theorem 6.12 below that the image of this homomorphism of cones is $\mathcal{C}_{\mathrm{pol}}$.

Lemma 6.4. *Suppose that $\mathbf{v} = (v^{(1)}, v^{(2)}, \ldots, v^{(n)})$ is a Coxeter necklace. Then each $v^{(i)}$ is a vertex of $Q(\mathbf{v})$.*

Proof. It suffices to show that $v^{(i)} \in Q(\mathbf{v})$. We will show that $v^{(i)}_{[1,j]} \le v^{(1)}_{[1,j]}$; the full set of inequalities follows by cyclic symmetry. Suppose $i \le j + 1$. Then

$$(v^{(i)} - v^{(1)})_{[1,j]} = \sum_{k\in[2,i]} (v^{(k)} - v^{(k-1)})_{[1,j]} \le 0$$

using the definition (6.1) and the fact $k - 1 \in [1, j]$. Suppose $i > j + 1$. Then

$$(v^{(1)} - v^{(i)})_{[1,j]} = \sum_{k\in[i+1,1]} (v^{(k)} - v^{(k-1)})_{[1,j]} \ge 0$$

using the definition (6.1) and the fact that $k - 1 \notin [1, j]$. □

It follows from Lemma 6.4 that the map $\mathbf{v} \to Q(\mathbf{v})$ is injective.

Proposition 6.5. *Suppose P is a generalized permutohedron in H with vertices $\{v_w \mid w \in S_n\}$. Then*

1. *the sequence $(v_{\mathrm{id}}, v_c, v_{c^2}, \ldots, v_{c^{n-1}})$ is a Coxeter necklace,*
2. *one has $v_{c^{i-1}} = v^{(i)}(P)$ for $i = 1, \ldots, n$, where $v^{(i)}$ is defined in (5.3).*

Proof. For (1), we show that all but the first coordinate of $v_c - v_{\mathrm{id}}$ is nonnegative, and a similar argument shows that the entire sequence is a Coxeter necklace. We have $c = 23\cdots1$ in one-line notation.

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By Theorem 3.2(3),

$$\begin{aligned}
v_{213\cdots n} - v_{123\cdots n} &\in \quad \mathbb{R}_{\geq 0}((-2,-1,-3,\ldots) - (-1,-2,-3,\ldots)) = \mathbb{R}_{\geq 0}(e_2 - e_1)\\
v_{2314\cdots n} - v_{213\cdots n} &\in \quad \mathbb{R}_{\geq 0}((-3,-1,-2,\ldots) - (-2,-1,-3,\ldots)) = \mathbb{R}_{\geq 0}(e_3 - e_1)\\
&\cdots\\
v_{23\cdots(n-1)n1} - v_{23\cdots(n-1)1n} &\in \quad \mathbb{R}_{\geq 0}((-n,-1,-2,\ldots,1-n) - (1-n,-1,-2,\ldots,-n))\\
&\qquad = \mathbb{R}_{\geq 0}(e_n - e_1).
\end{aligned}$$

It follows that all but the first coordinate of $v_c - v_{\mathrm{id}}$ is nonnegative.

For (2), we note that all edges of P incident to $v_{c^{i-1}}$ are in the same direction as an edge of the cone C_i. It follows that $P \subset v_{c^{i-1}} + C_i$, so that $v_{c^{i-1}} = v^{(i)}(P)$. □

6.2. Balanced digraphs

Definition 6.6. Let G be a real-weighted directed graph on the vertex set $[n]$. Let $m_{ij} \in \mathbb{R}$ denote the weight of the edge $i \to j$ of G (we assume that $m_{ij} = 0$ if G does not contain the edge $i \to j$). We say that G is *balanced* if

1. the weight of nonloop edges are nonnegative, that is, $m_{ij} \in \mathbb{R}_{\geq 0}$ when $i \neq j$;
2. the total outdegree is equal to the total indegree of each vertex; that is, for each i one has

$$\sum_{j=1}^{n} m_{ij} = \sum_{j=1}^{n} m_{ji}. \tag{6.2}$$

Remark 6.7. The set of decorated permutations (see Section 2) is the same as the set of balanced graphs satisfying the following two conditions: (a) every vertex has one nonzero incoming edge and one nonzero outgoing edge, and (b) all edges have weight ± 1 (only loop edges can have weight -1).

Lemma 6.8. *Suppose G is a balanced digraph. Then the quantity*

$$S_i(G) := \sum_{j \in [n]} (m_{j,j} + m_{j+1,j} + \cdots + m_{i,j})$$

does not depend on $i \in [n]$ (indices are taken modulo n in the sum $m_{j,j} + m_{j+1,j} + \cdots + m_{i,j}$).

Proof. We have $S_i - S_{i+1} = \sum_{j\in[n]} m_{j,i+1} - \sum_{j\in[n]} m_{i+1,j} = 0$. □

For a balanced digraph G, we define $S(G) = S_i(G)$ to be the sum of Lemma 6.8.

Note that the space of balanced graphs forms a cone in a natural way: $G = \alpha G' + \beta G''$ has weights given by $m_{ij} = \alpha m'_{ij} + \beta m''_{ij}$.

Proposition 6.9. *Every balanced graph G is a linear combination of balanced directed cycles (including loops and cycles of length 2), such that the coefficients of those cycles that are not loops are nonnegative.*

Proof. Let G be a balanced graph. We may assume that $m_{ii} = 0$ for all $i \in [n]$. Let $m_{ij} > 0$ be minimal amongst the positive weights. Then the unweighted graph underlying G must contain a directed cycle C containing the edge $i \to j$. All the weights m_e for e an edge of C satisfy $m_e \geq m_{ij}$. Thus, we may write $G = G' + C(m_{ij})$, where $C(m_{ij})$ is the balanced directed cycle, where all edges have weight m_{ij}, and G' is still a balanced graph. But G' has fewer edges (with nonzero weight) than G, so repeating this reduction, we deduce the proposition. □

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Let $\mathbf{v} = (v^{(1)}, v^{(2)}, \ldots, v^{(n)})$ be a balanced sequence of points in H. We define a weighted directed graph $G(\mathbf{v})$ by the formula

$$m_{ij} = \begin{cases} (v^{(i+1)} - v^{(i)})_j & \text{if } j \neq i, \\ v_i^{(i+1)} & \text{if } j = i. \end{cases} \tag{6.3}$$

Lemma 6.10. *The map $\mathbf{v} \mapsto G(\mathbf{v})$ is a bijection between Coxeter necklaces in H, and balanced graphs G satisfying $S(G) = k$. Furthermore, allowing k to vary, we obtain an isomorphism of cones.*

Proof. We first check that $G(\mathbf{v})$ is a balanced graph. By definition of a balanced sequence, $m_{ij} \in \mathbb{R}_{\geq 0}$. We have for each $i \in [n]$,

$$\sum_{j \in [n]} m_{ij} = \sum_{j \in [n]} (v^{(i+1)} - v^{(i)})_j + v_i^{(i)} = v_i^{(i)} = \sum_{j \in [n]} (v^{(j+1)} - v^{(j)})_i + v_i^{(i)} = \sum_{j \in [n]} m_{ji},$$

so $G(\mathbf{v})$ is balanced. Finally, using $v_j^{(i+1)} = m_{j,j} + \cdots + m_{i,j}$, one obtains $S(G) = \sum_j v_j^{(i+1)} = k$.

Conversely, suppose we are given a balanced graph satisfying $S(G) = k$. Then we define a sequence $\mathbf{v}$ by $v_j^{(i+1)} = m_{j,j} + \cdots + m_{i,j}$. It is easy to verify that $\mathbf{v}$ is a balanced sequence in H, and that this is inverse to $\mathbf{v} \mapsto G(\mathbf{v})$.

The last statement follows immediately from the linearity of (6.3). □

We write $\mathbf{v}(G)$ for the Coxeter necklace associated to a balanced digraph G.

Example 6.11. Let G be the balanced digraph on $[4]$, such that $m_{ij} = 1$ for all $i \neq j$. Then $S(G) = 6$ and $\mathbf{v}(G)$ is given by

$$v^{(1)} = (3, 2, 1, 0), \qquad v^{(2)} = (0, 3, 2, 1), \qquad v^{(3)} = (1, 0, 3, 2), \qquad v^{(4)} = (2, 1, 0, 3).$$

6.3. Parametrization

Theorem 6.12. *There are natural bijections between the following sets:*

1. *The set of all polypositroids $P \subset H_k$.*
2. *The set of all Coxeter necklaces in H_k.*
3. *The set of all balanced digraphs G with $S(G) = k$.*

Furthermore, these bijections are compatible with the respective cone structures on the three sets.

After Lemma 6.10, it suffices to show that the map $\mathbf{v} \to Q(\mathbf{v})$ is a bijection between Coxeter necklaces and polypositroids. We delay the proof of Theorem 6.12 to Section 6.5.

The following result follows easily from the definition of the cone of balanced digraphs.

Corollary 6.13. *The cone of balanced digraphs G satisfying $S(G) = 0$ has $n(n-1)$ facets, given by the inequalities $m_{ij} \geq 0$ for $i \neq j$. Thus, the cone $\mathcal{C}_{\mathrm{pol}}$ also has $n(n-1)$ facets.*

The group $H_0 = \mathbb{R}^{n-1}$ of translations preserves the affine hyperplane H_k and acts on the cone $\mathcal{C}_{\mathrm{pol}}$ of polypositroids via the formula $z : (a_{ij}) \mapsto (a_{ij} + (h_i - h_j)(z))$, where $z = (z_1, z_2, \ldots, z_n) \in H_0$. The lineality space of the cone $\mathcal{C}_{\mathrm{pol}}$ can be identified with $\mathbb{R}^{n-1}$. Let $\mathcal{C}'_{\mathrm{pol}} := \mathcal{C}_{\mathrm{pol}}/\mathbb{R}^{n-1}$ denote the quotient cone, which may be identified with the set of polypositroids inside H_0, modulo translations.

Corollary 6.14. *The cone $\mathcal{C}'_{\mathrm{pol}}$ is a pointed cone, with extremal rays corresponding to the polypositroids $Q(\mathbf{v}(G))$, where G is a balanced directed cycle (including cycles of length 2).*

Proof. By Theorem 6.12, we are equivalently considering the cone of balanced digraphs with $S(G) = 0$. The translation action of $\mathbb{R}^{n-1}$ on H_0 corresponds to changing the weight of the n loop edges of G. Ignoring the weights of the loops, the statement then follows from Proposition 6.9. □

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6.4. Face graphs

Let P be an alcoved polytope and F a face of P. We define the graph T_F of P at F to be the directed graph on $[n]$ with a directed edge $(i \to j)$ whenever F lies on the hyperplane $(h_j - h_i)(x) \leq a_{ji}$, or equivalently, on the hyperplane $x_{[i+1,j]} = f_{[i+1,j]}$. We say that a digraph T on $[n]$ is *noncrossing* if, when the graph is drawn inside a circle with the vertices $[n]$ arranged in clockwise order, there are no intersections in the interior of the circle. We say that a digraph T on $[n]$ is *alternating* if no vertex has both incoming and outgoing edges.

Lemma 6.15. *Suppose E is an edge of an alcoved polytope P, with a noncrossing graph T_E. Then E is parallel to $e_i - e_j$ for some i and j.*

Proof. Let us take a minimal subgraph $T \subset T_E$, such that the corresponding $(n-2)$ facets still define (the affine span of) E. We claim that T is a forest with two components. It is enough to show that the underlying undirected graph of T has no cycles. A cycle in T would correspond to a linear dependence in the equations defining E, contradicting the minimality of T. For the remainder of the proof, it is enough to think of T as an undirected forest.

Let the two components of T be T_1 and T_2. By the noncrossing assumption, it is clear that T_1 and T_2 are induced subgraphs of T on cyclic intervals $[i, j-1]$ and $[j, i-1]$.

Let $x = (x_1, \ldots, x_n)$ lie on E. We claim that if $k \notin \{i, j\}$, then x_k is fixed (that is, constant on E). For simplicity, we shall assume that x_k lies in a component T_1 of T which is an induced subgraph on a usual interval (no wraparound). First, note that an edge of the form $(r, r+1) \in T_1$ completely determines x_{r+1}. Also, if $(r, s) \in T$ is an edge, and all but one of the coordinates $\{x_{r+1}, x_{r+2}, \ldots, x_s\}$ is determined, then the last coordinate is also determined. By induction on the length of edges, we thus see that for each edge $(r, s) \in T_1$, all of $\{x_{r+1}, x_{r+2}, \ldots, x_s\}$ are determined, proving our claim that x_k is fixed for $k \notin \{i, j\}$.

Thus, only the coordinates x_i and x_j vary on E. Since $E \in H$, we deduce that E is parallel to $e_i - e_j$. □

Remark 6.16. The converse of Lemma 6.15 is false. For example, consider an alcoved polytope P in $H = \{x \in \mathbb{R}^5 \mid x_1 + x_2 + \cdots + x_5 = 0\}$, where $x_2 \leq 0$, $x_2 + x_3 \leq 0$, and $x_3 + x_4 \geq 0$ are all facets, and so that $x_2 = x_2 + x_3 = x_3 + x_4 = 0$ defines an edge E of P. Then T_E contains the directed edges $(1 \to 2), (1 \to 3), (2 \to 4)$ and is alternating but not noncrossing. However, the edge E is clearly in the direction $e_1 - e_5$.

Let us say that a Coxeter necklace $(v^{(1)}, \ldots, v^{(n)})$ is *generic* if every coordinate of $v^{(i+1)} - v^{(i)}$ is nonzero, for every i. This is equivalent to saying that all the nonloop edges of the graph $G(\mathbf{v})$ are nonzero.

Lemma 6.17. *Let $(v^{(1)}, \ldots, v^{(n)})$ be a generic Coxeter necklace in H. Then any face F of the alcoved polytope $Q = \bigcap_{i \in \mathbb{Z}/n\mathbb{Z}} (v^{(i)} + C_i)$ has a noncrossing and alternating graph T_F.*

Proof. Let F be a face, such that T_F is either not noncrossing or not alternating.

Suppose $[r, s]$ and $[r', s']$ are cyclic intervals so that the corresponding directed edges $(r-1) \to s$ and $(r'-1) \to s'$ in T_F are either crossing or form a directed path $(r-1) \to s = (r'-1) \to s'$. In the crossing case, we may assume that $r < r' \leq s < s' < r$ (interpreted in a cyclic manner). We have the following equations and inequalities

$$x_{[r,s]} = v^{(r)}_{[r,s]} \tag{6.4}$$

$$x_{[r',s']} = v^{(r')}_{[r',s']} \tag{6.5}$$

$$x_{[r,s']} \leq v^{(r)}_{[r,s']} \tag{6.6}$$

$$x_{[r',s]} \leq v^{(r')}_{[r',s]} \tag{6.7}$$

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for points $x = (x_1, x_2, \ldots, x_n)$ on the face F. By (6.4) and (6.7), we have

$$x_{[r,r'-1]} = x_{[r,s]} - x_{[r',s]} \geq v^{(r)}_{[r,s]} - v^{(r')}_{[r',s]},$$

and by (6.5) and (6.6), we have

$$x_{[r,r'-1]} = x_{[r,s']} - x_{[r',s']} \leq v^{(r)}_{[r,s']} - v^{(r')}_{[r',s']}.$$

Equating the two expressions for $x_{[r,r'-1]}$, we obtain

$$v^{(r)}_{[s+1,s']} \geq v^{(r')}_{[s+1,s']}.$$

This is impossible because $\mathbf{v}$ is generic balanced, implying that the coordinates in the positions $[s+1, s']$ of $v^{(r')} - v^{(r)}$ are all positive. □

6.5. *Proof of Theorem 6.12*

Let $\mathbf{v} = (v^{(1)}, \ldots, v^{(n)})$ be a Coxeter necklace in H. We show that $Q(\mathbf{v})$ is a polypositroid. First, suppose that $\mathbf{v}$ is generic. Then by Lemmas 6.15 and 6.17, $Q(\mathbf{v})$ is a generalized permutohedron, and thus a polypositroid.

Now suppose that $\mathbf{v}$ is not generic, and let $G(\mathbf{v})$ be the balanced digraph under the bijection of Lemma 6.10. Let G_ε be obtained from $G(\mathbf{v})$ by adding $\varepsilon > 0$ to every nonloop edge. It is immediate that $G(\varepsilon)$ is again a balanced digraph, and we define the Coxeter necklace $\mathbf{v}_\varepsilon$ by $G(\mathbf{v}_\varepsilon) = G_\varepsilon$. Then $\mathbf{v} = \lim_{\varepsilon \to 0} \mathbf{v}_\varepsilon$ is a limit of the generic balanced sequences $\mathbf{v}_\varepsilon$. For sufficiently small but nonzero ε, the combinatorial type of the polytope $Q(\mathbf{v}_\varepsilon)$ corresponding to $\mathbf{v}_\varepsilon$ does not change. The alcoved polytope Q is thus a deformation of such a $Q(\mathbf{v}_\varepsilon)$, in the sense of moving facets. Since $Q(\mathbf{v}_\varepsilon)$ is a generalized permutohedron, so is $Q(\mathbf{v})$ (see, for example [CL]).

Now suppose P is a polypositroid with vertices v_w. Since P is a generalized permutohedron, by Proposition 6.5 and Lemma 5.1, $\mathbf{v} = (v_{\mathrm{id}}, v_c, v_{c^2}, \ldots, v_{c^{n-1}})$ is balanced, and we have $\mathrm{env}(P) = Q(\mathbf{v})$. But P is also alcoved, so we have $P = \mathrm{env}(P) = Q(\mathbf{v})$. Thus, the map $\mathbf{v} \mapsto Q(\mathbf{v})$ is surjective. Finally, it follows from Lemma 6.4 that $\mathbf{v} \mapsto Q(\mathbf{v})$ is injective. This proves the equivalence of (1) and (2) in Theorem 6.12.

6.6. *Proof of Theorem 4.8*

The following result follows immediately from Proposition 6.5(1) and Theorem 6.12.

Proposition 6.18. *The alcoved envelope of a generalized permutohedron is a generalized permutohedron, and thus a polypositroid.*

Recall that $\pi_{\mathcal{I}}$ denotes the composition of the restriction map $f_P \mapsto f_P|_{\mathcal{I}}$ with the transformation (3.4) from $f_{[r,s]}$-coordinates to a_{ij} coordinates. Let P be a generalized permutohedron and $f_P|_{2^{[n]}} \in \mathcal{C}_{\mathrm{sub}}$ be its support function. Then $\pi_{\mathcal{I}}(f_P)$ represents the alcoved polytope $\mathrm{env}(P)$. By Proposition 6.18, we thus have $\pi_I(\mathcal{C}_{\mathrm{sub}}) \subseteq \mathcal{C}_{\mathrm{pol}}$. But if P is a polypositroid, then $\mathrm{env}(P) = P$. It follows that $\pi_{\mathcal{I}}(\mathcal{C}_{\mathrm{sub}}) = \mathcal{C}_{\mathrm{pol}}$.

7. Components of a polypositroid

A *noncrossing partition* $\tau = (\tau_1|\tau_2|\cdots|\tau_\ell)$ of $[n]$ is a partition of $[n]$, such that there do not exist $1 \leq i < j < k < l \leq n$, such that $i, k \in \tau_a$ and $j, l \in \tau_b$ for $a \neq b$. Let G be a graph on $[n]$. Then there exists a finest noncrossing partition $\tau(G) = (\tau_1|\tau_2|\cdots|\tau_\ell)$ of $[n]$, such that G is the disjoint union of the induced subgraphs $G|_{\tau_a}$, $a = 1, 2, \ldots, \ell$. Note that the graphs $G|_{\tau_a}$ need not be connected.

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If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, then $x|_{\tau_a} \in \mathbb{R}^{\tau_a}$ denotes the vector obtained by projecting x to the components indexed by τ_a. Given $\mathbf{v} = (v^{(1)}, \ldots, v^{(n)})$ and $\tau_a = \{i_1, i_2, \ldots, i_t\}$, we define

$$\mathbf{v}|_{\tau_a} := (v^{(i_1)}|_{\tau_a}, v^{(i_2)}|_{\tau_a}, \ldots, v^{(i_t)}|_{\tau_a}).$$

The following result should be compared to [ARW16, Theorem 7.6] in the positroid case.

Lemma 7.1. *Let* $\mathbf{v}$ *be a Coxeter necklace, and let* $\tau(G(\mathbf{v})) = (\tau_1|\tau_2|\cdots|\tau_\ell)$. *Then* $\mathbf{v}|_{\tau_a}$, $a = 1, 2, \ldots, k$ *are Coxeter necklaces satisfying* $G(\mathbf{v}|_{\tau_a}) = G|_{\tau_a}$, *and we have*

$$Q(\mathbf{v}) = Q(\mathbf{v}|_{\tau_1}) \times Q(\mathbf{v}|_{\tau_2}) \times \cdots \times Q(\mathbf{v}|_{\tau_\ell}),$$

where $Q(\mathbf{v}|_{\tau_a})$ *lies inside* $\mathbb{R}^{\tau_a}$.

Proof. The lemma holds more generally for τ any noncrossing partition, such that G is the disjoint union of the induced subgraphs $G|_{\tau_a}$, that is, τ need not be chosen finest. The sums $k_a := v^{(j)}_{\tau_a} = \sum_{i \in \tau_a} v^{(j)}_i$ (not to be confused with the projection $v^{(j)}|_{\tau_a}$) do not depend on $j \in [n]$. It follows that $Q(\mathbf{v})$ lies in the hyperplane $\sum_{i \in \tau_a} x_i = k_a$, and $Q(\mathbf{v}_{\tau_a})$ lies in the same hyperplane intersected with $\mathbb{R}^{\tau_a}$.

The first statement of the lemma is straightforward. For the second statement, we may assume by induction that $\tau = (\tau_1|\tau_2)$, where τ_a are cyclic intervals. The polytope $Q(\mathbf{v})$ is cut out by the inequalities $x_{[r,s]} \le v^{(r)}_{[r,s]}$. Suppose that $[r, s] \cap \tau_1$ and $[r, s] \cap \tau_2$ are both nonempty. For simplicity, we suppose that $[r, s] \cap \tau_1 = [r, t]$ and $[r, s] \cap \tau_2 = [t + 1, s]$. The assumption that there are no edges in $G(\mathbf{v})$ between τ_1 and τ_2 implies that $v^{(r)}|_{[t+1,s]} = v^{(t+1)}|_{[t+1,s]}$. Thus, $v^{(r)}_{[r,s]} = v^{(r)}_{[r,t]} + v^{(t+1)}|_{[t+1,s]}$. It follows that the inequality $x_{[r,s]} \le v^{(r)}_{[r,s]}$ is implied by the inequalities $x_{[r,t]} \le v^{(r)}_{[r,t]}$ and $x_{[t+1,s]} \le v^{(t+1)}|_{[t+1,s]}$. The latter inequalities are among those cutting out $Q(\mathbf{v}_{\tau_1})$ and $Q(\mathbf{v}_{\tau_2})$, respectively. It follows that $Q(\mathbf{v}) = Q(\mathbf{v}_{\tau_1}) \times Q(\mathbf{v}_{\tau_2})$. □

Proposition 7.2. *Let* $\mathbf{v}$ *be a Coxeter necklace. The dimension of the polypositroid* $Q(\mathbf{v})$ *is equal to* $n - \#\{\text{parts in } \tau(G)\}$.

Proof. By induction and Lemma 7.1, it suffices to show $\dim(Q(\mathbf{v})) = n - 1$ whenever $\tau(G) = ([n])$ has a single part.

Assume that $\tau(G) = ([n])$ and that $Q(\mathbf{v})$ has dimension less than $n - 1$. Then since $Q(\mathbf{v})$ is alcoved, it must lie in some hyperplane $(h_i - h_j)(x) = a_{ij}$. This implies that $v^{(j+1)}_{[j+1,i]} = a_{ij} = v^{(i+1)}_{[j+1,i]}$. But

$$(v^{(i+1)} - v^{(j+1)})_{[j+1,i]} = \sum_{k \in [j+2, i+1]} (v^{(k)} - v^{(k-1)})_{[j+1,i]} \le 0$$

with equality if and only if there are no edges in $G(\mathbf{v})$ from the vertices $[j + 1, i]$ to $[i + 1, j]$. The same argument shows that there are no edges from $[i + 1, j]$ to $[j + 1, i]$, so $\tau(G)$ must be a refinement of the partition $\tau = ([i + 1, j]|[j + 1, i])$, contradicting our assumption. □

8. Normal fans of polypositroids

8.1. Normal fans to generic simple alcoved polytopes

Recall that we say that an alcoved polytope is *generic* if every inequality $(h_i - h_j)(x) \le a_{ij}$ determines a facet. The f-vector $(f_0, f_1, \ldots, f_d)$ of a d-dimensional polytope P is given by

$$f_i := \#\{i\text{-dimensional faces of } P\}.$$

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Theorem 8.1. *The f-vectors of any two generic simple alcoved polytopes $P \subset \mathbb{R}^n$ are the same. The face numbers are given by*

$$f_i = \binom{n-1}{i}\binom{2n-i-2}{n-1} \qquad \text{for } i = 0, 1, \ldots, n-1. \tag{8.1}$$

The *root polytope* R is the convex hull of the vectors $\{h_i - h_j \mid i, j \in \{1, 2, \ldots, n\}\}$, which we think of as lying inside $(\mathbb{R}^n)^*/h_n$. If T is a directed tree on $[n]$ (or, more generally, a directed graph on $[n]$), we let $\Delta_T \subset (\mathbb{R}^n)^*$ denote the convex hull of the points $\{h_i - h_j \mid j \to i \text{ is an edge of } T\} \cup \{0\}$. A *local triangulation* of R is a triangulation, such that every simplex is one of the Δ_T.

Let P be a generic simple alcoved polytope. The normal fan $\mathcal{F}_P$ of P lies in $(\mathbb{R}^n)^*/h_n$. The condition that P is generic implies that every root $h_i - h_j \neq 0$ is an edge of $\mathcal{F}_P$. The condition that P is simple implies that each maximal cone C_v of P is spanned by $(n-1)$ roots. Thus, the collection of maximal cones of $\mathcal{F}_P$ induces a local triangulation of P. Let $C_T := \mathrm{span}_{\geq 0}\Delta_T$ be the cone spanned by Δ_T.

Proof of Theorem 8.1. The f-vector of P is given by counting the number of cones of each dimension of $\mathcal{F}_P$, which is the same as counting the number of simplices (with the origin as a vertex) of each dimension in the corresponding local triangulation of the root polytope R.

We claim that every local triangulation of R has the same number of simplices of each dimension. To see this, we note that since the type A root system is unimodular, every simplex Δ_T has the same volume (in fact, normalized volume 1). The number $\#\{\mathrm{Int}(m\Delta) \cap \mathbb{Z}^r\}$ of integer points lying in the interior of an integer scalar multiple of a normalized volume 1 simplex Δ with integer coordinates depends only on the scalar multiple m and the dimension $\dim(\Delta)$ of the simplex.

It follows easily from this that the Ehrhart polynomial of R can be written in terms of, and in fact determines, the number of simplices (with the origin as a vertex) in each dimension of a local triangulation. But clearly, the Ehrhart polynomial of R does not depend on the triangulation of R.

The cyclohedron (defined in (3.5)) is a generic and simple polypositroid. Thus, every generic simple polypositroid has the same f-vector as the cyclohedron. According to [Sim], the f-vector of the cyclohedron is given by (8.1). For example, the 2-dimensional cyclohedron is a hexagon with face numbers $(f_0, f_1, f_2) = (6, 6, 1)$. □

8.2. Matching ensembles

Let P be a generic simple alcoved polytope. Recall that in Section 6.4, we have defined the graph T_F for any face F of P. When $F = v$ is a vertex of P, the graph T_v is a tree that we call a vertex tree. Let $\mathcal{T}(P)$ denote the set of vertex trees of P. The data of $\mathcal{T}(P)$ are equivalent to the knowledge of the normal fan $\mathcal{F}_P$. Thus, the fan $\mathcal{F}_P$ is complete, and the maximal cones C_v of $\mathcal{F}_P$ are indexed by vertices v, such that C_v is the positive span of the vectors $h_i - h_j$ for $j \to i$ an edge of T_v.

The first part of the following result is similar to [Po09, Lemma 13.2].

Lemma 8.2. *Let P be a generic simple alcoved polytope, and v a vertex of P. Then the tree T_v is alternating. Furthermore, if P is a polypositroid, then T_v is in addition noncrossing.*

Proof. If $j \to i$ and $k \to j$ both belong to T_v, then $h_i - h_k \in C_v$ and is not one of the edges of C_v, contradicting the assumption that all roots are edges of $\mathcal{F}_P$. We conclude that T_v is alternating.

Now suppose that P is a polypositroid. Since P is generic and simple, the polytope P' obtained from a small perturbation of the facets of P will have the same combinatorial type as P. We can pick such a P' to be a polypositroid $Q(\mathbf{v})$ for a generic Coxeter necklace $\mathbf{v}$. It follows from Lemma 6.17 that the trees T_v are noncrossing. □

Suppose T is an alternating tree $[n]$. A matching of (I, J) in T is a collection of edges of T which form a matching of I with J, such that vertices in I are sources, and the vertices in J are sinks. Say that two directed alternating trees T, T' on $[n]$ are *compatible* if there do not exist disjoint subsets $I, J \subset [n]$ of the same cardinality, such that both T and T' contain matchings of (I, J), and these matchings disagree.

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Lemma 8.3 (cf. [Po09, Lemma 12.6]). *Let T, T' be distinct directed alternating trees on $[n]$. The intersection $C_T \cap C_{T'}$ is a common face of both C_T and $C_{T'}$ if and only if T and T' are compatible.*

Proof. Suppose T and T' are not compatible. Let $I, J \subset [n]$ be disjoint, such that T (respectively, T') contains a matching M (respectively, M') from I to J, such that $M \neq M'$. We assume that I and J are chosen to be minimal, so that $M \cap M' = \emptyset$. Let

$$x = \sum_{j \in J} h_j - \sum_{i \in I} h_i = \sum_{(i \to j) \in M} h_j - h_i = \sum_{(i \to j) \in M'} h_j - h_i.$$

Clearly, $x \in C_T \cap C_{T'}$. The minimal face of C_T containing x is C_M. The minimal face of $C_{T'}$ containing x is $C_{M'}$. Since $M \neq M'$, we conclude that $C_T \cap C_{T'}$ is not a common face.

Conversely, suppose that T and T' are compatible. Let $F = T \cap T'$ be the intersection, a directed forest on $[n]$. Define a partial order $\prec$ on the connected components (denoted A) of F, by letting $A \prec A'$ if there is a (necessarily unique) sequence $A = A_0, A_1, \ldots, A_\ell = A'$ of distinct components of F, such that T has a (unique) directed edge f_i joining A_i to A_{i+1} for $i \in [0, \ell - 1]$. Similarly, define $\prec'$ using T'. We claim that $A \prec A'$ if and only if $A' \prec' A$. Assuming otherwise, the sequence of components from A to A' for T and from A' to A for T' can be assumed to be distinct except for A and A'. Using the directed edges $f_i \in T$ from A to A' and $g_i \in T'$ from A' to A, together with some of the edges in F, one obtains an alternating cycle of even length, such that (picking an orientation) the clockwise edges belong to T, and the counterclockwise edges belong to T'. This immediately contradicts the compatibility of T and T'.

Now, let $f : [n] \to \mathbb{R}$ be a function with the following properties: it is constant with value $f(A)$ on the components A of F, and such that $f(A) < f(A')$ if and only if $A \prec A'$ if and only if $A' \prec' A$. Assume that $f(n) = 0$. Then f extends to a linear function $\phi_f : (\mathbb{R}^n)^*/h_n \mapsto \mathbb{R}$, by setting $h_i \mapsto f(i)$. It follows by construction that $\phi_f(C_T) \geq 0$, $\phi_f(C_{T'}) \leq 0$, and $\phi_f(C_F) = 0$. It follows that $C_T \cap C_{T'} = C_F$ is a common face of both cones. □

A *matching field* on $[n]$ is a collection $\mathcal{E} = \{M_{I,J}\}$ of matchings, one for each pair (I, J) of disjoint subsets of $[n]$ of equal size, such that for each $I' \subset I$ and $J' \subset J$, where I' is matched to J' in $M_{I,J}$, we have $M_{I',J'}$ is the restriction of $M_{I,J}$ to (I', J'). If $M_{I',J'}$ is a restriction of $M_{I,J}$, we shall say that $M_{I,J}$ contains $M_{I',J'}$.

We shall call a matching field *noncrossing*, if every matching $M_{I,J}$ is noncrossing when drawn on the circle.

Theorem 8.4. *For each generic simple alcoved polytope P, there is a unique matching field $\mathcal{E}(P)$, such that $\mathcal{E}(P)$ and the set of vertex trees $\mathcal{T}(P)$ are related by the condition: $T \in \mathcal{T}(P)$ if and only if all matchings in T belong to the $\mathcal{E}(P)$.*

Furthermore, if P is a polypositroid, then $\mathcal{E}(P)$ consists of noncrossing trees.

Proof. Let P be a generic simple alcoved polytope, and $\mathcal{T}(P)$ be its set of vertex trees. We claim that for each pair (I, J) of disjoint subsets of $[n]$ of equal cardinality, some tree $T \in \mathcal{T}(P)$ contains a matching of (I, J). To see this, consider the point $x_{I,J} = \sum_{j \in J} h_j - \sum_{i \in I} h_i$. Since $\{C_T \mid T \in \mathcal{T}(P)\}$ are the maximal cones of a complete fan, it belongs to C_T for some $T \in \mathcal{T}(P)$. But T is alternating by Lemma 8.2, and it follows that T contains a unique matching of (I, J). It follows from Lemma 8.3 that $\mathcal{T}(P)$ determines a unique matching ensemble $\mathcal{E}(P)$. Furthermore, $\mathcal{T}(P)$ is exactly the set of trees T, such that all matchings in T belong to $\mathcal{E}(P)$: if T' is another tree satisfying this condition, then, by Lemma 8.3, $C_{T'}$ can be added to the complete fan $\{C_T \mid T \in \mathcal{T}(P)\}$, which is a contradiction.

The last sentence follows from Lemma 8.2. □

Definition 8.5 (cf. [OY, SZ]). Let $\mathcal{E}$ be a matching field. Then we say that $\mathcal{E}$ satisfies the *linkage axiom* if

1. for any disjoint (I, J) of equal size, and $j' \in [n] \setminus (I \cup J)$, there is an edge $(i, j) \in M_{I,J}$, such that the matching $M'_{I,J'} := M_{I,J} \setminus \{(i, j)\} \cup \{(i, j')\}$ belongs to $\mathcal{E}$, where $J' = J \setminus \{j\} \cup \{j'\}$;
2. for any disjoint (I, J) of equal size, and $i' \in [n] \setminus (I \cup J)$, there is an edge $(i, j) \in M_{I,J}$, such that the matching $M'_{I',J} := M_{I,J} \setminus \{(i, j)\} \cup \{(i', j)\}$ belongs to $\mathcal{E}$, where $I' = I \setminus \{i\} \cup \{i'\}$;

If $\mathcal{E}$ satisfies the linkage axiom, we say that $\mathcal{E}$ is a *matching ensemble*.

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Proposition 8.6. *Let P be a generic simple alcoved polytope, and let $\mathcal{E}(P)$ be the appearing in Theorem 8.4. Then $\mathcal{E}(P)$ satisfies the linkage axiom, and is a matching ensemble.*

Proof. Let $k = |I| = |J|$. Consider the vector

$$x = \left(1 + \frac{1}{k}\right) \sum_{i \in I} h_i + \sum_{j \in J \cup \{j'\}} h_j.$$

Then x lies in the cone C_T for some $T \in \mathcal{T}(P)$. Let $\tilde{T} = T|_{I \cup J \cup \{j'\}}$ denote the induced subgraph on $I \cup J \cup \{j'\}$. It is not difficult to see that $\tilde{T}$ must be connected, and thus itself a tree. Let $A_1, A_2, \ldots, A_r \subset I \cup J$ be the (vertex sets of the) connected components of the forest $\tilde{T} \setminus \{j'\}$ obtained by removing j'. Looking at the coefficients of $h_t, t \in A_s$ in x, we deduce that $|A_s \cap I| = |A_s \cap J|$ for each $s = 1, 2, \ldots, r$. It follows that the matching $M_{I,J} \in \mathcal{E}(P)$ restricts to a matching on $(A_s \cap I, A_s \cap J)$ for each $s = 1, 2, \ldots, r$.

Now let $i \in (A_1 \cap I)$ be the vertex in A_1 connected to j' and let $(i, j) \in M_{I,J}$ be the edge of $M_{I,J}$ incident to i. Then T also contains the matching $M_{I,J} \setminus \{(i, j)\} \cup \{(i, j')\}$, and, by Theorem 8.4, so does $\mathcal{E}(P)$. This establishes condition (1) of the linkage axiom for $\mathcal{E}(P)$. Condition (2) is similar. □

Conjecture 8.7. *Every noncrossing matching ensemble appears as $\mathcal{E}(P)$ for some generic simple polypositroid P.*

Remark 8.8. In some way, matching ensembles are analogous to matroids, and matching ensembles of the form $\mathcal{E}(P)$ are analogous to *realizable* matroids. So Conjecture 8.7 is similar in spirit to the result of [ARW17] that positive oriented matroids are positroids, that is, they are realizable.

The following examples support Conjecture 8.7.

Example 8.9. Let $n = 3$. In this case, every alcoved polytope is automatically a polypositroid. Indeed, there is a single matching field on $\{1, 2, 3\}$, it satisfies the linkage axiom, and it is noncrossing. Thus, there is only one possible normal fan for a generic simple polypositroid.

Example 8.10. Let $n = 4$. Let P be a generic simple polypositroid. We use Theorem 8.4 to understand the possible choices for $\mathcal{T}(P)$. By the noncrossing condition, a matching field $\mathcal{E}(P)$ is uniquely determined except for matchings on $(I, J) = (\{1, 3\}, \{2, 4\})$ and $(I, J) = (\{2, 4\}, \{1, 3\})$, each of which there are two choices of matchings, giving four possibilities for $\mathcal{E}(P)$, all of which satisfy the linkage axiom. By an explicit calculation, for example, by computing whether putative vertices v_T lie inside P, we find that the matching $M_{\{1,3\},\{2,4\}}$ (respectively, $M_{\{2,4\},\{1,3\}}$) depends on the sign of $a_{41} + a_{23} - a_{43} - a_{21}$ (respectively, $a_{12} + a_{34} - a_{14} - a_{32}$) (if $a_{41} + a_{23} - a_{43} - a_{21} = 0$, then P is not simple). Suppose that P arises from the balanced graph G via Theorem 6.12. Then we have

$$\begin{aligned} a_{41} + a_{23} - a_{43} - a_{21} &= m_{43} + m_{13} - m_{24} - m_{34} \\ a_{12} + a_{34} - a_{14} - a_{32} &= m_{14} + m_{24} - m_{31} - m_{41} \end{aligned} \tag{8.2}$$

(using (6.2), the right hand side (RHS) can be written in a number of equivalent ways). It is easy to construct generic balanced G, such that the RHS has any of the four possible ordered pairs of signs. For example, a balanced directed cycle $(1 \to 4 \to 3 \to 1)$ (respectively, $(4 \to 3 \to 2 \to 4)$) makes the first (respectively, second) quantity in (8.2) positive and the second (respectively, first) 0. It follows that there are exactly four normal fans of generic simple polypositroids for $n = 4$.

We consider the cyclohedron defined in (3.5).

Proposition 8.11. *Let P be a cyclohedron. Then the set $\mathcal{T}(P)$ of vertex trees of P is the set of noncrossing, alternating trees on $[n]$ with the following additional property: there is a cyclic rotation $i \prec i+1 \prec \cdots \prec n \prec 1 \prec \cdots \prec i-1$ of the usual order on $[n]$, such that every edge (i, j) of T_v satisfies $i \prec j$.*

Proof. Let r, s, r', s' be four indices in cyclic order. Then it follows from (3.6) that $f_{[r,s]} + f_{[r',s']} > f_{[r,s']} + f_{[r',s]} - k$, so that in any vertex tree T_v, the directed edges $(r-1) \to s$ and $(r'-1) \to s'$

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Figure 1. *Noncrossing, decreasing, alternating trees on* $\{1, 2, 3, 4\}$.

cannot both be present. The set of trees satisfying this condition is exactly the set of trees stated in the proposition. To complete the proof, it suffices to note from (8.1) that the cyclohedron has $\binom{2n-2}{n-1}$ vertices, and that the number of noncrossing, decreasing (every edge $(i \to j)$ satisfies $j < i$), alternating trees on $[n]$ is equal to the Catalan number $\frac{1}{n}\binom{2n-2}{n-1}$ (see [Po09] or Remark 8.21 below). □

For example, let $n = 4$. Then the cyclohedron has 20 vertices, and each vertex tree is a rotation of one of the trees in Figure 1.

Remark 8.12. The decomposition in Proposition 8.11 of $\mathcal{T}(P)$ into the n cyclic rotations of the set of noncrossing, alternating, decreasing trees has the following geometric interpretation: the cones $C, c(C), c^2(C), \ldots, c^{n-1}(C)$ cover all of H_0 and intersect in lower-dimensional faces.

8.3. Proof of Theorem 4.9

Proposition 8.13. *Let P be a generic simple alcoved polytope given by the inequalities (3.2). Then P is a polypositroid if and only if, for any four indices i, j, k, l in cyclic order, we have $a_{ik} + a_{jl} > a_{il} + a_{jk}$.*

Proof. Suppose P is a generic simple polypositroid. By the uniqueness part of Theorem 8.4 (or directly from the proof), there is a vertex tree T_v of P, which contains a noncrossing matching of $(\{k, l\}, \{i, j\})$. Since i, j, k, l are in cyclic order, the matching must match i with l and match j with k. We thus have

$$\begin{aligned} a_{il} + a_{jk} &= (h_i - h_l)(v) + (h_j - h_k)(v) \\ &= (h_i - h_k)(v) + (h_j - h_l)(v) \\ &\leq a_{ik} + a_{jl}. \end{aligned}$$

Equality cannot occur, for otherwise, T_v will contain a cycle. Conversely, if P is a generic simple alcoved polytope and $a_{ik} + a_{jl} > a_{il} + a_{jk}$ holds for any four indices i, j, k, l in cyclic order, then we deduce that $\mathcal{E}(P)$ is noncrossing, so $\mathcal{T}(P)$ consists of noncrossing trees, and, by Lemma 6.15, we conclude that P is a polypositroid. □

Now, the inequalities $a_{ik} + a_{jl} > a_{il} + a_{jk}$ define an open subcone C of $\mathcal{C}_{\text{alc}}$, each point of which represents a generic alcoved polytope. It follows from Proposition 8.11 that the cyclohedron is a generic simple polypositroid and thus C is nonempty. An open dense subset of $C' \subseteq C$ corresponds to generic alcoved polytopes that are simple. Applying Proposition 8.13, we see that these polytopes are generic simple polypositroids. The closure of C' is the closed cone cut out by (4.3). The corresponding limits of generic simple polypositroids are nonempty (possibly not generic, possibly not simple) polypositroids, finishing the proof of the Theorem 4.9.

8.4. Duality for alternating trees

A noncrossing tree T on $[n]$ is called *circular-alternating* if for each vertex v, the edges incident to v alternate between incoming and outgoing as they are read in order when T_v is drawn on a circle. Thus, for example, if v is incident to u_1, u_2, u_3, u_4 with $u_1 < v < u_2 < u_3 < u_4$, then the edges (v, u_1), (v, u_4), (v, u_3), (v, u_2) alternate in direction.

Let $\mathcal{T}_{\text{alt}}$ denote the set of alternating, noncrossing trees, and let $\mathcal{T}_{\text{cir}}$ denote the set of noncrossing, circular-alternating trees. For $T \in \mathcal{T}_{\text{alt}}$, let T' be obtained from T as follows: place the numbers $1', 1, 2', 2, \ldots, n', n$ in clockwise order around a circle, and draw T using the numbers $1, 2, \ldots, n$. Then

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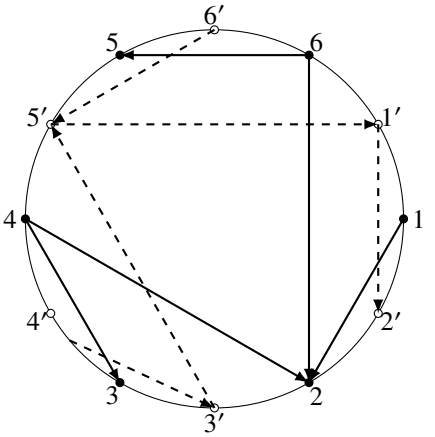

Figure 2. *The bijection of Proposition 8.14: when T is the graph consisting of the solid arrows, $T' = \varphi(T)$ is the graph consisting of the dashed arrows.*

T' is the unique tree on the numbers $1', 2', \ldots, n'$, such that each directed edge $c' \to d'$ of T' intersects a unique directed edge $a \to b$ of T, and furthermore, a, d', b, c' are in clockwise order.

Proposition 8.14. *The map $T \mapsto T'$ gives a bijection $\varphi : \mathcal{T}_{\mathrm{alt}} \to \mathcal{T}_{\mathrm{cir}}$.*

Proof. The tree T cuts the disk up into n pieces, each containing exactly one of $1', 2', \ldots, n'$. Let D be one of these components. Then D is bounded by an arc of the circle and a number of edges of T. When read in order around the boundary of D, these edges of T alternate in direction. It follows that $T' \in \mathcal{T}_{\mathrm{cir}}$.

In the other direction, let D' be one of the components that $T' \in \mathcal{T}_{\mathrm{cir}}$ divides the disk into. Then the edges of T' along the boundary of D' are all in the *same* direction. It follows that there is unique $T \in \mathcal{T}_{\mathrm{alt}}$, such that $\varphi(T) = T'$. □

For a tree T, let C'_T denote the cone in $\mathbb{R}^n$ spanned by $e_i - e_j$ for each directed edge $j \to i$ in T.

Proposition 8.15. *For $T \in \mathcal{T}_{\mathrm{alt}}$, the cones C_T and $C'_{\varphi(T)}$ are dual cones.*

Proof. We have

$$(h_b - h_a)(e_d - e_c) = \begin{cases} 0 & \text{if } \overline{ab} \text{ and } \overline{d'a'} \text{ are noncrossing,} \\ 1 & \text{if } a, d', b, c' \text{ are in clockwise order,} \\ -1 & \text{if } a, d', b, c' \text{ are in anticlockwise order.} \end{cases}$$

The result then follows from the definition of φ. □

Example 8.16. A noncrossing alternating tree $T \in \mathcal{T}_{\mathrm{alt}}$ and the corresponding noncrossing, circular-alternating tree $\mathcal{T}' = \varphi(T)$ is given in Figure 2. One can check that the cones $C_T = \mathrm{span}_{\geq 0}(h_3 - h_4, h_2 - h_4, h_2 - h_6, h_2 - h_1, h_5 - h_6)$ and $C'_{T'} = \mathrm{span}_{\geq 0}(e_3 - e_4, e_5 - e_3, e_1 - e_5, e_2 - e_1, e_5 - e_6)$ are dual, in agreement with Proposition 8.15.

Let T be a directed tree on $[n]$. An edge is *directed away from n* if it is directed away from n as part of some path connected to n. Define two statistics on T by

$$\mathrm{up}(T) = \#\{\text{edges directed away from } n\}$$
$$\mathrm{des}(T) = \#\{\text{edges } i \to j \text{ with } i > j\}.$$

The edges counted by $\mathrm{des}(T)$ are called descent edges.

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Proposition 8.17. *Let $T \in \mathcal{T}_{\text{alt}}$ and $T' = \varphi(T) \in \mathcal{T}_{\text{cir}}$. Suppose that $e \in T$ is the unique edge intersecting $e' \in T'$. Then e is directed away from n in T if and only if e' is a descent edge in T'. In particular, we have* $\text{up}(T) = \text{des}(T')$.

Let P be a generic simple polypositroid. Let $\mathcal{T}(P) = \{T_v\}$ be the collection of its vertex trees: by Lemma 8.2, $\mathcal{T}(P) \subset \mathcal{T}_{\text{alt}}$.

We now describe the 1-skeleton of P in terms of the trees T_v. Suppose $E = (v, v')$ is an edge of P. The forest T_E has two components and one has $T_v = T_E \cup \{e\}$ and $T_{v'} = T_E \cup \{e'\}$ for distinct directed edges e, e'. The graph $T_E \cup \{e, e'\}$ has a unique (nondirected) cycle containing both e and e'.

Lemma 8.18. *The edges e and e' have the same direction along this cycle.*

Proof. The edges e and e' correspond to facets intersecting the edge E, and the direction corresponds to a choice of one infinite direction along the affine span of E, which we assume to be parallel to $e_i - e_j$. The direction is determined by which of the two components T_1 and T_2 of T_E the source (and hence sink) of e (respectively, e') lies in. Indeed, if e goes from T_1 to T_2, and $i \in T_1$ while $j \in T_2$, then the facet corresponding to e bounds the coordinate x_j above. It follows that the ray in the direction of v' emitting from v goes in the direction $\mathbb{R}_{\geq 0}(e_i - e_j)$.

But the two infinite directions corresponding to e and e' are opposite, so e must go from T_1 to T_2 (without loss of generality), and e' must go from T_2 to T_1. But this implies that e and e' have the same direction along the cycle containing them both. □

For a vertex v of a generalized permutohedron P, define the tree T'_v as follows (cf. [PRW]): T'_v has a directed edge $j \to i$ if there is an edge incident with v which goes in the direction $\mathbb{R}_{\geq 0}(e_i - e_j)$. When P is a polypositroid, we thus have two directed trees T_v and T'_v on $[n]$ for each vertex $v \in P$.

Theorem 8.19. *Let P be a generic simple polypositroid. Then for each vertex v of P, we have $T'_v = \varphi(T_v)$.*

Proof. Suppose $e = (a \to b)$ is a directed edge of T_v, such that $T_v \backslash \{e\}$ has two components $T_1 \ni a$ and $T_2 \ni b$. Let c be the cyclic minimum of T_1 and d the cyclic minimum of T_2. It follows from the discussion in the proof of Lemma 8.18 that T'_v has a directed edge from c' to d'. One can check that $c' \to d'$ intersects only $a \to b$, and that the four vertices are in the stated order. □

Let $h_P(t)$ denote the h-polynomial of a simple d-dimensional polytope P. It is given by the equality $\sum_{i=0}^{d} f_i(P)t^i = h_P(t+1)$.

Corollary 8.20. *Suppose P is a generic simple polypositroid. The h-polynomial of P is*

$$h_P(t) = \sum_{T'_v \in \mathcal{T}(P)} t^{\text{des}(T'_v)} = \sum_{T_v \in \mathcal{T}(P)} t^{\text{up}(T_v)}.$$

Proof. The first equality is shown in [PRW, Theorem 4.2], and the second follows from Proposition 8.17, or it can be proved in the same way as the first. □

Remark 8.21. The bijection of Theorem 8.19 gives a bijection between noncrossing, decreasing, alternating trees on $[n]$, and rooted plane binary trees on $[n]$ equipped with the depth-first search labeling.

8.5. Coarsenings of braid arrangements

Recall that the *braid arrangement* $\mathcal{B}_n \subset (\mathbb{R}^n)^*/h_n$ is the central arrangement which is the union of all hyperplanes of the form $y_i - y_j = 0$, $i \neq j \in [n]$, where $y = (y_1, y_2, \ldots, y_n)$ is identified with the linear function $y_1x_1 + y_2x_2 + \cdots + y_nx_n \in (\mathbb{R}^n)^*$. We shall consider the hyperplane arrangement $\mathcal{B}_n$ as a complete fan, the *braid fan*. The maximal cones of $\mathcal{B}_n$ are indexed by $w \in S_n$:

$$C_w = \{y = (y_1, y_2, \ldots, y_n) \in (\mathbb{R}^n)^* \mid y_{w(1)} \leq y_{w(2)} \leq \cdots \leq y_{w(n)}\}.$$

and the rays of $\mathcal{B}_n$ are the $h_S = \sum_{i \in S} x_i \in (\mathbb{R}^n)^*/h_n$ for $S \in 2^{[n]} - \{\emptyset, [n]\}$.

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The normal fan to the permutohedron $P_n \subset H$ is the braid fan $\mathcal{B}_n$. More generally, any generalized permutohedron P that is sufficiently generic has $\mathcal{B}_n$ as its normal fan (see [PRW, Proposition 3.2]).

Now, let P be a generic simple polypositroid and $\mathcal{F} = \mathcal{F}(P) \subset (\mathbb{R}^n)^*/h_n$ be its normal fan. The rays of $\mathcal{F}$ are the $h_{[r,s]}$, $[r,s] \in \mathcal{I}$. While there are many possibilities for $\mathcal{F}$, as we saw in Theorem 8.1, all such $\mathcal{F}$ have the same f-vector. By definition, P is a generalized permutohedron, so $\mathcal{F}$ is a coarsening of the fan $\mathcal{B}_n$. In particular, each maximal cone $C_T, T \in \mathcal{T}(P)$ in $\mathcal{F}$ is a union of a number of the cones C_w.

Proposition 8.22. *Let $T \in \mathcal{T}_{\mathrm{alt}}$ and $w \in S_n$. We have $C_w \subset C_T$ if and only if for each edge $j \to i$ of $T' = \varphi(T)$, we have $w^{-1}(i) > w^{-1}(j)$.*

Proof. By Proposition 8.15, the inclusion $C_w \subset C_T$ is equivalent to the condition that for all $y \in C_w$ and $x \in C'_{T'}$ we have $y(x) \geq 0$. This is equivalent to the condition that $y(e_i - e_j) = y_i - y_j \geq 0$ for edges $j \to i$ of T'. □

Corollary 8.23. *Let $T \in \mathcal{T}_{\mathrm{alt}}$. We have $C_T = C_w$ for some $w \in S_n$ if and only if $\varphi(T) \in \mathcal{T}_{\mathrm{cir}}$ is a path. Furthermore, for such T, we have $T \in \mathcal{T}(P)$ for any generic simple polypositroid.*

Proof. Let $T' \in \mathcal{T}_{\mathrm{cir}}$. If the underlying graph of T' is a path, then T' itself is a directed path. In such a case, the condition of Proposition 8.22 uniquely determines $w \in S_n$, and conversely, w being uniquely determined implies that T' is a directed path. The last sentence follows from Theorem 8.4: the sources (respectively, the sinks) of T form cyclic intervals when T' is a path. □

Example 8.24. Let $n = 3$. In this case, the normal fan of a generic simple polypositroid is the braid arrangement. There are six trees in $\mathcal{T}_{\mathrm{alt}}$:

and six in $\mathcal{T}_{\mathrm{cir}}$:

Since the underlying graph of every $T' \in \mathcal{T}_{\mathrm{cir}}$ is a path, there is a unique $w = w_{T'} \in S_n$ satisfying the condition of Proposition 8.22. This gives a bijection between $\mathcal{T}_{\mathrm{alt}}$ and S_n, identifying $C_T, T \in \mathcal{T}_{\mathrm{alt}}$, and $C_w, w \in S_n$.

Example 8.25. Let $n = 4$. We have $|\mathcal{T}_{\mathrm{alt}}| = 24$, consisting of 8 trees that are stars and 16 trees that are paths. For a generic simple polypositroid P, we have $|\mathcal{T}(P)| = f_0(P) = \binom{6}{3} = 20$ by Theorem 8.1. There are 16 trees in $T \in \mathcal{T}_{\mathrm{alt}}$, such that $\varphi(T)$ is a directed path. By Corollary 8.23, these trees belong to $\mathcal{T}(P)$, for any P.

There are eight trees in $\mathcal{T}_{\mathrm{alt}}$, such that $\varphi(T)$ is a (circular-alternating) star. These eight trees are the four cyclic rotations of the following two trees

$$(2 \to 1 \leftarrow 4 \to 3) \qquad \text{and} \qquad (3 \to 4 \leftarrow 1 \to 2).$$

For each of these trees, C_T is a union of two of the cones C_w. For example, take $T = (2 \to 1 \leftarrow 4 \to 3)$ with dual tree $T' = (2, 4 \to 3 \to 1)$. According to Proposition 8.22, we have $C_{2\to 1\leftarrow 4\to 3} = C_{2431} \cup C_{4231}$. Similarly, we obtain

$$C_{4\to 3\leftarrow 2\to 1} = C_{4213} \cup C_{2413}, \qquad C_{4\to 1\leftarrow 2\to 3} = C_{2413} \cup C_{2431}, \qquad C_{2\to 3\leftarrow 4\to 1} = C_{4213} \cup C_{4231}.$$

Thus, we have

$$C_{2\to 1\leftarrow 4\to 3} \cup C_{4\to 3\leftarrow 2\to 1} = C_{4\to 1\leftarrow 2\to 3} \cup C_{2\to 3\leftarrow 4\to 1}.$$

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Each $\mathcal{T}(P)$ contains either both $(2 \to 1 \leftarrow 4 \to 3)$ and $(4 \to 3 \leftarrow 2 \to 1)$ or both $(4 \to 1 \leftarrow 2 \to 3)$ and $(2 \to 3 \leftarrow 4 \to 1)$. Switching between these two choices corresponds to switching the matching $M_{\{2.4\},\{1,3\}}$ in $\mathcal{E}(P)$.

9. Integer points in polypositroids

We assume in this section that $H = \{x \in \mathbb{R}^n \mid x_1 + x_2 + \cdots + x_n = k\}$, where k is an integer. A polytope $P \subset H$ is an *integer polytope* if its vertices have integer coordinates. By translating P and H, we may and will assume that $S := P \cap \mathbb{Z}^n \subset \mathbb{N}^n$, so that to each integer point $p = (p_1, p_2, \ldots, p_n) \in P$, one may associate a multiset I_p of size k which contains p_1 1's, p_2 2's, and so on. Thus, if P is the matroid polytope of a matroid M, then the multisets I_p are honest sets, equal to the bases of M.

If $I = \{i_1 \le i_2 \le \cdots \le i_k\}$ and $J = \{j_1 \le j_2 \le \cdots \le j_k\}$ are two multisets consisting of elements in $\{1, 2, \ldots, n\}$, we define two multisets $\mathrm{sort}_1(I, J)$ and $\mathrm{sort}_2(I, J)$ of the same size as follows. Let $I \cup J = \{a_1 \le a_2 \le \cdots \le a_{2k}\}$. Then $\mathrm{sort}_1(I, J) := \{a_1, a_3, \ldots, a_{2k-1}\}$ and $\mathrm{sort}_2(I, J) := \{a_2, a_4, \ldots, a_{2k}\}$. For example, suppose $I = \{1, 1, 3, 4, 4, 5\}$ and $J = \{1, 2, 2, 2, 3, 4\}$. Then $\mathrm{sort}_1(I, J) = \{1, 1, 2, 3, 4, 4\}$ and $\mathrm{sort}_2(I, J) = \{1, 2, 2, 3, 4, 5\}$.

The following characterization of integer alcoved polytopes is given in [LP07].

Theorem 9.1 [LP07, Theorem 3.1]. *Suppose $P \subset H$ is an integer polytope, such that $S := P \cap \mathbb{Z}^n \subset \mathbb{N}^n$. Then P is an alcoved polytope if and only if for any $p, p' \in S$, there exist $q, q' \in S$ so that $I_q = \mathrm{sort}_1(I_p, I_{p'})$ and $I_{q'} = \mathrm{sort}_2(I_p, I_{p'})$.*

If a collection S of nonnegative integer points satisfies the condition in Theorem 9.1, then we call S *sort-closed*.

Murota [Mur] studies certain collections of lattice points called *M-convex sets*, which are essentially equivalent to the *discrete polymatroids* of Herzog and Hibi [HH]. We use the terminology of the latter. A *base polymatroid* is a generalized permutohedron P, such that all the values $f_P(S)$ of the support function are nonnegative (see, for example [CL, Section 3]). Any generalized permutohedron can be translated so that the nonnegativity condition holds. A *discrete (base) polymatroid* is a collection of multisubsets of $[n] = \{1, 2, \ldots, n\}$ satisfying an exchange criterion. The exchange criterion can be formulated in the language of generalized permutohedra as follows.

Theorem 9.2 [HH, Theorem 2.3]. *Suppose $P \subset H$ is an integer polytope. Let $S := P \cap \mathbb{Z}^n$. Then P is a generalized permutohedron if and only if for any $p, q \in S$, we have:*

whenever $p_i > q_i$, we can find j, so that $p_j < q_j$ and $p_i - e_i + e_j \in S$.

If a collection S of integer points satisfies the condition in Theorem 9.2, we say S satisfies the Exchange Lemma. Combining Theorems 9.1 and 9.2, we obtain the following characterization of integer polypositroids:

Theorem 9.3. *Suppose $P \subset H$ is an integer polytope, such that $S := P \cap \mathbb{Z}^n \subset \mathbb{N}^n$. Then P is an integer polypositroid if and only if S is sort-closed, and S satisfies the Exchange Lemma.*

Example 9.4. Let $P = P(\mathbf{v})$ for the Coxeter necklace $\mathbf{v}$ of Example 6.11. Then the set $S := P \cap \mathbb{Z}^4$ consists of the integer vectors (p_1, p_2, p_3, p_4) satisfying $0 \le p_i \le 3$ and $1 \le p_i + p_{i+1} \le 5$, with indices taken modulo 4. In particular, the integer points $(3, 0, 3, 0)$ and $(0, 3, 0, 3)$ belong to S. If we removed these two integer points from S, the Exchange Lemma would still be satisfied. However, S would not be sort-closed, for example, by considering $p = (3, 1, 2, 0)$ and $q = (2, 0, 3, 1)$.

In the case that P consists of 0-1 vectors, Theorem 9.3 characterizes positroids as those collections $M \subset \binom{[n]}{k}$ of k-element subsets that are both sort-closed and satisfies the Exchange Lemma.

Corollary 9.5. *Positroids are exactly the sort-closed matroids.*

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Remark 9.6. Corollary 9.5 can also be deduced directly from the characterization of positroids as matroids associated to points $X \in \mathrm{Gr}(k,n)_{\geq 0}$ in the totally nonnegative Grassmannian. Namely, the Plücker coordinates $\Delta_I(X)$ of such a point satisfy inequalities that give a sort-closed matroid (see [Lam, Proposition 8.7]).

Part II Coxeter polypositroids

Generalized permutohedra are defined by specifying the possible directions of edges. Alcoved polytopes are defined by specifying the possible directions of normal vectors to facets. The set of allowed edge directions and the set of allowed facet normal directions are related by the linear transformation $e_i \mapsto h_i$. We give this linear transformation a root-system theoretic interpretation, and develop the theory of Coxeter polypositroids.

10. Coxeter elements

10.1. Root systems

First, we recall some terminology and a few well-known facts related to root systems and Weyl groups (see [Bou, Hum] for more details).

Let $V \simeq \mathbb{R}^r$ be a vector space of dimension $r \geq 2$ equipped with a symmetric positive definite bilinear form (x, y). Let $R \subset V$ be an irreducible and reduced crystallographic *root system* of rank r. For a root $\alpha \in R$, the corresponding *coroot* is $\alpha^\vee = 2\alpha/(\alpha,\alpha)$, and the *reflection* $s_\alpha \in GL(V)$ with respect to α is given by

$$s_\alpha : x \mapsto x - (\alpha^\vee, x)\,\alpha, \quad \text{for } x \in V.$$

The *Weyl group* $W \subset GL(V)$ is the group generated by the reflections s_α, $\alpha \in R$. Let us fix a choice of *positive roots* $R^+ \subset R$ and the corresponding choice of *simple roots* $\alpha_1, \ldots, \alpha_r$ in R and simple coroots $\alpha_1^\vee, \ldots, \alpha_r^\vee$. The *Cartan matrix* $A = (A_{ij})$ is given by

$$A_{ij} = (\alpha_i^\vee, \alpha_j) \in \mathbb{Z}. \tag{10.1}$$

Let $s_i = s_{\alpha_i}$ be the *simple reflections*. It is well-known that all possible choices of positive roots are conjugate to each other by the action of the Weyl group W.

Let $\omega_1, \ldots, \omega_r \in V$ be the basis of V dual to the basis of simple coroots $\alpha_1^\vee, \ldots, \alpha_r^\vee$, that is, $(\alpha_j^\vee, \omega_i) = \delta_{ij}$ for any $i, j \in \{1, \ldots, r\}$. The vectors $\omega_1, \ldots, \omega_r$ are called *fundamental weights*. Let $\Lambda \subset V$ denote the weight lattice spanned by $\omega_1, \ldots, \omega_r$.

Remark 10.1. Many of our results hold even for noncrystallographic root systems. However, for the connections to cluster algebras in Section 18, we must use a crystallographic root system.

10.2. Coxeter elements

A *standard Coxeter element* $c = s_{i_1} s_{i_2} \cdots s_{i_r} \in W$ is the product of the simple reflections $s_1, \ldots, s_r$ written in some order $s_{i_1}, \ldots, s_{i_r}$. More generally, a *Coxeter element* $c' = s'_{i_1} s'_{i_2} \cdots s'_{i_r} \in W$ is a similar product for some (possibly different) choice of simple reflections $s'_1, \ldots, s'_r$. In other words, Coxeter elements are Weyl group conjugates $c' = wcw^{-1}$, $w \in W$, of standard Coxeter elements c. Moreover, any two Coxeter elements are conjugates of each other. Thus, all Coxeter elements have the same order, called the *Coxeter number* h.

The eigenvalues of a Coxeter element are $e^{2\pi\sqrt{-1}\, m_i/h}$, where $m_1, \ldots, m_r \in \{1, \ldots, h-1\}$ are the *exponents* of the root system. In particular, 1 is not an eigenvalue of c. This implies the following claim.

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Lemma 10.2. *For any Coxeter element c, the transformation $I - c$ is an invertible element of $GL(V)$, and the inverse is given by $(I-c)^{-1} = -\frac{1}{h}\sum_{j=1}^{h-1} j\, c^j$.*

Proof. Since all eigenvalues of c are h-th roots of unity, not including the identity, we have $I + c + c^2 + \cdots + c^{h-1} = 0$. Thus

$$(I-c)(\sum_{j=1}^{h-1} j\, c^j) = (\sum_{j=1}^{h-1} c^j) - (h-1)I = -hI.$$

□

We say that a choice of positive roots R^+ is *compatible* with c if c is a standard Coxeter element with respect to R^+.

Let $\Gamma = \{1, c, c^2, \ldots, c^{h-1}\} \subset W$ be the subgroup generated by c. Given R^+ compatible with c, and a reduced factorization $c = s_1 s_2 \cdots s_r$, we obtain a total ordering of the root system R: set $\beta_1 = \alpha_1$, $\beta_2 = s_1\alpha_2$, …, $\beta_r = s_1 s_2 \cdots s_{r-1}\alpha_r$. Then define β_i for $i \in \mathbb{Z}$ recursively by $\beta_{i+r} := c\beta_i$. For each $i = 1, 2, \ldots, r$, the roots $\beta_i, \beta_{i+r}, \ldots, \beta_{i+(h-1)r}$ is a Γ-orbit in R.

Proposition 10.3.

1. *The set of $h \cdot r$ vectors $\{\beta_i \mid 1 \le i \le hr\}$ is exactly the set of all roots in R without repetitions. In particular, $|R| = hr$.*
2. *For each $i = 1, 2, \ldots, r$, there exists a unique integer $M(i) \in [1, h-1]$, such that*

$$\beta_i, c\beta_i, \ldots, c^{M(i)-1}\beta_i \in R^+, \text{ and } c^{M(i)}\beta_i, c^{M(i)+1}\beta_i, \ldots, c^{h-1}\beta_i \in R^-.$$

Proof. Follows from [Bou, Chapter VI, Section 1, n° 11, Proposition 33]. □

The first part of the following result is [KiTh, Theorem 3.6]. It is stated there for simply-laced Weyl groups but holds in the multiply-laced types as well.

Proposition 10.4. *Let R_1^+ and R_2^+ be two positive systems compatible with c. Then R_1^+ and R_2^+ are related by a sequence of elementary transformations $R^+ \mapsto (R^+)'$ of the following form: suppose s_i is a simple generator for R^+ and c has a reduced factorization either starting or ending in s_i, then set $(R^+)' = s_i \cdot R^+$.*

If $c = s_1 \cdots s_r$ and $(R')^+ = s_1 R^+$, then the total ordering of R coming from $((R')^+, c)$ is the cyclic shift $(\beta_2, \beta_3, \ldots, \beta_{hr}, \beta_1)$.

For the remainder of this section, we fix a Coxeter element c and a choice of positive roots R^+ compatible with c. We extend the definition of the fundamental weights ω_i by defining ω_i for $i \in \mathbb{Z}$ recursively by $\omega_{i+r} := c\omega_i$.

Proposition 10.5. *We have $(I-c)\omega_i = \beta_i$ for all $i \in \mathbb{Z}$.*

Proof. According to the definitions, $s_j(\omega_i) = \omega_i - \delta_{ij}\,\alpha_j$. Repeatedly applying the simple reflections $s_r, s_{r-1}, \ldots, s_1$ to ω_i, we get $c(\omega_i) = s_1 \cdots s_r(\omega_i) = \omega_i - s_1 \cdots s_{i-1}(\alpha_i)$. Thus

$$(I-c)(\omega_i) = s_1 \cdots s_{i-1}(\alpha_i) = \beta_i, \quad \text{for } i = 1, \ldots, r.$$

The statement now follows from Proposition 10.3. □

For convenience, when $\beta \in R$, we use the notation $\tilde{\beta}$ to denote $(I-c)^{-1}\beta \in \tilde{R}$. Thus, $\omega_i = \tilde{\beta}_i$ for $i = 1, 2, \ldots, hr$.

Let $w_0 \in W$ be the longest element, and let $i \mapsto i^\star$ denote the bijection on $\{1, 2, \ldots, r\}$, determined by $w_0\alpha_i = -\alpha_{i^\star}$. Recall that for $i = 1, 2, \ldots, r$, we have defined a positive integer $M(i)$ in Proposition 10.3.

Lemma 10.6. *Fix $k \in I$. We have*

1. *$(c^m\beta_i^\vee, \omega_k) \ge 0$, for $0 \le m < M(k^\star)$ and any $i = 1, \ldots, r$;*
2. *$(c^m\beta_i^\vee, \omega_k) = 0$, if $M(i) < M(k^\star)$ and $M(i) \le m < M(k^\star)$;*

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3. $(c^m\beta_i^\vee, \omega_k) = 0$, *if* $M(k^\star) \le M(i)$ *and* $M(k^\star) \le m < M(i)$;
4. $(c^m\beta_i^\vee, \omega_k) \le 0$, *for* $M(k^\star) \le m < h$ *and any* $i = 1, \dots, r$.

Define $i \prec_c j$ if i and j are connected in the Dynkin diagram, and s_i precedes s_j in c. Orient the Coxeter diagram of W so that an edge (i, j) is oriented $j \to i$ if $i \prec_c j$. We will use the following formulae from [YZ].

Lemma 10.7. *We have*

1. $c^{M(i)}\beta_i = -\beta_{i^*}$;
2. $M(i) + M(i^\star) = h$;
3. *If* $j \to i$, *then we have*

$$M(i) - M(j) = \begin{cases} 1 & \text{if } i^\star \to j^\star \\ 0 & \text{if } j^\star \to i^\star \end{cases}.$$

Proof of Lemma 10.6. Fix i and j. We prove $(c^m\beta_i^\vee, \omega_j) \ge 0$ for $0 \le m < M(j^\star)$. If $m < M(i)$, then by definition, $c^m\beta_i \in R^+$, so $c^m\beta_i^\vee \in (R^\vee)^+$, and the inequality is clear. Thus, we may assume that $M(i) < M(j^\star)$ and $M(i) \le m < M(j^\star)$. In particular, we are assuming that $i \ne j$.

Define the support $S(\beta^\vee) \subseteq I$ of a coroot $\beta^\vee \in R^\vee$ to be the (positive or negative) simple coroots that occur in the expansion of $\beta^\vee$ into simple coroots. For a nonnegative integer a, let $P_a(i) \subseteq I$ be the set of vertices $j \in I$ that can be reached from $i \in I$ by a path where at most a edges are in the wrong direction. One can check that

$$S(\beta_k^\vee) \subset P_0(k)$$

and by induction on a, we have for $a \ge 0$,

$$S(c^a\beta_k^\vee) \subset P_a(k).$$

By Lemma 10.7(1), it follows that we have

$$S(c^m\beta_i^\vee) \subset P_{m-M(i)}(i^\star)$$

for $m \ge M(i)$. But, by Lemma 10.7(3), $M(j^\star) - M(i)$ is bounded above by the number of edges directed in the wrong direction on the path from $i^\star$ to j. Thus, if $m < M(j^\star)$, we have $j \notin P_{m-M(i)}(i^\star)$ and $(c^m\beta_i^\vee, \omega_j) = 0 \ge 0$. This proves statements (1) and (2). Statements (3) and (4) are similar. □

Example 10.8. We consider the root system of type A_{n-1}. Let $V = H_0 = \{x \mid x_1 + x_2 + \cdots + x_n = 0\} \subset \mathbb{R}^n$ and $R = \{e_i - e_j \mid i \ne j\}$. Then W is the symmetric group S_n. We take as positive simple roots $\alpha_i = e_{i+1} - e_i$. Then the linear functional $(\cdot, \omega_k) : V \to \mathbb{R}$ can be identified with the function $h_n - h_k \in (\mathbb{R}^n)^*$.

Now choose the Coxeter element $c = s_1 s_2 \cdots s_{n-1}$, which coincides with our choice in Section 5. The Dynkin diagram is oriented as follows:

$$1 \leftarrow 2 \leftarrow \cdots \leftarrow (n-1).$$

We have $\beta_i = s_1 \cdots s_{i-1}(e_{i+1} - e_r) = e_{i+1} - e_1$ for $i = 1, 2, \dots, n-1$. We have $c(e_i) = e_{i+1}$, where $e_{n+1} := e_1$. We compute that $i^\star = n - i = M(i)$. It is straightforward to verify Lemma 10.6 directly, noting also that for $R = A_{n-1}$, we have $\beta = \beta^\vee$.

Example 10.9. We consider the root system of type B_r with $V = \mathbb{R}^r$ and simple roots $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{r-1} = e_{r-1} - e_r, \alpha_r = e_r$, and Coxeter element $c = s_1 s_2 \cdots s_r$. Thus, $\beta_1 = e_1 - e_2, \beta_2 = e_1 - e_3, \dots, \beta_{r-1} = e_1 - e_r, \beta_r = e_1$. We have $c(e_1) = e_2, c(e_2) = e_3, \dots$, and $c(e_r) = -e_1$. We have $M(i) = r$ for all i and $i^\star = i$. Thus, $M(i) + M(i^\star) = 2r = h$, the Coxeter number. Since all $M(i) = r$

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are equal, Lemma 10.6 follows immediately from $c^m\beta_i^\vee \in (R^+)^\vee$ for $0 \le m < r$ and $c^m\beta_i^\vee \in (R^-)^\vee$ for $n \le m < 2r = h$.

11. Generalized W-permutohedra

A *W-permutohedron* P is a convex polytope in the space V which is the convex hull of an orbit $W(x)$ of the Weyl group W, for some $x \in V$ not lying in any of the hyperplanes $H_\alpha := \{x \in V \mid (x, \alpha) = 0\}$, $\alpha \in R$. One key property of W-permutohedra is that every edge of P is parallel to some coroot $\alpha^\vee \in R^\vee$.

Definition 11.1. A *generalized W-permutohedron* P is a convex polytope in the space V, such that every edge $[u, v]$ of P is parallel to a coroot $\alpha^\vee \in R^\vee$, that is, $u - v = a\alpha^\vee$, for some $a \in \mathbb{R}_{>0}$.

Note that the notion of a generalized W-permutohedron is unchanged when we replace the root system R by the dual root system $R^\vee$. Furthermore, the class of generalized W-permutohedra is preserved by the action of W.

The normal fan to a W-permutohedron is the W-Coxeter fan, the fan associated to the hyperplane arrangement consisting of all hyperplanes H_α as $\alpha \in R$ varies. The maximal cones of the W-Coxeter fan are indexed by $w \in W$. The 1-dimensional cones, or rays, of the W-Coxeter fan are generated by the set $W \cdot \{\omega_1, \dots, \omega_r\}$ of vectors lying in the W-orbit of a fundamental weight. Generalized W-permutohedra can be equivalently defined as convex polytopes whose normal fan refines the W-Coxeter fan (see, for example [PRW, Theorem 15.3]). In particular, any facet of a generalized W-permutohedron has a normal vector that lies in the W-orbit of a fundamental weight. Thus, a generalized W-permutohedron is given by a collection of inequalities

$$(x, \omega) \le a_\omega, \qquad \omega \in W \cdot \{\omega_1, \dots, \omega_r\}, \tag{11.1}$$

where we always assume that a_ω has been taken minimal. Let $C^W_{\text{sub}} = \{(a_\omega)\}$ denote the cone of vectors (a_ω) arising from generalized W-permutohedra. This cone is studied in [ACEP].

Let the *dominant cone*

$$C = \mathbb{R}_{\ge 0}\langle \alpha_1^\vee, \dots, \alpha_r^\vee \rangle \subset V \tag{11.2}$$

be the cone of nonnegative linear combinations of the simple (equivalently, positive) roots in R. For example, with $R = A_{n-1}$ and the conventions of Example 10.8, this agrees with the cone (5.1), intersected with $H_0 \simeq V$. The cone C is the dual of the *dominant Weyl chamber* $D = \mathbb{R}_{\ge 0}\langle \omega_1, \dots, \omega_r \rangle$. Namely, $D = \{y \in V \mid (x, y) \ge 0 \text{ for any } x \in C\}$ and $C = \{x \in V \mid (x, y) \ge 0 \text{ for any } y \in D\}$. The following result follows from the statement that the normal fan of a generalized W-permutohedra is a refinement of the W-Coxeter fan.

Theorem 11.2. *A polytope P is a generalized W-permutohedron if and only if it has the following form*

$$P = \bigcap_{w \in W} (v_w + w(C)),$$

where $v_w \in V$, $w \in W$, is a collection of points in V, such that, for any $u, w \in W$,

$$v_u \in v_w + w(C), \quad \textit{or, equivalently,} \quad v_u - v_w \in w(C) \cap (-u(C)).$$

The points $v_w, w \in W$ are exactly all vertices of the polytope P (possibly with repetitions).

Define the *dominance order* $\le_C$ as the partial order on points V given by $x \le_C y$ if $y - x \in C$. Similarly, for $w \in W$, the *w-dominance order* $\le_{w(C)}$ is the partial order on V given by $x \le_{w(C)} y$ if $y - x \in w(C)$, for $w \in W$. The following result easily follows from Theorem 11.2.

Corollary 11.3. *A polytope P is a generalized W-permutohedron if and only if for each $w \in W$, P has a unique minimum element $v_w \in P$ in the w-dominance order $\le_{w(C)}$.*

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[PRW, Theorem 15.3] also implies that the condition on the points $v_w, w \in W$ can be reformulated, as follows. It is enough to require the condition $v_u - v_w \in w(C) \cap (-u(C))$ only for the pairs $u, w \in W$, such that $u = w\, s_i$, for a simple reflection s_i. In this case, the cone $w(C) \cap (-w\, s_i(C))$ is the 1-dimensional cone spanned by the coroot $w(\alpha_i^\vee)$:

$$w(C) \cap (-w\, s_i(C)) = \mathbb{R}_{\geq 0}\big\langle w(\alpha_i^\vee)\big\rangle.$$

Theorem 11.4. *A polytope P is a generalized W-permutohedron if and only if it has the following form*

$$P = \bigcap_{w \in W} (v_w + w(C)),$$

where $v_w \in V$, $w \in W$ is a collection of points in V, such that, for any $w \in W$ and $i = 1, \ldots, r$,

$$v_w - v_{w s_i} = a\, w(\alpha_i^\vee) \quad \text{for } a \in \mathbb{R}_{\geq 0}.$$

12. Twisted (W, c)-alcoved polytopes

In [LP18], we studied the W-alcoved polytopes: polytopes in V with the property that all facet normals belong to R. Here, we introduce a twisted variant of W-alcoved polytopes that depends on the choice of a Coxeter element c.

12.1. Coxeter twisted roots

Definition 12.1. Define the *c-twisted root system* $\tilde{R}$ as the image $\tilde{R} = (I - c)^{-1}(R)$ of the root system R under the transformation $(I - c)^{-1}$. We call the elements of $\tilde{R}$ the *c-twisted roots*, or simply *twisted roots* if the Coxeter element is understood.

We have that $\tilde{R} = -\tilde{R}$. Note also that $\tilde{R}$ does not depend on the choice of positive system R^+.

According to Lemma 10.5, the twisted root system $\tilde{R}$ is exactly the set of weights that lie in the Γ-orbits of the vectors $\omega_1, \omega_2, \ldots, \omega_r$:

$$\tilde{R} = (I - c)^{-1}(R) = \{c^t(\omega_i) \mid t = 0, \ldots, h-1;\ i = 1, \ldots, r\} = \{\omega_i \mid i = 1, \ldots, hr\}. \tag{12.1}$$

Note that $\tilde{R} = \Gamma \cdot \{\omega_1, \ldots, \omega_r\} \subset W \cdot \{\omega_1, \ldots, \omega_r\}$.

Definition 12.2. A (W, c)-twisted alcoved polytope is a polytope $P \subset V$ whose facets are normal to twisted roots. More precisely, P is a nonempty set with the presentation

$$P = P(a_\omega) = \{x \in V \mid (x, \omega) \leq a_\omega \text{ for } \omega \in \tilde{R}\}.$$

Here, a_ω are arbitrary real numbers which we always assume to be chosen minimal.

Example 12.3. We continue Example 10.8. Applying Lemma 10.2, the twisted roots are given by

$$\tilde{R} = \left\{ -\frac{1}{n}\left(\sum_{k=1}^{n-1} k(e_{i+k} - e_{j+k}) \right) \mid i \neq j \right\} \subset V.$$

For example, $(I - c)^{-1}\alpha_1 = \frac{1}{5}(-4, 1, 1, 1, 1)$. In $(\mathbb{R}^n)^*/h_n$, this is equal to $h_n - h_1 = (0, 1, 1, 1, 1)$. Identifying $\tilde{R}$ with a subset of $(\mathbb{R}^n)^*/h_n$, we get $\tilde{R} = \{h_i - h_j \mid i \neq j\}$. The notion of (W, c)-twisted alcoved polytope here agrees with our notion of alcoved polytope in Definition 3.3.

If W and c are understood to be fixed, we may simply use the name "twisted alcoved polytope."

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Remark 12.4. Suppose that c and c' are two Coxeter elements. Then there exists $w \in W$ so that $c' = wcw^{-1}$. The c-twisted roots $\tilde{R}_c$ and the c'-twisted roots $\tilde{R}_{c'}$ are related by $\tilde{R}_{c'} = w \cdot \tilde{R}_c$. We have that $P \subset V$ is a (W, c')-twisted alcoved polytope if and only if $w \cdot P \subset V$ is a (W, c)-twisted alcoved polytope.

Definition 12.5. For a compact subset $Q \subset V$, the (W, c)-*twisted alcoved envelope* $\mathrm{env}(Q)$ is the smallest (W, c)-twisted alcoved polytope containing Q.

Clearly, $\mathrm{env}(P) = P$ if and only if P is a (W, c)-twisted alcoved polytope. Note that the intersection of generalized W-permutohedra may not be a generalized W-permutohedron. Thus, the "generalized W-permutohedron envelope" is not a well-defined operation.

Recall that the dominant cone C was defined in (11.2). Let $Q \subset V$ be a compact subset. For $i = 0, 1, \ldots, h-1$, let $v_i \in V$ be the minimum point in dominance order $\leq_{c^i(C)}$, such that $Q \subset v_i + c^i(C)$. Thus, every facet of $v_i + c^i(C)$ touches Q.

Proposition 12.6. *The (W, c)-twisted alcoved envelope of Q is given by the following intersection:*

$$\mathrm{env}(Q) = \mathrm{env}_c(Q) = \bigcap_{i=0}^{h-1} (v_i + c^i(C)).$$

Proof. Since Q is compact, for each $\omega \in \tilde{R}$, there is a minimal number a_ω, such that Q belongs to the half space $H^+_\omega := \{x \in V \mid (x, \omega) \leq a_\omega\}$. Then $\mathrm{env}(Q) = \bigcap_{\omega \in \tilde{R}} H^+_\omega$. The convex cone C has inward pointing normals given by $\omega_1, \omega_2, \ldots, \omega_r$. Thus, $v_0 + C = \bigcap_{j=1}^r H^+_{-\omega_j}$, and similarly, we have $v_i + c^i(C) = \bigcap_{j=1}^r H^+_{-c^i\omega_j}$. The claim follows from (12.1) and the fact that $\tilde{R} = -\tilde{R}$. □

12.2. Faces of twisted alcoved polytopes

We call a (W, c)-twisted alcoved polytope P *generic* if it is full-dimensional and every twisted root $\omega \in \tilde{R}$ defines a facet $F = \{(x, \omega) = a_\omega\} \cap P$ of P.

Now let P be a (W, c)-twisted alcoved polytope and F a face of P. Let $S(F) \subset \tilde{R}$ be the set of twisted roots ω, such that F lies on the hyperplane $(x, \omega) = a_\omega$ and P lies in the halfspace $(x, \omega) \leq a_\omega$. The set $S(F)$ is a (W, c)-analogue of the graph T_F of Section 6.4.

Suppose that β and β' are two distinct roots. We say that β and β' are *alternating* if $(\beta, \beta') \geq 0$. Note that β and β' are alternating if and only if $-\beta$ and $-\beta'$ are. Also note that this notion of alternating agrees with the notion used in the definition of alternating trees in Section 6.4. Suppose that $\omega, \omega' \in \tilde{R}$ are distinct twisted roots. Then we say that ω and ω' are alternating if $(I - c)\omega$ and $(I - c)\omega'$ are.

Lemma 12.7. *Suppose that P is a generic simple (W, c)-twisted alcoved polytope and v a vertex of P. Then $S(v) \subset \tilde{R}$ is a basis of V and that consists of pairwise alternating twisted roots.*

Proof. That $S(v)$ is a basis follows from the assumption that P is generic and simple. Suppose $\tilde{\beta}, \tilde{\beta}' \in S(v)$ are not alternating. Then $(\beta, \beta') < 0$ so there is a root of the form $\beta'' = b\beta + c\beta'$ with $b, c \in \mathbb{R}_{>0}$. Thus, there is a twisted root of the form $\tilde{\beta}'' = b\tilde{\beta} + c\tilde{\beta}'$. Clearly, $a_{\tilde{\beta}''} = \max_{x \in P}(x, \tilde{\beta}'') \leq b a_{\tilde{\beta}} + c a_{\tilde{\beta}'}$. We have

$$a_{\tilde{\beta}''} \geq (v, \tilde{\beta}'') = (v, b\tilde{\beta} + c\tilde{\beta}') = b a_{\tilde{\beta}} + c a_{\tilde{\beta}'} \geq a_{\tilde{\beta}''},$$

implying that $(v, \tilde{\beta}'') \in a_{\tilde{\beta}''}$ and thus $\tilde{\beta}'' \in S(v)$, contradicting the statement that $S(v)$ is a basis. □

13. (W, c)-polypositroids

Definition 13.1. A convex polytope $P \subset V$ is called a (W, c)-*polypositroid* if it is both a generalized W-permutohedron and a (W, c)-twisted alcoved polytope. In other words, we require that

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(1) every edge $[u, v]$ of P is parallel to a coroot $\alpha^\vee \in R^\vee$, i.e., $u - v = a\alpha^\vee$, $a \in \mathbb{R}_{>0}$;
(2) every facet of P is orthogonal to a twisted root $(I - c)^{-1}(\beta) \in \tilde{R}$, $\beta \in R$.

By Example 12.3, Definition 13.1 agrees with the notion of "polypositroid in H_0" of Definition 3.8 with the choices of Example 10.8. If W and c are understood to be fixed, we may simply use the name "polypositroid." Let us hasten to point out that the notion of a (W, c)-polypositroid *does not* depend on a choice of R^+.

Remark 13.2. Suppose c and c' are two Coxeter elements. Then there exists $w \in W$ so that $c' = wcw^{-1}$. We have that P is a (W, c')-polypositroid if and only if $w \cdot P$ is a (W, c)-polypositroid.

Remark 13.3. The notion of (W, c)-polypositroid is unchanged if the root and coroot vectors are dilated by scalars. In particular, the notion of (W, c)-polypositroid is identical for a root system R and its dual $R^\vee$.

14. Coxeter necklaces

Consider the cone

$$A := \mathbb{R}_{\geq 0}\langle \beta_1^\vee, \beta_2^\vee, \ldots, \beta_r^\vee \rangle.$$

Since for $i = 1, 2, \ldots, r$ we have $\beta_i^\vee \in R^+$ and $c^{-1}(\beta_i^\vee) \in R^-$, we have $A \subseteq C \cap (-c(C))$. But, in general, these two cones are not equal to each other. Note also that A depends only on c and R^+, and not on the choice of reduced factorization of c (though the enumeration of the set $\{\beta_1^\vee, \beta_2^\vee, \ldots, \beta_r^\vee\}$ does depend on the reduced factorization).

Definition 14.1. A (W, R^+, c)-*Coxeter necklace* is a sequence $(v_0, v_1, \ldots, v_{h-1})$ of points in V, such that for any $i = 1, \ldots, h$,

$$v_i - v_{i-1} \in c^i(A).$$

Here, we set $v_{i+h} := v_i$ and thus, $v_h := v_0$.

With the choices made in Example 10.8, Definition 14.1 agrees with the notion of Coxeter necklace in Definition 6.1.

Proposition 14.2. *Suppose that* $\mathbf{v} = (v_0, v_1, \ldots, v_{h-1})$ *is a* (W, R^+, c)*-Coxeter necklace. Then each point* v_i *is a vertex of the* (W, c)*-twisted alcoved polytope*

$$Q(\mathbf{v}) := \bigcap_{i=0}^{h-1} (v_i + c^i(C)).$$

In particular, $\mathrm{env}_c(\mathbf{v}) = Q(\mathbf{v})$.

Proof. It suffices to show that $v_i \in Q(\mathbf{v})$, or equivalently, $v_j - v_i \in c^i(C)$, for any $i, j \in \{0, \ldots, h-1\}$. Rotating by a power of c, it is enough to show that $v_j - v_0 \in C$. Equivalently, we need to show that $(v_j - v_0, \omega_k) \geq 0$, for any j and any k.

We have $v_j - v_0 = (v_1 - v_0) + (v_2 - v_1) + \cdots + (v_j - v_{j-1})$. The vector $v_{m+1} - v_m$ is a nonnegative linear combination of the coroots $c^m \beta_i^\vee$, for $i = 1, \ldots, r$. According to Lemma 10.6, if $j \leq M(k^\star)$, then $(\beta^\vee, \omega_k) \geq 0$, for all coroots $\beta^\vee$ involved in the expression of $v_{m+1} - v_m$, for $m = j-1, j-2, \ldots, 0$. This implies that $(v_j - v_0, \omega_k) \geq 0$.

On the other hand, if $j \geq M(k^\star)$, we can express $v_j - v_0$ in a different way as $v_j - v_0 = v_j - v_h = -((v_{j+1} - v_j) + (v_{j+2} - v_{j+1}) + \cdots + (v_h - v_{h-1}))$. Now, according to Lemma 10.6, we have $(\beta^\vee, \omega_k) \leq 0$, for any coroot $\beta^\vee$ involved in the expression of $v_{m+1} - v_m$, for $m = j, j+1, \ldots, h-1$. Thus, again, we get $(v_j - v_0, \omega_k) \geq 0$. □

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Lemma 14.3. *Let R_1^+ and R_2^+ be two choices of positive roots for R compatible with c. Then there is a bijection $\mathbf{v} \mapsto \mathbf{v}'$ between (W, R_1^+, c)-Coxeter necklaces and (W, R_2^+, c)-Coxeter necklaces, such that $Q(\mathbf{v}) = Q(\mathbf{v}')$.*

Proof. By Proposition 10.4, we may assume that $R_2^+ = s_1 \cdot R_1^+$, and $c = s_1 s_2 \cdots s_r$ is a reduced factorization in R_1^+. Letting $s_i' = s_1 s_i s_1$ be the simple generators of R_2^+, we have $c = s_2' \cdots s_r' s_1'$. Let $\beta_1, \beta_2, \ldots$ be the enumeration of R associated to the factorization $c = s_1 s_2 \cdots s_r$. Then the enumeration of R associated to $c = s_2' \cdots s_r' s_1'$ is $\beta_2, \beta_3, \ldots$.

Let $v_i' = v_i + [\beta_{ir+1}^\vee](v_{i+1} - v_i)$, where $[\beta_{ir+1}^\vee](v_{i+1} - v_i)$ denotes the coefficient of $\beta_{ir+1}^\vee$ in the vector $v_{i+1} - v_i$ which lies in the cone $c^i(A)$ spanned by $\beta_{ir+1}^\vee, \beta_{ir+2}^\vee, \ldots, \beta_{(i+1)r}^\vee$. It is clear that $\mathbf{v} \mapsto \mathbf{v}'$ is a bijection from (W, R_1^+, c)-Coxeter necklaces to (W, R_2^+, c)-Coxeter necklaces. We claim that $Q(\mathbf{v}) = Q(\mathbf{v}')$. It suffices to show that each v_j' belongs to $Q(\mathbf{v})$, and we can reduce to showing that $v_j' - v_0 = (v_j - v_0) + a\beta_{jr+1}^\vee \in A$, where $a \in \mathbb{R}_{\geq 0}$. The proof is identical to that of Proposition 14.2, using Lemma 10.6. □

Lemma 14.4. *Let c and $c' = wcw^{-1}$ be two Coxeter elements. Then $\mathbf{v}$ is a (W, R^+, c)-Coxeter necklace if and only if $w \cdot \mathbf{v}$ is a $(W, w \cdot R^+, c')$-Coxeter necklace. We have $w \cdot Q(\mathbf{v}) = Q(w \cdot \mathbf{v})$, where $Q(\mathbf{v})$ denotes a (W, c)-twisted alcoved polytope and $Q(w \cdot v)$ denotes a (W, c')-twisted alcoved polytope.*

Proof. Clear from the definitions. □

We call a Coxeter necklace $\mathbf{v} = (v_0, v_1, \ldots, v_{h-1})$ *generic* if each difference $v_i - v_{i-1}$ belongs to the interior of the cone $c^i(A)$.

Lemma 14.5. *Suppose that $\mathbf{v}$ is a generic (W, R^+, c)-Coxeter necklace. Then for each twisted root ω, the face $\{x \in Q(\mathbf{v}) \mid (x, \omega) = 0\}$ contains at most one of the vertices $v_0, v_1, \ldots, v_{h-1}$. In particular, $Q(\mathbf{v})$ is a generic (W, c)-twisted alcoved polytope.*

Proof. By acting with c, we may assume that $\omega = \omega_k$, where $k \in \{1, 2, \ldots, r\}$. In the proof of Proposition 14.2, we note that we have the strict inequalities $(\beta_k^\vee, \omega_k) > 0$ and $(\beta_{k-r}^\vee, \omega_k) < 0$. □

Whereas Theorem 11.2 describes a generalized W-permutohedron as an intersection of $|W|$ cones, our next result describes a (W, c)-polypositroid as an intersection of h cones.

Proposition 14.6. *Fix a Coxeter element c. Suppose P is a W-generalized permutohedron. Then for any choice of R^+, we have*

$$\mathrm{env}_c(P) = Q(\mathbf{v}) = \bigcap_{i=0}^{h-1} (v_i + c^i(C)),$$

where $\mathbf{v} = (v_0, \ldots, v_{h-1})$ is a (W, R^+, c)-Coxeter necklace. In particular, this holds for P a (W, c)-polypositroid, in which case, $\mathrm{env}_c(P) = P$.

Proof. By Proposition 12.6, any (W, c)-twisted alcoved polytope has the form $P = \bigcap_{i=0}^{h-1} (v_i + c^i(C))$ for some uniquely determined points v_i, and thus this holds for $\mathrm{env}_c(P)$.

Since P is also a W-generalized permutohedron, by Corollary 11.3, we must have $v_i = v_{c^i}$, the vertex of P that is the minimum in $c^i(C)$-dominance order. Let us show that the conditions $v_i - v_{i-1} \in c^i(A)$ hold. It is enough to show this for $i = 0$ (the general case is obtained by the action of c^i). Consider the following sequence of vertices that connect $v_0 = v_{id}$ with $v_1 = v_c$:

$$v_{id},\ v_{s_1},\ v_{s_1 s_2},\ \ldots,\ v_{s_1 s_2 \cdots s_r} = v_c.$$

According to Theorem 11.4, $v_{s_1 s_2 \cdots s_i} - v_{s_1 s_2 \cdots s_{i-1}} = a_i\, s_1 \cdots s_{i-1}(\alpha_i^\vee)$, for $a_i \in \mathbb{R}_{\geq 0}$. Thus, $v_1 - v_0 = \sum_{i=1}^r a_i\, \beta_i^\vee \in A$. We conclude that $\mathbf{v} = (v_0, v_1, \ldots, v_{h-1})$ is a Coxeter necklace, as required. □

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15. Balanced arrays

Definition 15.1. A *W-balanced array* is a collection $(m_\alpha)_{\alpha \in R}$ of nonnegative real numbers satisfying the equality

$$\sum_{\alpha} m_\alpha \alpha^\vee = 0. \tag{15.1}$$

A *W-balanced pair* is a pair $((m_\alpha), z)$ consisting of a W-balanced array (m_α) and a vector $z \in V$.

Now let $c = s_1 s_2 \cdots s_r$ be a standard Coxeter element with respect to R^+ and $\beta_1, \beta_2, \ldots,$ be the corresponding ordering of roots. We define a *Coxeter necklace*

$$\mathbf{v}((m_\alpha), z) = (v_0, v_1, \ldots, v_{h-1}, v_h = v_0)$$

by setting $v_0 = z$ and

$$v_i = v_{i-1} + \sum_{k=(i-1)r+1}^{ir} m_{\beta_k} \beta_k^\vee \tag{15.2}$$

for $i = 1, 2, \ldots, h-1$. The equality $v_h = v_0$ follows from (15.1) and Proposition 10.3.

Proposition 15.2. *The map $((m_\alpha), z) \mapsto \mathbf{v}((m_\alpha), z)$ is a bijection between W-balanced pairs and (W, R^+, c)-Coxeter necklaces, for any choice of (R^+, c).*

Proof. In (15.2), the m_β can be recovered from $\mathbf{v}((m_\alpha), z)$ because $\{\beta^\vee_{(i-1)r+1}, \ldots, \beta^\vee_{ir}\}$ is a basis. The result easily follows. □

Proposition 15.3. *Fix a W-balanced pair $((m_\alpha), z)$. Then the (W, R^+, c)-Coxeter necklace $\mathbf{v}((m_\alpha), z)$ depends on (R^+, c) and not on the reduced word of c.*

Proof. Changing the reduced word of c replaces $\{\beta_1, \ldots, \beta_r\}$ by a permutation of the same set. □

For a balanced pair $((m_\alpha), z)$, we also define a *Coxeter belt*

$$\mathbf{u}((m_\alpha), z) := (u_0, u_1, \ldots, u_{hr-1}, u_{hr} = u_0)$$

by setting $u_0 = z$ and

$$u_i = u_{i-1} + m_{\beta_i} \beta_i^\vee$$

for $i = 1, 2, \ldots, hr-1$. Note that given $((m_\alpha), z)$, the Coxeter belt depends on R^+, on c, and on a reduced word for c.

Suppose R^+ has simple roots $\alpha_1, \ldots, \alpha_r$. Then $s_1 R^+$ is also a positive system, and its simple roots are $-\alpha_1, s_1\alpha_2, \ldots, s_1\alpha_r$.

Proposition 15.4. *The (W, R^+, c)-Coxeter belt for the W-balanced pair $((m_\alpha), z)$ with respect to $(R^+, c = s_1 s_2 \cdots s_r)$ is the same, up to a cyclic shift, as the $(W, s_1 R^+, c)$-Coxeter belt for the balanced pair $((m_\alpha), z' = u_1)$ with respect to $(s_1 R^+, c = (s_1 s_2 s_1) \cdots (s_1 s_r s_1) s_1)$.*

Proof. Follows from Proposition 10.4. □

Given a balanced pair $((m_\alpha), z)$, let $Q((m_\alpha), z) := Q(\mathbf{v}((m_\alpha), z))$.

Corollary 15.5. *Let $((m_\alpha), z)$ be a balanced pair. Then $\mathrm{env}_c(\mathbf{u}((m_\alpha), z)) = Q((m_\alpha), z)$, and each point of the Coxeter belt $\mathbf{u}((m_\alpha), z)$ is a vertex of $Q((m_\alpha), z)$.*

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Proof. By Proposition 14.2, we have $\mathrm{env}_c(\mathbf{v}((m_\alpha), z)) = Q((m_\alpha), z)$. Since the Coxeter necklace $\mathbf{v}((m_\alpha), z)$ is a subset of the Coxeter belt $\mathbf{u}((m_\alpha), z)$, we have $\mathrm{env}_c(\mathbf{u}((m_\alpha), z)) \supseteq \mathrm{env}_c(\mathbf{v}((m_\alpha), z))$. To establish equality, it suffices to show that each u_i belongs to $Q((m_\alpha), z)$. By combining the action of c with Proposition 15.4, it suffices to show that $u_1 \in Q(\mathbf{v}((m_\alpha), z))$. This follows from Lemma 14.3.

The claim that every point on the Coxeter belt is a vertex of $Q((m_\alpha), z)$ follows from applying Proposition 14.2 to the (W, s_1R^+, c)-Coxeter necklace appearing in Proposition 15.4. □

Definition 15.6. A (W, c)-*balancedtope* is a polytope of the form $Q((m_\alpha), z)$.

By Propositions 15.3 and 15.4 and Corollary 15.5, up to changing z (or equivalently, up to a translation), the (W, c)-balancedtope $Q((m_\alpha), z)$ does not depend on the choice of reduced word of c, or on the choice of R^+.

16. Prepolypositroids

Let $\mathbb{R}^{\tilde{R}} = \{(a_\omega)_{\omega \in \tilde{R}}\}$ denote the vector space whose coordinates are labeled by the set $\tilde{R}$ of twisted roots. Let $\mathcal{C}_{\mathrm{pre}}^{W,c} \subset \mathbb{R}^{\tilde{R}}$ denote the cone cut out by the inequalities

$$a_{c^{m-1}\omega_k} + a_{c^m\omega_k} \geq \sum_{k \to i} -A_{ik} a_{c^m\omega_i} + \sum_{i \to k} -A_{ik} a_{c^{m-1}\omega_i} \tag{16.1}$$

for $k \in \mathbb{Z}$ and $1 \leq m \leq r$. Recall that $j \to i$ if i and j are connected in the Dynkin diagram and i occurs before j in all reduced words of c. The twisted roots appearing in (16.1) all belong to the set $\{c^{m-1}\omega_k = \tilde{\beta}_{k+(m-1)r}, \tilde{\beta}_{k+(m-1)r+1}, \ldots, \tilde{\beta}_{k+mr} = c^m\omega_k\}$.

Proposition 16.1. *The inequalities (16.1) depend on c and not on the choice of reduced word of c, or on the choice of* R^+.

Proof. That the inequalities (16.1) do not depend on the reduced word of c is apparent. For the second part, using Proposition 10.4, we need to check what happens if we replace R^+ by s_1R^+, where $c = s_1s_2\cdots s_r$. If $j \to_{c,R^+} i$, then we also have $j \to_{c,s_1R^+} i$ unless one of i, j is equal to 1, in which case, the relation reverses. This is exactly the required condition for the inequalities (16.1) to be preserved when R^+ is changed to s_1R^+. □

Example 16.2. Let us take (W, R^+, c), as in Examples 10.8 and 12.3. In the notation of Part I, we have $a_{ij} = a_{h_i - h_j}$. The inequalities (16.1) are of three types: (1) $a_{i,i+1} + a_{i+1,i+2} \geq a_{i,i+2}$, (2) $a_{i+1,i} + a_{i+2,i+1} \geq a_{i+2,i}$, and (3) $a_{i,j} + a_{i+1,j+1} \geq a_{i,j+1} + a_{j,j+1}$, where $i, i+1, j, j+1$ are distinct. The inequalities (1) and (2) are special cases of the triangle inequality (4.2), while (3) is a special case of (4.3). It will follow from Theorem 17.3, and can be verified directly, that the inequalities (4.2) and (4.3) are consequences of the smaller set of inequalities (16.1). Indeed, the $n(n-1) = |R|$ inequalities in (16.1) are exactly the facet inequalities appearing in Corollary 6.13.

Example 16.3. Let us take (W, R^+, c) as in Example 10.9. Using Proposition 10.5, we have $\omega_1 = e_1, \omega_2 = e_1 + e_2, \ldots, \omega_{r-1} = e_1 + e_2 + \cdots + e_{r-1}$, and $\omega_r = \frac{1}{2}(e_1 + e_2 + \cdots + e_r)$. The inequalities (16.1) are:

$$\begin{aligned}
a_{c^{m-1}\omega_1} + a_{c^m\omega_1} &\geq a_{c^{m-1}\omega_2} \\
a_{c^{m-1}\omega_i} + a_{c^m\omega_i} &\geq a_{c^m\omega_{i-1}} + a_{c^{m-1}\omega_{i+1}} \qquad \text{for } i = 2, \ldots, r-2 \\
a_{c^{m-1}\omega_{r-1}} + a_{c^m\omega_{r-1}} &\geq a_{c^m\omega_{r-2}} + a_{c^{m-1}\omega_r} \\
a_{c^{m-1}\omega_r} + a_{c^m\omega_r} &\geq 2a_{c^m\omega_{r-1}}.
\end{aligned}$$

Definition 16.4. A (W, c)-*prepolypositroid* is a (W, c)-twisted alcoved polytope cut out by the halfspaces $(x, \omega) \leq a_\omega$, $\omega \in \tilde{R}$, where $a_\omega \in \mathcal{C}_{\mathrm{pre}}^{W,c}$.

We call $\mathcal{C}_{\mathrm{pre}}^{W,c}$ the *cone of* (W, c)-*prepolypositroids*.

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Theorem 16.5. *There are natural isomorphisms between the following three cones:*

1. *the cone of (W, c)-prepolypositroids.*
2. *the cone of W-balanced pairs $((m_\alpha), z)$;*
3. *the cone of (W, R^+, c)-Coxeter necklaces for any choice of R^+.*

Furthermore, if a (W, c)-prepolypositroid P arises from $(a_\omega) \in \mathcal{C}_{\mathrm{pre}}^{W,c}$, then each $a_\omega \in \mathbb{R}$ is minimal, that is, is a value of the support function of P.

Proposition 15.2 gives the isomorphism between (2) and (3). For the remainder of this section, our aim is to show that $((m_\alpha), z) \mapsto Q((m_\alpha), z)$ is a bijection between balanced pairs and (W, c)-prepolypositroids.

Proposition 16.6. *We have*

$$c^{m-1}\omega_k + c^m\omega_k = \sum_{k \to i} -A_{ik}c^m\omega_i + \sum_{i \to k} -A_{ik}c^{m-1}\omega_i \tag{16.2}$$

for any k and any m.

Proof. By Proposition 16.1, to verify the claim, we can assume that $c = s_1 s_2 \cdots s_r$ and verify the equality

$$\omega_1 + c\omega_1 = \sum_j -A_{j1}\omega_j,$$

where the sum is over all j connected to 1 in the Dynkin diagram. The left hand side is equal to $2\omega_1 - \alpha_1$, and we check that

$$(\alpha_j^\vee, 2\omega_1 - \alpha_1) = \begin{cases} -A_{j1} & \text{if } j \text{ is connected to } i \text{ in the Dynkin diagram} \\ 0 & \text{otherwise.} \end{cases}$$

□

Given a (W, c)-balancedtope $Q((m_\alpha), z)$, we define a collection (a_ω) of real numbers, one for each twisted root $\omega \in \tilde{R}$ by

$$a_\omega = a_\omega(Q((m_\alpha), z)) := \max((x, \omega) \mid x \in Q((m_\alpha), z)). \tag{16.3}$$

Proposition 16.7. *For any (W, c)-balancedtope $Q((m_\alpha), z)$, the collection (a_ω) satisfies the inequalities (16.1).*

Proof. It follows from Lemma 10.7 that the inequalities (16.1) are equivalent to the set of "negated" inequalities

$$a_{-c^{m-1}\omega_k} + a_{-c^m\omega_k} \geq \sum_{k \to i} -A_{ik}a_{-c^m\omega_i} + \sum_{i \to k} -A_{ik}a_{-c^{m-1}\omega_i} \tag{16.4}$$

for all k and m. By Proposition 16.1, to verify the claim, we can assume that $c = s_1 s_2 \cdots s_r$ and verify just one of the inequalities (16.4), say

$$a_{-\omega_1} + a_{-c\omega_1} \geq \sum_j -A_{j1}a_{-\omega_j},$$

where the sum is over all j connected to 1 in the Dynkin diagram. By Proposition 16.6, the claim is translation invariant. Thus, we may assume that $v_0 = 0$, that is, $z = 0$.

The maximum of $-\omega_1, \ldots, -\omega_r$ on $Q((m_\alpha), z)$ occurs at the vertex v_0. The maximum of $-c\omega_1$ occurs at vertex v_1, where the value taken is greater than or equal to $(-c\omega_1, v_0) = 0$. The required inequality now follows from $a_{-c\omega_1} \geq 0$. □

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By Proposition 16.7, we have an injective and linear map $((m_\alpha), z) \mapsto (a_\omega)$ from the cone of W-balanced pairs to the cone $\mathcal{C}_{\mathrm{pre}}^{W,c}$ of (W,c)-prepolypositroids. The cone of balanced pairs has exactly $|R|$ facets, given by $m_\alpha = 0$, for $\alpha \in R$. Each equality $a_{-c^{m-1}\omega_k} + a_{-c^m\omega_k} = \sum_{k\to i} -A_{ik}a_{-c^m\omega_i} + \sum_{i\to k} -A_{ik}a_{-c^{m-1}\omega_i}$ defines a face of $\mathcal{C}_{\mathrm{pre}}^{W,c}$.

Lemma 16.8. *The map $((m_\alpha), z) \mapsto (a_\omega)$ sends the facet $\{m_\alpha = 0\}$ of the cone of W-balanced pairs to the face $\{a_{-c^{m-1}\omega_k} + a_{-c^m\omega_k} = \sum_{k\to i} -A_{ik}a_{-c^m\omega_i} + \sum_{i\to k} -A_{ik}a_{-c^{m-1}\omega_i}\}$ of $\mathcal{C}_{\mathrm{pre}}^{W,c}$, where α satisfies $\tilde{\alpha} = (I-c)^{-1}\alpha = c^{m-1}\omega_k$.*

Proof. In the end of the proof of Proposition 16.7, it suffices to note that $a_{-c\omega_1} = m_{\alpha_1}$. Indeed, for $i \in [2,r]$, we have

$$(\beta_i^\vee, c\omega_1) = (c^{-1}s_1s_2\cdots s_{i-1}\alpha_i^\vee, \omega_1) = (s_r\cdots s_i\alpha_i^\vee, \omega_1) = 0,$$

but $(\beta_1^\vee, c\omega_1) = (\alpha_1^\vee, \omega_1 - \alpha_1) = -1$. □

It follows that each equality $\{a_{-c^{m-1}\omega_k} + a_{-c^m\omega_k} = \sum_{k\to i} -A_{ik}a_{-c^m\omega_i} + \sum_{i\to k} -A_{ik}a_{-c^{m-1}\omega_i}\}$ defines a facet of $\mathcal{C}_{\mathrm{pre}}^{W,c}$.

Proof of Theorem 16.5. We compare (1) and (2). It follows from Lemma 16.8 that each equality $a_{-c^{m-1}\omega_k} + a_{-c^m\omega_k} = \sum_{k\to i} -A_{ik}a_{-c^m\omega_i} + \sum_{i\to k} -A_{ik}a_{-c^{m-1}\omega_i}$ defines a facet of the cone of (W,c)-prepolypositroids. It then follows from the same lemma that the map $((m_\alpha), z) \mapsto (a_\omega)$ is a linear isomorphism, completing the proof of the isomorphism between the three cones.

The last sentence of Theorem 16.5 follows from (16.3). □

17. From prepolypositroids to polypositroids

17.1. Alcoved envelope of generalized W-permutohedra

Theorem 17.1.

1. *The cone of generalized W-permutohedra $\mathcal{C}_{\mathrm{sub}}^W$ projects to the cone of (W,c)-prepolypositroids $\mathcal{C}_{\mathrm{pre}}^{W,c}$ by projecting the vector $(a_{w\omega_i}) \in \mathbb{R}^{W\cdot\{\omega_1,\ldots,\omega_r\}}$ of (11.1) to $(a_{c^m\omega_i}) \in \mathbb{R}^{\tilde{R}}$.*
2. *The (W,c)-twisted alcoved envelope of a generalized W-permutohedron is a (W,c)-prepolypositroid.*
3. *The vertices $(v_{\mathrm{id}}, v_c, v_{c^2}, \ldots, v_{c^{h-1}})$ of a generalized W-permutohedron are a (W,c)-Coxeter necklace.*

Proof. (3) was established in the proof of Proposition 14.6. (1) and (2) thus follow from Theorem 16.5. □

Conjecture 17.2. *The maps in Theorem 17.1(1) are surjective.*

17.2. Type A

Theorem 17.3. *Suppose R is of type A. Then every (W,c)-prepolypositroid is also a generalized W-permutohedron. Thus, the class of (W,c)-prepolypositroids is identical to the class of (W,c)-polypositroids.*

Proof. Let c be the Coxeter element of Example 10.8. Then by Theorems 16.5 and 6.12, the class of (W,c)-prepolypositroids is exactly the class of polypositroids. Thus, the result holds in this case. Now, let $c' = wcw^{-1}$ be an arbitrary Coxeter element. Since the class of generalized W-permutohedra is preserved under the action of W, the result holds by Remark 13.2. □

It follows from Theorem 17.3 that Conjecture 17.2 holds in type A.

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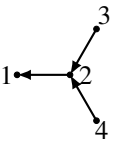
17.3. A prepolypositroid that is not a polypositroid

We give an example of a (W, c)-prepolypositroid that is not a (W, c)-polypositroid. Let $R = D_4$. We take as positive simple roots

$$\alpha_1 = (1, -1, 0, 0), \qquad \alpha_2 = (0, 1, -1, 0), \qquad \alpha_3 = (0, 0, 1, -1), \qquad \alpha_4 = (0, 0, 1, 1),$$

and let $c = s_1 s_2 s_3 s_4$, so that the Dynkin diagram is oriented



The ordering of the 24 roots of R is given by (here, $r = 4$ and $h = 6$)

$(1, -1, 0, 0)$,	$(1, 0, -1, 0)$,	$(1, 0, 0, -1)$,	$(1, 0, 0, 1)$,
$(0, 1, -1, 0)$,	$(1, 1, 0, 0)$,	$(0, 1, 0, 1)$,	$(0, 1, 0, -1)$,
$(1, 0, 1, 0)$,	$(0, 1, 1, 0)$,	$(0, 0, 1, -1)$,	$(0, 0, 1, 1)$,
$(-1, 1, 0, 0)$,	$(-1, 0, 1, 0)$,	$(-1, 0, 0, 1)$,	$(-1, 0, 0, -1)$,
$(0, -1, 1, 0)$,	$(-1, -1, 0, 0)$,	$(0, -1, 0, -1)$,	$(0, -1, 0, 1)$,
$(-1, 0, -1, 0)$,	$(0, -1, -1, 0)$,	$(0, 0, -1, 1)$,	$(0, 0, -1, -1)$,

and we see that $M(k) = 3$ for $k = 1, 2, 3, 4$. The twisted roots $\tilde{R}$ are, in the same order,

$(1, 0, 0, 0)$,	$(1, 1, 0, 0)$,	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$,	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$,
$(0, 1, 0, 0)$,	$(0, 1, 1, 0)$,	$(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$,	$(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$,
$(0, 0, 1, 0)$,	$(-1, 0, 1, 0)$,	$(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$,	$(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$,
$(-1, 0, 0, 0)$,	$(-1, -1, 0, 0)$,	$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$,	$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$,
$(0, -1, 0, 0)$,	$(0, -1, -1, 0)$,	$(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$,	$(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$,
$(0, 0, -1, 0)$,	$(1, 0, -1, 0)$,	$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$,	$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$.

We consider the Coxeter necklace

$$\mathbf{v} = (v_0 = (0, 0, 0, 0), v_1 = v_2 = (1, 0, 0, 1), v_3 = v_4 = (1, 1, 1, 1), v_5 = (0, 0, 1, 1)).$$

The polytope $Q(\mathbf{v})$ is a (W, c)-prepolypositroid, and the a_ω are given by

$$(1, 2, 1, 2, 1, 2, 1, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0)$$

in the same order as $\tilde{R}$. The inequalities (16.1) can be verified directly. For example, we have

$$2 + 2 = a_{(1,1,0,0)} + a_{(0,1,1,0)} \geq a_{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})} + a_{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})} + a_{(0,1,0,0)} \geq 1 + 2 + 1.$$

$$1 + 1 = a_{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})} + a_{(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})} \geq a_{(0,1,1,0)} = 2.$$

Now, one computes (for example, by [GJ]) that $Q(\mathbf{v})$ has seven vertices

$$\{(0, 0, 1, 1), (1, 0, 1, 2), (1, 0, 0, 1), (1, 1, 1, 1), (1, 0, 1, 0), (0, 0, 0, 0), (0, 1, 0, 1)\}$$

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and that there is an edge connecting $(1,0,1,2)$ and $(1,0,1,0)$. Indeed, one can check that this edge is the intersection of the three facets indexed by $(1,0,0,0)$, $(0,0,1,0)$, and $(0,-1,0,0)$. This edge is in the direction $(0,0,0,1)$, which is not a root direction. Thus, $Q(\mathbf{v})$ is not a generalized W-permutohedron.

It turns out that $Q(\mathbf{v})$ **is** the (W,c)-twisted alcoved envelope of a generalized permutohedron, namely, one with vertices

$$\{(0,0,1,1),(1,0,0,1),(1,1,1,1),(1,0,1,0),(0,0,0,0),(0,1,0,1)\},$$

consistent with Conjecture 17.2.

18. Prepolypositroids and finite type cluster algebras

We briefly recall some basic facts concerning finite type cluster algebras, following [YZ]. Let $\mathcal{A}(W,R^+,c)$ denote the finite type cluster algebra with principal coefficients of type R and associated to a compatible pair (R^+,c). Usually, the choice of positive system R^+ is not made explicit in the theory of cluster algebras, but for our purposes, it is necessary.

The cluster algebra $\mathcal{A}(W,R^+,c)$ is a commutative subring of the field of rational functions $\mathbb{C}(x_1,x_2,\ldots,x_r,y_1,y_2,\ldots,y_r)$, where x_i (respectively, y_i) are called *initial mutable cluster variables* (respectively, *coefficient variables*). The choice of c determines an initial exchange matrix B, given by the formula [YZ, (1.4)]. The cluster algebra $\mathcal{A}(W,R^+,c)$ contains a distinguished set of *cluster variables*, and associated to each cluster variable is a *g-vector* which belongs to V.

Theorem 18.1 [YZ, Theorems 1.4 and 1.10]. *The cluster variables*[2] $x_{\tilde{\beta}}$ *of* $\mathcal{A}(W,R^+,c)$ *are labeled by the set*

$$\Pi(c) := \{c^m\omega_i \mid i=1,2,\ldots,r \text{ and } 0\le m\le M(i)\},$$

where $M(i)$ *is defined in Proposition 10.3. Furthermore,* $\tilde{\beta}$ *is the g-vector of* $x_{\tilde{\beta}}$.

It follows from Lemma 10.7 that $\pm\omega_i\in\Pi(c)$. The cluster variables $x_{\tilde{\beta}}$ are arranged into *clusters*. We say that $\tilde{\beta}$ and $\tilde{\gamma}$ (or $x_{\tilde{\beta}}$ and $x_{\tilde{\gamma}}$) are (R^+,c)-*compatible*, or simply *compatible*, if they belong to the same cluster; the clusters are exactly the collections of r pairwise-compatible cluster variables. The (W,R^+,c)-*cluster fan* (also called the c-*Cambrian fan*) is the complete fan in V with cones $\mathrm{span}_{\ge 0}(\tilde{\gamma}_1,\tilde{\gamma}_2,\ldots,\tilde{\gamma}_s)$, where $\{\tilde{\gamma}_1,\tilde{\gamma}_2,\ldots,\tilde{\gamma}_s\}\subset\Pi(c)$ is a set of pairwise (R^+,c)-compatible vectors. A polytope with normal fan equal to the (W,R^+,c)-cluster fan is called a (W,R^+,c)-*generalized associahedron*. The following result combines work of Hohlweg et al. [HLT], Reading and Speyer [RS], and Yang and Zelevinsky [YZ, Remark 6.1].

Theorem 18.2. *The* (W,R^+,c)*-cluster fan is a refinement of the* W*-Coxeter fan. Furthermore,* (W,R^+,c)*-generalized associahedra exist and are generalized* W*-permutohedra.*

The W-Coxeter fan has maximal cones C_w labeled by Weyl group elements $w\in W$. By Theorem 18.2, every maximal cone of the (W,R^+,c)-cluster fan is a union of the cones C_w. A (W,R^+,c)-*singleton* [HLT] is a Weyl group element w, such that C_w is itself a maximal cone of the (W,R^+,c)-cluster fan. Hohlweg et al. [HLT, Theorem 1.2] characterize the set of (W,R^+,c)-singletons as prefixes (up to commutation relations) of a particular reduced word of w_0 that depends on R^+ and c.

The following result should be compared with Corollary 8.23.

Proposition 18.3. *Let* P *be a generic simple* (W,c)*-polypositroid. For any choice of* R^+*, and any* (W,R^+,c)*-singleton* w*, the cone* C_w *is a maximal cone of the normal fan* $\mathcal{N}(P)$.

Proof. By definition, the normal fan $\mathcal{N}(P)$ is a coarsening of the W-Coxeter fan. It follows that there exists a maximal cone C of $\mathcal{N}(P)$ that contains the simplicial cone C_w. But each generating ray of C_w

[2]The cluster variables in [YZ] are denoted $x_{\gamma;c}$. We have suppressed the dependence on c in our notation and use $\tilde{\beta}$ instead of γ.

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must be a generating ray of C, for otherwise, P would not be generic. But then we must have $C = C_w$, since both are simplicial cones. □

The cluster variables are related by *exchange relations*. A distinguished subset of the exchange relations are called *primitive exchange relations* in [YZ].

Theorem 18.4 [YZ, Theorem 1.5]. *The primitive exchange relations of $\mathcal{A}(W, R^+, c)$ are of the form*

$$x_{-\omega_k} x_{\omega_k} = y_k \prod_{k \to i} x_{\omega_i}^{-A_{ik}} \prod_{i \to k} x_{-\omega_i}^{-A_{ik}} + 1; \tag{18.1}$$

for $k = 1, 2, \ldots, r$ and

$$x_{c^{m-1}\omega_k} x_{c^m \omega_k} = \prod_{k \to i} x_{c^m \omega_i}^{-A_{ik}} \prod_{i \to k} x_{c^{m-1}\omega_i}^{-A_{ik}} + Y, \tag{18.2}$$

for $k = 1, 2, \ldots, r$ and $1 \le m \le M(k)$, where Y is some monomial in the y_i.

The relations (18.1) and (18.2) are homogeneous with respect to the g-vector grading. In particular, we have for each $k = 1, 2, \ldots, r$,

$$-\omega_k + \omega_k = 0; \tag{18.3}$$

and for each $1 \le m \le M(k)$,

$$c^{m-1}\omega_k + c^m \omega_k = \sum_{k \to i} -A_{ik} c^m \omega_i + \sum_{i \to k} -A_{ik} c^{m-1} \omega_i. \tag{18.4}$$

The latter we recognize as a special case of Proposition 16.7.

More generally, we say that $\tilde{\beta}, \tilde{\gamma} \in \Gamma(c)$ are an *exchangeable pair* if we have a (necessarily unique) exchange relation that exchanges $x_{\tilde{\beta}}$ for $x_{\tilde{\gamma}}$. This exchange relation takes the form

$$x_{\tilde{\beta}} x_{\tilde{\gamma}} = \prod_{\tilde{\delta} \in E(\tilde{\beta}, \tilde{\gamma})} x_{\tilde{\delta}}^{c_{\beta,\gamma;\delta}} + \text{ other monomial}, \tag{18.5}$$

where $c_{\beta,\gamma;\delta} > 0$, and $E(\beta, \gamma) \subset \Pi(c)$ consists of elements that are pairwise compatible, and compatible with both $\tilde{\beta}$ and $\tilde{\gamma}$. Here, the key point is that one of the two monomials on the RHS of the exchange relation (18.5) does not involve any of the coefficient variables, known as *sign-coherence*. For a description of all the exchange relations in a principal coefficient finite-type cluster algebra, see [ST]. Note that in type A, any incompatible pair $(\tilde{\beta}, \tilde{\gamma})$ is automatically exchangeable, but this is not the case in general type.

The following is the main result of this section.

Theorem 18.5. *Let P be a (W, c)-prepolypositroid defined by the inequalities $(x, \omega) \le a_\omega$, where $(a_\omega) \in \mathcal{C}_{\text{pre}}^{W,c}$. Then for any choice of R^+ compatible with c, and any pair $(x_{\tilde{\beta}}, x_{\tilde{\gamma}})$ of exchangeable cluster variables, we have*

$$a_{\tilde{\beta}} + a_{\tilde{\gamma}} \ge \sum_{\tilde{\delta} \in E(\tilde{\beta}, \tilde{\gamma})} c_{\beta,\gamma;\delta} a_{\tilde{\delta}}, \tag{18.6}$$

where $E(\tilde{\beta}, \tilde{\gamma})$ and $c_{\beta,\gamma;\delta}$ are defined in (18.5).

Proof. We begin by noting that (18.5) is homogeneous with respect to the g-vector grading, so

$$\tilde{\beta} + \tilde{\gamma} = \sum_{\tilde{\delta} \in E(\tilde{\beta}, \tilde{\gamma})} c_{\beta,\gamma;\delta} \tilde{\delta} \tag{18.7}$$

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(see [ST, P^4]). In [P^4, Proposition 2.22], it is shown that any linear dependence (18.7) is a positive sum of linear dependencies of the form (18.4). Replacing a linear dependence (18.4) by the corresponding inequality (16.1), we deduce that the inequality (18.6) is a positive sum of the inequalities (16.1). □

Example 18.6. Pick (W, R^+, c) as in Example 10.8. Then the inequalities (18.7) are all of the form (4.2) or (4.3).

The following is a variant of the noncrossing condition of Lemma 8.2.

Corollary 18.7. *Let P be a generic simple (W, c)-prepolypositroid and F be a face of P. Let $S(F) \subset \tilde{R}$ be as defined in Section 12.2. Then for any choice of R^+ compatible with c, and any exchangeable pair $(\tilde{\beta}, \gamma) \in \Gamma(c)$, we have that $S(F)$ can contain at most one of $\tilde{\beta}$ and $\tilde{\gamma}$.*

Proof. Suppose we have an exchangeable pair $\tilde{\beta}, \tilde{\gamma} \in S(F)$. Let $x \in F$. Then, by (18.7), we have

$$a_{\tilde{\beta}} + a_{\tilde{\gamma}} = (x, \tilde{\beta} + \tilde{\gamma}) = (x, \sum_{\tilde{\delta} \in E(\tilde{\beta}, \tilde{\gamma})} c_{\beta,\gamma;\delta} \tilde{\delta}) \leq \sum_{\tilde{\delta} \in E(\tilde{\beta}, \tilde{\gamma})} c_{\beta,\gamma;\delta} a_{\tilde{\delta}}.$$

By (18.6), we must have equality, giving $\tilde{\delta} \in S(F)$ for $\tilde{\delta} \in E(\tilde{\beta}, \tilde{\gamma})$. But then $S(F)$ contains twisted roots that are not linearly independent, contradicting the assumption that P is generic simple. □

Corollary 18.7 has the following defect: while (W, c)-prepolypositroids depend only on the choice of c, the notion of an exchangeable pair $(\tilde{\beta}, \tilde{\gamma})$ depends additionally on a choice of R^+. We thus pose the following question:

Question 18.8. Which pairs of c-twisted roots are exchangeable for *some* choice of R^+ compatible with c?

Corollary 18.9. *Let P be a generic simple (W, c)-prepolypositroid, and R^+ be a positive system compatible with c. Then removing the facets $(x, \omega) \leq a_\omega$ indexed by facet normals $\omega \notin \Pi(c)$ gives a (W, R^+, c)-generalized associahedron.*

Proof. Removing the stated facets gives the polytope cut out by the inequalities $(x, \omega) \leq a_\omega$ for $\omega \in \Pi(c)$. These a_ω satisfy

$$a_{c^{m-1}\omega_k} + a_{c^m \omega_k} > \sum_{k \to i} -A_{ik} a_{c^m \omega_i} + \sum_{i \to k} -A_{ik} a_{c^{m-1}\omega_i}, \tag{18.8}$$

for each $1 \leq m \leq M(k)$. According to [P^4, Theorem 2.23], these inequalities cut out the *deformation cone* of the (W, R^+, c)-generalized associahedron. In other words, the inequalities $(x, \omega) \leq a_\omega$, $\omega \in \Pi(c)$ define a (W, R^+, c)-generalized associahedron. □

On the other hand, not every maximal cone in the normal fan of a generic simple (W, c)-prepolypositroid P is a maximal cone in some (W, R^+, c)-cluster fan, as the following example shows.

Example 18.10. We continue the example from Section 17.3. By slightly perturbing the W-balanced pair associated to $\mathbf{v}$, we obtain a generic simple (W, c)-prepolypositroid $Q(\mathbf{v}')$ whose normal fan is a refinement of that of $Q(\mathbf{v})$ but not a refinement of the W-Coxeter fan. There is a vertex v of $Q(\mathbf{v}')$ with $S(v)$ given by

$$S(v) = \{(1, 0, 0, 0), (0, 0, 1, 0), (-\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}), (0, -1, 0, 0)\}.$$

The roots in $S(v)$ are pairwise c-noncrossing in the sense of Section 19.1. The dual cone is spanned by the vectors

$$\{(0, -1, 0, 1), (1, 0, 0, -1), (0, 0, 1, 1), (0, 0, 0, -2)\},$$

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the last of which is not in a direction of a root. Since the (W, R^+, c)-cluster fan is a refinement of the W-Coxeter fan (Theorem 18.2), the vertex cone C_v in the normal fan of P is not a maximal cone for any cluster fan associated to (W, c).

19. Normal fans of (W, c)-prepolypositroids

19.1. Coxeter noncrossing roots

Recall that a pair of distinct roots $(\beta, \gamma) \in R$ is said to be *alternating* if $(\beta, \gamma) = (\gamma, \beta) \geq 0$. Let us say that $(\beta, \gamma) \in R$ are *c-noncrossing* if either $(\gamma^\vee, \tilde{\beta}) = 0$ or $(\beta^\vee, \tilde{\gamma}) = 0$. We say that $(\tilde{\beta}, \tilde{\gamma}) \in \tilde{R}$ are alternating (respectively, c-noncrossing) if $(\beta, \gamma) \in R$ are. It is straightforward to see that with the choices in Example 10.8, "alternating" and "c-noncrossing" agrees with the corresponding notions in Part I.

Lemma 19.1. *Two roots β, γ are alternating (respectively, c-noncrossing) if and only if $c\beta, c\gamma$ are alternating (respectively, c-noncrossing).*

Lemma 19.2. *Let c be a Coxeter element and $c' = wcw^{-1}$. Then (β, β') are c-noncrossing if and only if $(w\beta, w\beta')$ are c'-noncrossing.*

Conjecture 19.3. *Let P be a generic simple (W, c)-prepolypositroid, and suppose that $(\tilde{\beta}, \tilde{\gamma}) \in \tilde{R}$ span a 2-dimensional face of the normal fan of P. Then (β, γ) must be alternating and c-noncrossing.*

By Lemma 12.7, the alternating part of Conjecture 19.3 holds. By Lemma 8.2, Conjecture 19.3 holds when R is of type A. We will show in Proposition 19.12 that the condition "alternating and c-noncrossing" is essentially the same as cluster compatibility. Thus, Conjecture 19.3 is consistent with Corollary 18.7, since exchangeable pairs of cluster variables are incompatible (and the converse holds in type A).

Remark 19.4. The notion of c-noncrossing depends only on the choice of c, and not of R^+. Furthermore, (β, γ) is c-noncrossing if and only if $(-\beta, \gamma)$, $(\beta, -\gamma)$, and $(-\beta, -\gamma)$ are. This is consistent with our usage of "noncrossing" in type A for directed edges: two directed edges are noncrossing if the underlying undirected edges are.

19.2. Reflection factorizations

For $w \in W$, write $\ell_R(w)$ for the length of the shortest factorization of w into reflections $s_\gamma \in W$, $\gamma \in R$. We define a partial order $\leq_R$ on W by

$$u \leq v \qquad \Leftrightarrow \qquad \ell_R(v) = \ell_R(u) + \ell_R(vu^{-1}).$$

Note that ℓ_R and $\leq_R$ do not depend on the choice of R^+. It is wellknown that for any Coxeter element c, we have $\ell_R(c) = r$. We refer the reader to [Bes, BW] for general background on reflection factorizations and reflection order.

19.3. Bipartite positive systems

We say that (R^+, c) is *bipartite*, or c (respectively, R^+) is bipartite with respect to R^+ (respectively, c), if R^+ is compatible with c, and, in addition,

$$c = \tau_+\tau_-, \qquad \tau_+ = s_{i_1} \cdots s_{i_t}, \qquad \text{and} \qquad \tau_- = s_{i_{t+1}} \cdots s_{i_r}, \tag{19.1}$$

where the partition $I = \{1, 2, \ldots, r\} = I_+ \sqcup I_-$, with $I_+ := \{i_1, \ldots, i_t\}$ and $I_- := \{i_{t+1}, \ldots, i_r\}$ makes the Dynkin diagram bipartite.

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Lemma 19.5. *Let c be a fixed Coxeter element. Then a choice of R^+ bipartite with respect to c exists. There are exactly $2h$ bipartite $(R')^+$ with respect to c, and they are of the form $\tau_+\tau_-\tau_+\cdots\tau_-\tau_+R^+$ or $\tau_-\tau_+\cdots\tau_-\tau_+R^+$.*

Proof. There are exactly two orientations of the Dynkin diagram that correspond to bipartite (R^+, c). The claim thus follows from [KiTh, Theorem 3.6]. This theorem is stated for simply-laced root systems, but the statement and proof are valid also for multiply-laced Weyl groups. □

When (R^+, c) is bipartite, the ordering of Proposition 10.3 induces an ordering of R of the form $(A_1 \prec A_2 \prec \cdots \prec A_{2h} = A_0)$, where $|A_i| = t$ or $|A_i| = r - t$ depending on whether i is odd or i is even. The ordering within each A_i depends on the choice of a reduced word of c, but the sets A_i themselves do not. We have

$$R^+ = \bigsqcup_{i=1}^{h} A_i \qquad R^- = \bigsqcup_{i=h+1}^{2h} A_i \qquad R_{\geq -1} = \bigsqcup_{i=0}^{h+1} A_i.$$

By Lemma 19.5, the ordering of R corresponding to another bipartite $(R')^+$ is of the form $(A_k \prec A_{k+1} \prec A_{k+2} \prec \cdots \prec A_{k-1})$.

We say that $\beta, \gamma \in R$ are *c-opposed* if $\beta \in A_i$ and $\gamma \in A_{i+h}$ for some choice of bipartite R^+ with respect to c. The notion of c-opposed does not depend on the choice of bipartite R^+.

If R^+ and c are fixed, we write $\alpha \prec_{R^+,c} \alpha'$ (or simply $\alpha \prec \alpha'$) if α precedes α' in the ordering of R from Proposition 10.3.

Lemma 19.6 [BW, Lemma 3.9]. *Suppose that $\beta \prec_{R^+,c} \gamma$ are distinct positive roots. Then the following are equivalent:*

1. $s_\beta s_\gamma \leq_R c^{-1}$;
2. $(\beta^\vee, \tilde{\gamma}) = 0$.

Lemma 19.7 [BW, Lemma 5.6]. *Suppose that $\beta \prec_{R^+,c} \gamma$ are distinct positive roots. Then:*

1. *if $s_\beta s_\gamma \leq_R c^{-1}$, we have $(\beta, \gamma) \geq 0$;*
2. *if $s_\gamma s_\beta \leq_R c^{-1}$, we have $(\beta, \gamma) \leq 0$.*

Lemma 19.8. *Suppose that $\beta, \gamma \in R$ are c-opposed and $\beta \neq -\gamma$. Then:*

1. $(\beta, \gamma) = (\gamma^\vee, \tilde{\beta}) = (\beta^\vee, \tilde{\gamma}) = 0$;
2. $s_\beta s_\gamma = s_\gamma s_\beta \leq_R c^{-1}$.

Proof. We have $A_{i+h} = -A_i$. Suppose that β and γ are c-opposed. Then we may choose bipartite R^+ so that $\beta \in A_1$ and $\gamma \in A_{1+h}$. Then $\beta = \alpha_i$ and $\gamma = -\alpha_j$, where $i, j \in I_+$. Since i, j are not adjacent, $(\beta, \gamma) = 0$ follows. Also, $\tilde{\beta} = \omega_i$ and $\tilde{\gamma} = -\omega_j$, so (1) follows. (2) is also clear from (19.1). □

Proposition 19.9. *Let (R^+, c) be bipartite and $\beta \prec_{R^+,c} \gamma$ be distinct positive roots. Then $s_\beta s_\gamma \leq_R c^{-1}$ if and only if β, γ are alternating c-noncrossing.*

Proof. If $s_\beta s_\gamma \leq_R c^{-1}$, then by Lemma 19.7, we have $(\beta, \gamma) \geq 0$, and by Lemma 19.6, we have $(\beta^\vee, \tilde{\gamma}) = 0$. Thus, (β, γ) is alternating c-noncrossing.

Conversely, suppose that (β, γ) is alternating and c-noncrossing. If $(\beta^\vee, \tilde{\gamma}) = 0$, then by Lemma 19.6, we have $s_\beta s_\gamma \leq c^{-1}$, and we are done. Next, suppose that we have $(\gamma^\vee, \tilde{\beta}) = 0$. We claim that $(\beta, \gamma) = 0$. To see this, we assume that R^+ has been chosen so that $\beta \in A_1$ while $\gamma \in \bigcup_{i=1}^{h} A_i$. Thus, $\beta = \alpha_i$ and $\tilde{\beta} = \omega_i$. The condition $(\gamma^\vee, \omega_i) = 0$ implies that $(\gamma^\vee, \alpha_i) \leq 0$, and the alternating condition gives $(\gamma^\vee, \alpha_i) \geq 0$. Thus, $(\beta, \gamma) = 0$, establishing our claim.

We are thus in the situation that $(\gamma^\vee, \tilde{\beta}) = 0$ and $(\beta, \gamma) = 0$. If $\beta, \gamma \in A_i$ for some i (that is, they are close together in the ordering), then by Lemma 19.16, we have $\prod_{\delta \in A_i} s_\delta \prod_{\delta' \in A_{i+1}} s_{\delta'} = c^{-1}$, so we know that $s_\beta s_\gamma \leq c^{-1}$, and we are done. Now, if $\beta \in A_i$, then $-\beta \in A_{i+h}$. Thus, we may find a different

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positive system $(R')^+$, bipartite with respect to c, so that $\gamma \prec_{(R')^+,c} -\beta$, and $\gamma, -\beta \in (R')^+$. By Lemma 19.6, we have $s_\gamma s_\beta \leq c^{-1}$, and since $s_\gamma s_\beta = s_\beta s_\gamma$, we are done. □

Corollary 19.10. *Suppose that β, γ are distinct roots, such that $\beta \neq -\gamma$. Then the following are equivalent:*

1. *β and γ are c-noncrossing;*
2. *either $s_\beta s_\gamma \leq_R c^{-1}$ or $s_\gamma s_\beta \leq_R c^{-1}$.*

Proof. Suppose (1) holds. Replacing γ by $-\gamma$ does not change either of the conditions (1) or (2). Thus, we may assume that β, γ are alternating c-noncrossing. If β, γ are c-opposed, we apply Lemma 19.8. Otherwise, we pick any bipartite R^+ containing β and γ and apply Proposition 19.9.

Conversely, suppose (2) holds, for concreteness, let us assume that $s_\beta s_\gamma \leq_R c^{-1}$. If β, γ are c-opposed, we apply Lemma 19.8. Otherwise, we can find R^+, such that either $\beta \prec_{R^+,c} \gamma$ are both positive roots, or $-\beta \prec_{R^+,c} \gamma$ are both positive roots. By Proposition 19.9, in both cases, (2) holds. □

19.4. Cluster compatibility and Coxeter noncrossing

Let

$$R_{\geq -1} := R^+ \cup \{-\alpha_1, \ldots, -\alpha_r\}$$

denote the set of *almost simple roots*. The notion of (cluster) compatibility of a pair of almost simple roots $\beta, \beta' \in R_{\geq -1}$ is defined in [FZ03], and it is related to the notion of compatibility of a pair $\omega, \omega' \in \Pi(c)$ by [YZ, (5.6)]. Namely, let $\psi : \Pi(c) \to R_{\geq -1}$ be defined by

$$\psi(\omega) = \begin{cases} -\alpha_i & \text{if } \omega = \omega_i \text{ for some } i = 1, 2, \ldots, r \\ c^{-1}\omega - \omega = c^{-1}(I - c)\omega & \text{otherwise.} \end{cases} \tag{19.2}$$

Then

$$\omega, \omega' \in \Pi(c) \text{ are compatible if and only if } \psi(\omega), \psi(\omega') \in R_{\geq -1} \text{ are.}$$

Theorem 19.11 [BW, Theorem 8.3]. *Suppose that $\beta \prec_{R^+,c} \gamma$ are two distinct positive roots. Then (β, γ) are (W, R^+, c)-compatible if and only if $s_\beta s_\gamma \leq_R c^{-1}$.*

Proposition 19.12. *Suppose that (R^+, c) is bipartite and $\beta, \gamma \in R^+$. Then (β, γ) is alternating c-noncrossing if and only if they are (W, R^+, c)-compatible.*

Proof. Let us suppose that $\beta \prec \gamma$. By Theorem 19.11, we must show that β, γ are alternating c-noncrossing if and only if $s_\beta s_\gamma \leq_R c^{-1}$. This follows from Proposition 19.9. □

Proposition 19.12 says that the condition "alternating and c-noncrossing" is an approximation to the notion of cluster compatibility that does not depend on a choice of R^+.

19.5. Coxeter noncrossing trees

An r-tuple $T = (\gamma_1, \ldots, \gamma_r)$ of roots is called a *tree* if they form a basis of V.

Definition 19.13. A *c-noncrossing tree* is an ordered sequence $T = (\gamma_1, \ldots, \gamma_r)$ of roots, such that

$$s_{\gamma_1} s_{\gamma_2} \cdots s_{\gamma_r} = c^{-1}.$$

Let $\mathcal{T}_{W,c}$ denote the set of c-noncrossing trees.

The terminology is justified by the following result.

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Lemma 19.14. *A c-noncrossing tree $T = (\gamma_1, \ldots, \gamma_r)$ is an ordered basis of V.*

Proof. Suppose the roots $\gamma_1, \ldots, \gamma_r$ span a proper subspace $W \subsetneq V$. Then any vector in the orthogonal complement $W^\perp \subset V$ (with respect to $(\cdot, \cdot)$) will be invariant under c^{-1}. This contradicts the fact that c^{-1} does not have the eigenvalue one (see Lemma 10.2). □

Remark 19.15. For a c-noncrossing tree $T = (\gamma_1, \ldots, \gamma_r)$, define the operation of *i-th sign reversal*

$$T \mapsto (\gamma_1, \ldots, \gamma_r);$$

and the operation of *j-th conjugation*

$$T \mapsto (\gamma_1, \ldots, \gamma_{j-1}, s_{\gamma_j}(\gamma_{j+1}), \gamma_j, \gamma_{j+2}, \ldots, \gamma_r).$$

These operations transform c-noncrossing trees into c-noncrossing trees. It follows from the results of Deligne [Del] and Bessis [Bes] that any two c-noncrossing trees are related by repeated application of sign reversal and conjugation. Furthermore, the conjugation actions give an action of the braid group.

Lemma 19.16. *Let $(\beta_1, \beta_2, \ldots, \beta_{hr})$ denote the ordering of R of Proposition 10.3. Then for any i, we have that $(\beta_i, \beta_{i+1}, \ldots, \beta_{i+r-1})$ is a c-noncrossing tree.*

Proof. By Proposition 10.4, it suffices to show this for $i = 1$. Suppose $c = s_1 s_2 \cdots s_r$. We calculate

$$s_{\beta_1} \cdots s_{\beta_r} = s_1(s_1 s_2 s_1)(s_1 s_2 s_3 s_2 s_1) \cdots (s_1 \cdots s_r \cdots s_1) = s_r s_{r-1} \cdots s_1 = c^{-1}.$$ □

According to Corollary 19.10, the "c-noncrossing" condition characterizes when a pair of roots can belong to a c-noncrossing tree. However, in general, this pairwise condition is insufficient to characterize c-noncrossing trees.

Example 19.17. Let $R = B_3$ with simple roots $\alpha_1 = (1, -1, 0)$, $\alpha_2 = (0, 1, -1)$, and $\alpha_3 = (0, 0, 1)$, and choose $c = s_1 s_2 s_3$, as in Example 10.9. Take the three roots

$$\gamma_1 = (-1, 0, -1), \qquad \gamma_2 = (-1, 1, 0), \qquad \gamma_3 = (0, 1, -1),$$

with corresponding c-twisted roots

$$\tilde{\gamma}_1 = (0, 0, -1), \qquad \tilde{\gamma}_2 = (-1, 0, 0), \qquad \tilde{\gamma}_3 = (0, 1, 0).$$

The roots $\gamma_1, \gamma_2, \gamma_3$ are pairwise c-noncrossing (and also alternating). However, all three roots are long, so no ordering of them can give a reflection factorization of c^{-1}.

Definition 19.18. Let $T = (\gamma_1, \ldots, \gamma_r)$ be a c-noncrossing tree. Then the *dual tree* $T' = \varphi(T)$ is given by

$$T' := (s_{\gamma_1} \cdots s_{\gamma_{r-1}} \gamma_r, \ldots, s_{\gamma_1} \gamma_2, \gamma_1).$$

Also, define $T'' = \varphi^{-1}(T)$ by

$$T'' := (\gamma_r, s_{\gamma_r} \gamma_{r-1}, \ldots, s_{\gamma_r} \cdots s_{\gamma_2} \gamma_1).$$

Proposition 19.19. *Let T be a c-noncrossing tree. Then the trees $T' = \varphi(T)$ and $T'' = \varphi^{-1}(T)$ are c-noncrossing tree. The maps φ and φ^{-1} are inverse bijections from $\mathcal{T}_{W,c}$ to $\mathcal{T}_{W,c}$.*

Proof. Let $T = (\gamma_1, \ldots, \gamma_r)$ and $T' = (\gamma'_1, \ldots, \gamma'_r)$. We have

$$c^{-1} = s_{\gamma_1} s_{\gamma_2} \cdots s_{\gamma_r} = s_{\gamma_2}(s_{\gamma_2} s_{\gamma_1} s_{\gamma_2}) s_{\gamma_3} \cdots s_{\gamma_r} = \cdots = s_{\gamma'_1} \cdots s_{\gamma'_r}.$$

The proof for T'' is similar, and it is straightforward to see that φ and φ^{-1} are inverse. □

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For a tree $T = (\gamma_1, \dots, \gamma_r)$, let $C_T \subset V$ denote the cone spanned by $\tilde{\gamma}_1, \dots, \tilde{\gamma}_r$, and let $C'_T \subset V$ denote the cone spanned by $\gamma_1^\vee, \dots, \gamma_r^\vee$. The following result is a general root-system theoretic version of Proposition 8.15.

Proposition 19.20. *Let T be a c-noncrossing tree and $T' = \varphi(T)$. Then the two cones C_T and $C'_{T'}$ are dual.*

Proof. Let $T' = (\delta_1, \dots, \delta_r)$, and let $\kappa_1, \kappa_2, \dots, \kappa_r$ be the dual basis to $\delta_1^\vee, \dots, \delta_r^\vee$, that is, $(\delta_i^\vee, \kappa_j) = \delta_{ij}$. Then

$$c \cdot \kappa_j = s_{\delta_r} \cdots s_{\delta_1} \cdot \kappa_j = \kappa_j - s_{\delta_r} \cdots s_{\delta_{j+1}} \delta_j.$$

Thus, $(I - c)\kappa_j = s_{\delta_r} \cdots s_{\delta_{j+1}} \delta_j$, so the dual cone to $C'_{T'}$ is given by $C_{\varphi^{-1}(T')}$. □

Corollary 19.21. *Suppose that P is a simple (W, c)-twisted alcoved polytope, and all maximal cones of the normal fan of P are of the form C_T for a c-noncrossing tree T. Then P is a generalized W-permutohedron and thus a (W, c)-polypositroid.*

Question 19.22. Let P be a generic simple (W, c)-polypositroid. Are all maximal cones of the normal fan of P of the form C_T for a c-noncrossing tree T?

Question 19.22 has an affirmative answer in type A (see Section 19.7).

19.6. Cluster cones

A (W, c)-*cluster cone* is a maximal cone $C \subset V$ in the (W, R^+, c)-cluster fan, for some choice of positive roots $R^+ \subset R$. It follows from the results of Brady-Watt [BW] that some cluster cones are of the form C_T for a c-noncrossing tree T, though we do not know whether this is true in general.

Proposition 19.23. *Suppose that (R^+, c) is bipartite. Let $C = \mathrm{span}(\tilde{\gamma}_1, \dots, \tilde{\gamma}_r)$ be a cluster cone, such that $\{\tilde{\gamma}_1, \dots, \tilde{\gamma}_r\} \cap \{\omega_1, \dots, \omega_r\} = \emptyset$. Then there is an ordering of $(\tilde{\gamma}_1, \dots, \tilde{\gamma}_r)$ that gives a c-noncrossing tree T.*

Proof. As a simplicial complex on the set of rays, the (W, R^+, c)-cluster fan is isomorphic, via the bijection (19.2) to the cluster complex of [FZ03] defined on the set of almost positive roots $R_{\geq -1}$. The condition $\{\tilde{\gamma}_1, \dots, \tilde{\gamma}_r\} \cap \{\omega_1, \dots, \omega_r\} = \emptyset$ is equivalent to the condition that the almost positive roots $\psi(\tilde{\gamma}_i)$ are positive.

According to [BW, Note 4.2 and Theorem 8.3], a sequence $\delta_1 \prec \delta_2 \cdots \prec \delta_r$ of positive roots forms a simplex in the cluster complex if and only if $(\delta_1, \dots, \delta_r)$ is a c-noncrossing tree. The claim follows. □

Remark 19.24. Reading and Speyer [RS] have found a *linear* isomorphism from the cluster fan of [FZ03] (with rays the almost positive roots), to the (W, R^+, c)-cluster fan, called the g-vector fan in [RS].

19.7. Type A

We make explicit the relation between reflection factorizations of c^{-1} and noncrossing trees (in type A).

Lemma 19.25. *Let T be a noncrossing (undirected) tree on $[n]$. Then there is an ordering $e_1, \dots, e_{n-1}$ of the edges so that $s_{e_1} \cdots s_{e_{n-1}} = c = (12 \cdots n)$. Varying the possible orderings gives the same reduced factorization of c up to commutation relations.*

Proof. Let us draw T in the interior of a disk with the vertices arranged in clockwise order on the boundary. Let $f_1^{(i)}, f_2^{(i)}, \dots, f_s^{(i)}$ be edges incident to the vertex i, in counterclockwise order. An ordering $e_1, \dots, e_{n-1}$ of all the edges of T satisfies $s_{e_1} \cdots s_{e_{n-1}} = c$ if and only if $\{f_1^{(i)}, \dots, f_s^{(i)}\}$ appear in the same order in $e_1, \dots, e_{n-1}$ for any vertex i. Since T contains no cycles, it is not difficult to see

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that such an ordering $e_1, \dots, e_{n-1}$ exists. The last statement follows from: s_e and $s_{e'}$ commute if they have no vertex in common. □

Example 19.26. Let T be the (solid) tree of Figure 2, with the five edges

$$(1,2), (2,4), (2,6), (3,4), (5,6).$$

Then around the vertices 2, 4, and 6, we obtain the counterclockwise orderings

$$(1,2) < (2,6) < (2,4), \qquad (3,4) < (2,4), \qquad (5,6) < (2,6).$$

One may check that

$$(123456) = (56)(12)(26)(34)(24) = (12)(56)(26)(34)(24) = (56)(12)(34)(26)(24) = \cdots,$$

consistent with Lemma 19.25.

Part III Membranes

In this part, we discuss membranes, which are certain triangulated 2-dimensional surfaces embedded into $\mathbb{R}^n$. They can be viewed as a polypositroidal version of the plabic graphs from [Po06].

20. Root loops and root membranes

Let R be an irreducible reduced crystallographic root system of rank r in a Euclidean vector space $V \simeq \mathbb{R}^r$. We identify the weight lattice $\Lambda \subset V$ with $\Lambda \simeq \mathbb{Z}^r$ (see Section 10.1).

Definition 20.1. A *plane graph* is a planar graph with a particular drawing on the plane without crossing edges, considered up to a homeomorphism.

A *cactus* G is a finite connected undirected plane graph with at least two vertices, such that every face of G (except the outer face) is a triangle, that is, every face has exactly three distinct vertices connected by three edges. In other words, a cactus is either a single edge, a triangulated disk, or a wedge of smaller cacti along their boundary vertices.

If a cactus G is a wedge of smaller cacti, then we say that G is *decomposable*. Otherwise, we say that G is *indecomposable*.

For a cactus G, there is a unique (up to a cyclic shift) sequence $B = (b_1, \dots, b_m)$ of *boundary vertices* connected by *boundary edges* $\{b_1, b_2\}, \{b_2, b_3\}, \dots, \{b_m, b_1\}$ obtained by walking along the boundary of G in the clockwise direction.

An example of a (decomposable) cactus G is shown in Figure 5.

Remark 20.2. If G is a decomposable cactus, then the sequence B has repeated entries. The way a cactus decomposes into indecomposable cacti is given by a noncrossing set partition of $[m]$ without singleton blocks. Blocks of this noncrossing set partition correspond to boundary vertices of connected components of the dual plane graph G^*, cf. Remark 21.3.

Definition 20.3. (1) An *R-loop* $L = (\lambda^{(1)}, \dots, \lambda^{(m)})$ is a sequence of weights $\lambda^{(a)} \in \Lambda$, cyclically indexed by elements $a \in \mathbb{Z}/m\mathbb{Z}$, such that $\lambda^{(a+1)} - \lambda^{(a)} \in R$, for any $a \in \mathbb{Z}/m\mathbb{Z}$.

(2) An *R-membrane* $M = (G, f)$ with *boundary loop* $L = (\lambda^{(1)}, \dots, \lambda^{(m)})$ is a cactus G on a vertex set Vert with the sequence of boundary edges $B = (b_1, \dots, b_m)$ together with a (not necessarily injective) embedding map $f : \text{Vert} \to \Lambda$, such that,

- $f(u) - f(v) \in R$, for any edge $\{u, v\}$ of G, and
- $f(b_a) = \lambda^{(a)}$, for any $a \in \mathbb{Z}/m\mathbb{Z}$

(in particular, we require that $f(u) \neq f(v)$ for any edge $\{u, v\}$ of G).

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Equivalently, an R-membrane is a cactus G together with a graph homomorphism[3] f from G to the graph on the vertex set Λ with edges $\{\lambda, \mu\}$ for $\lambda - \mu \in R$.

(3) An *R-line segment* is a line segment $\mathrm{conv}(\lambda, \mu) \subset V$, where $\lambda - \mu \in R$ and an *R-triangle* is a triangle $\mathrm{conv}(\lambda, \mu, \nu) \subset V$, where $\lambda, \mu, \nu \in \Lambda$, such that $\lambda - \mu, \mu - \nu, \nu - \lambda \in R$.

Example 20.4. The sequence $L = (\omega_1, \omega_2 - \omega_1, \omega_3 - \omega_1, \omega_3, 2\omega_4, \omega_1 + \omega_3 - \omega_2)$ is a B_4-loop, in the notation of Examples 10.9 and 16.3. Let G have boundary vertices $b_1, b_2, b_3, b_4, b_5, b_6$ arranged clockwise, interior vertex b, and triangular (clockwise-oriented) faces $(b, b_1, b_2), (b, b_2, b_3), (b, b_3, b_6), (b, b_6, b_1), (b_3, b_4, b_6), (b_4, b_5, b_6)$. Let $f(b_i) = \lambda^{(i)}$ and $f(b) = \omega_3 - \omega_2$. Then (G, f) is a B_4-membrane.

An R-loop L can be viewed as a closed piecewise-linear curve $\langle L \rangle$ in V, and R-membranes M with boundary loop L can be viewed as 2-dimensional simplicial complexes embedded into V as surfaces $\langle M \rangle$ composed of R-triangles and R-line segments with a given boundary curve $\langle L \rangle$, as follows.

For an R-loop $L = (\lambda^{(1)}, \ldots, \lambda^{(m)})$, let $\langle L \rangle \subset V$ be the closed piecewise-linear curve given by the union of R-line segments

$$\langle L \rangle := \bigcup_{a \in \mathbb{Z}/m\mathbb{Z}} [\lambda^{(a)}, \lambda^{(a+1)}].$$

Let $M = (G, f)$ be an R-membrane with boundary loop L. For a face Δ of G with vertices u, v, w, let $\langle \Delta \rangle := \mathrm{conv}(f(u), f(v), f(w)) \subset V$ be the corresponding R-triangle. The triangulated surface $\langle M \rangle \subset V$ associated with the membrane M is given by the union

$$\langle M \rangle := \langle L \rangle \cup \bigcup_{\Delta \text{ face of } G} \langle \Delta \rangle .$$

Definition 20.5. Let $\mathrm{val} : \{R\text{-triangles}\} \to \mathbb{R}_{>0}$ be any positive real function, or valuation,[4] on the set of all R-triangles. We say that an R-membrane M with boundary loop L is *minimal,* with respect to the valuation val, if its surface area

$$\mathrm{Area}\, M := \sum_{\Delta \text{ face of } G} \mathrm{val}\langle \Delta \rangle$$

has minimal possible value among all membranes with the same boundary loop L.

Remark 20.6. Our membranes should not to be confused with those of Keel and Tevelev [KeTe] appearing in the study of the Chow quotient of the Grassmannian.

Remark 20.7. The famous *Plateau's problem* originally raised by Lagrange is the problem in geometric measure theory concerning the existence of a minimal surface with a given boundary. It was solved by Jesse Douglas [Dou] and Tibor Radó [Rad]. We view the problem about characterization of minimal membranes M with a given boundary loop L as a discrete version of Plateau's problem. Unlike the situation with its continuous counterpart, the existence of a minimal membrane is trivial. There can be many minimal membranes with a given boundary. However, we think that the characterization of minimal membranes might provide a better understanding of the continuous Plateau's problem.

[3] A *graph homomorphism* $f : G_1 \to G_2$ from a graph $G_1 = (V_1, E_1)$ to another graph $G_2 = (V_2, E_2)$ is a map $f : V_1 \to V_2$, such that, for any edge $\{u, v\} \in E_1$, we have $\{f(u), f(v)\} \in E_2$.

[4] There are several natural valuations on R-triangles. For example, $\mathrm{val}\langle \Delta \rangle$ can be the Euclidean area of triangle $\langle \Delta \rangle \subset V$, it can be the area of the projection of $\langle \Delta \rangle$ to some plane, or it can be $\mathrm{val}\langle \Delta \rangle = 1$ for all R-triangles.

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21. Membranes of type A

Let us now specialize the definitions from the previous section to type A. Let

$$R = \{e_i - e_j \mid i, j \in [n],\ i \neq j\} \subset \mathbb{R}^n$$

be the A_{n-1} root system embedded in $\mathbb{R}^n$, and let $\Lambda \simeq \mathbb{Z}^n \subset \mathbb{R}^n$. These are the root and weight lattices of $\mathrm{GL}(n)$. Recall that $e_1, \dots, e_n$ denote the standard coordinate vectors in $\mathbb{Z}^n$. We assume the valuation[5] of any R-triangle is $\mathrm{val}\langle\Delta\rangle = 1$.

In this case, we call R-loops and R-membranes simply *loops* and *membranes*. Let us formulate their definitions.

Definition 21.1. A *loop* $L = (\lambda^{(1)}, \dots, \lambda^{(m)})$ is a cyclically ordered sequence of integer vectors $\lambda^{(a)} \in \mathbb{Z}^n$, such that $\lambda^{(a+1)} - \lambda^{(a)} = e_{i_a} - e_{j_a}$, for $a \in \mathbb{Z}/m\mathbb{Z}$, for some sequence of roots $e_{i_1} - e_{j_1}, \dots, e_{i_m} - e_{j_m}$.

A *membrane* $M = (G, f)$ with *boundary loop* $L = (\lambda^{(1)}, \dots, \lambda^{(m)})$ is a cactus G on a vertex set Vert with the sequence of boundary vertices $B = (b_1, \dots, b_m)$ together with a map $f : \mathrm{Vert} \to \mathbb{Z}^n$, such that

- for any edge $\{u, v\}$ of G, there exists indices $i \neq j$, such that $f(u) - f(v) = e_i - e_j$,
- $f(b_a) = \lambda^{(a)}$, for any $a \in \mathbb{Z}/m\mathbb{Z}$.

Let $\langle L\rangle, \langle M\rangle \subset \mathbb{R}^n$ denote the images of a loop L and a membrane M in $\mathbb{R}^n$.

Note that here we do not require m and n to be equal. Also note that the embedding $\langle M\rangle \subset \mathbb{R}^n$ of a membrane lies on some affine hyperplane $H = H_k := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 + \dots + x_n = k\}$, where $k \in \mathbb{Z}$. Clearly, a loop L is determined (up to affine translation) by a sequence of roots $e_{i_1} - e_{j_1}, \dots, e_{i_m} - e_{j_m}$ with equal multisets of indices $\{i_1, \dots, i_m\} = \{j_1, \dots, j_m\}$.

All faces Δ in a membrane are of one of the following two types:

- *black triangles* embedded into $\mathbb{R}^n$ as triangles $\langle\Delta\rangle$ of the form $\mathrm{conv}(-e_i, -e_j, -e_k)$, up to an affine translation; and
- *white triangles* embedded as triangles $\langle\Delta\rangle$ of the form $\mathrm{conv}(e_i, e_j, e_k)$, up to an affine translation.

For a membrane $M = (G, f)$, let G^* be the graph, which is the plane dual of the cactus G. The graph G^* is drawn in a disk so that

1. There are m marked points on the boundary of the disk (labelled $1, \dots, m$ clockwise) to which boundary edges of G^* are attached. Only one edge of G^* can be attached to a marked point on the disk. But it is allowed that both ends of an edge are attached to two different marked points on the boundary. (The marked point on the boundary of the disk labelled a corresponds to the boundary edge $\{b_a, b_{a+1}\}$ of the cactus G. Note that we do not regard these m marked boundary points as vertices of G^*. The vertices of G^* are located strictly inside the disk.)
2. The vertices of G^* are 3-valent. The vertices of G^* are colored in two colors: black and white. (The vertices of G^* correspond to triangles of the membrane M. They are colored according to the colors of triangles in M. Note that the m marked boundary points of the disk are not colored.)
3. The faces F_v of G^* (associated with vertices v of G) are labelled by the vectors $f(v) \in \mathbb{Z}^n$. For a pair of faces F_u and F_v sharing an edge, we have $f(u) - f(v) = e_i - e_j$, for some $i \neq j$.

Plane graphs G^* satisfying conditions (1), (2) above are plabic graphs in the sense of [Po06, Section 11], that are additionally 3-valent[6].

Definition 21.2. A *plabic graph* is the plane dual G^* of a cactus G with all vertices colored in two colors: black and white (this graph may contain edges between vertices of the same color).

A *$\mathbb{Z}^n$-labelled plabic graph* is a pair (G^*, f), where G^* is the plane dual of a cactus G, such that $M = (G, f)$ is a membrane. Equivalently, a $\mathbb{Z}^n$-labelled plabic graph is a pair (G^*, f) satisfying conditions (1), (2), and (3) above.

[5] In type A, the Euclidean area of any R-triangle equals $\sqrt{3}$. It is convenient to rescale it to 1.

[6] "Plabic" is an abbreviation for "planar bicolored." In this work, we will consider only 3-valent plabic graphs without boundary leaves, which we simply call "plabic graphs." The study of more general plabic graphs can be reduced to these.

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Clearly, by the definition, membranes $M = (G, f)$ are in bijection with $\mathbb{Z}^n$-labelled plabic graphs (G^*, f).

Remark 21.3. For a membrane $M = (G, f)$, the cactus G is indecomposable if and only if the plabic graph G^* is connected.

Let us give another description of membranes and $\mathbb{Z}^n$-labelled plabic graphs. We recall the definition of strands (or trips) in plabic graphs.

Definition 21.4 [Po06, Section 13]. For a plabic graph G^*, a *strand* in G^* is a directed walk along the edges of G^* that satisfies the following "rules of the road":

- Turn right at a black vertex.
- Turn left at a white vertex.

Each strand is either a walk between two marked points on the boundary of the disk, or a closed walk.

The *strand permutation* $\pi : [m] \to [m]$ of a plabic graph G^* is given by $\pi(s) = t$, if the strand that starts at the marked point s on the boundary of the disk ends at the marked point t.

Let $\mathrm{Strand}(G^*)$ be the set of all strands in G^*. For every edge $\{a, b\}$ of G^*, there are two strands in $\mathrm{Strand}(G^*)$ that pass through the edge: one passing in the direction $a \to b$ and the other passing in the direction $b \to a$. We call such a pair of strands an *intersecting* pair of strands. If these two intersecting strands happen to be the same strand, we call it a *self-intersecting* strand.

Theorem 21.5. *Let G^* be a fixed plabic graph, and let F_0 be a fixed reference face of G^*. The set of all $\mathbb{Z}^n$-labelled plabic graphs (G^*, f), and thus all membranes (G, f), are in bijection with the following data:*

1. *An integer vector in $\mathbb{Z}^n$, which is the label of the reference face F_0.*
2. *A map $g : \mathrm{Strand}(G^*) \to \{1, \ldots, n\}$ that satisfies the condition:*

$$g(S) \neq g(T), \quad \text{for any pair of intersecting strands } S \text{ and } T. \tag{21.1}$$

Explicitly, the strand labelling g is obtained from the face labelling f by the following condition: if $S \in \mathrm{Strand}(G^*)$ is the strand passing through some edge $\{a, b\}$ of G^* in the direction $a \to b$, and F_u and F_v are the two adjacent faces of G^* located, respectively, to the left and to the right of the edge $a \to b$, and if $f(u) - f(v) = e_i - e_j$, then $g(S) = i$.

In particular, (21.1) implies that, for any membrane (G, f), the plabic graph G^* cannot have self-intersecting strands. Changing the label of the reference face F_0 accounts for affine translations of membranes in $\mathbb{R}^n$. Up to affine translations, membranes correspond just to the strand labelling g satisfying condition (2).

Proof. First, it is easy to check, using the rules of the road, that the description of the strand labelling g in terms of face labelling f is locally consistent, that is, for any vertex of G^*, the label of some strand passing through this vertex obtained using its incoming edge to the vertex coincides with the label obtained using its outgoing edge. This implies the global consistence of the strand labelling g: For any strand S, the label $g(S)$ obtained using any edge of S does not depend on a choice of the edge.

Condition (2) for the strand labelling g follows from the fact, that, for a pair of strands S and T intersecting at an edge of G^* with two faces F_u and F_v adjacent to the edge, we have $g(S) = i$ and $g(T) = j$, where $f(u) - f(v) = e_i - e_j \neq 0$.

Conversely, let v_0 be the vertex of the cactus G corresponding to the reference face F_0 of G^*. Let $f(v_0) \in \mathbb{Z}^n$ be any integer vector, and let g be any strand labelling satisfying condition (2). For any vertex v of the cactus G, we can construct the vector $f(v) \in \mathbb{Z}^n$ by picking a path P in G from v_0 to v and using the relationship between f and g, for all edges of the path P. The rules of the road imply that the label $f(v)$ does not change if we locally modify the path P along a (triangular) face of the cactus G.

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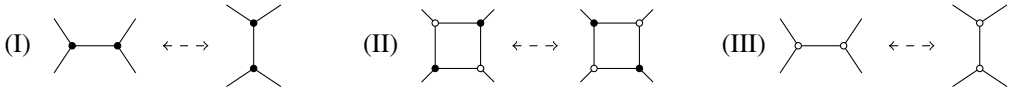

Figure 3. *Moves of plabic graphs: (I) contraction-uncontraction of black vertices, (II) square move, and (III) contraction-uncontraction of white vertices.*

This implies the independence of this construction for $f(v)$ from a choice of path P. Clearly, this function f, constructed from $f(v_0)$ and g, gives a valid membrane (G, f). □

By Theorem 21.5, the strands of a $\mathbb{Z}^n$-labeled plabic graph are not self-intersecting.

Corollary 21.6. *The strand permutation π of a $\mathbb{Z}^n$-labeled plabic graph is a derangement, that is, a permutation in S_m, such that $\pi(s) \neq s$ for any $s \in [m]$.*

In [Po06], strand permutations were *decorated* permutations with two types of fixed points (see Section 2). Here, we do not allow plabic graphs to have boundary leaves, so their strand permutations do not have fixed points.

22. Moves of plabic graphs and membranes

In [Po06, Section 12], the three types of local moves of plabic graphs were defined, which are shown below on Figure 3.

It is easy to see from the rules of the road that we have:

Lemma 22.1 [Po06, Lemma 13.1]. *Any two plabic graphs connected with each other by a sequence of local moves of types (I), (II), or (III) have the same strand permutations.*

The local moves of plabic graphs can be converted into local moves of membranes, as follows.

Lemma 22.2. *Let $M = (G, f)$ be a membrane. Let F_u be a square face of G^* with vertices of alternating colors as we go around F_u, that is, a face of G^* on which one can perform a square move (II). Let F_v, F_w, F_z, F_t be the four adjacent faces of G^* in the clockwise order. Then* $\mathrm{conv}(f(u), f(v), f(w), f(z), f(t))$ *is a square pyramid in $\mathbb{R}^n$, such that*

- *The pyramid has one square face (the base) and four faces given by equilateral triangles. All edges of the pyramid have equal lengths.*
- *$f(u)$ is the apex of the pyramid.*
- *The base is the square,* $\mathrm{conv}(f(v), f(w), f(z), f(t))$ *with vertices arranged as we go around the base.*

For a pyramid, as in the lemma above, let $\widetilde{f(u)} \in \mathbb{R}^n$ be the reflection of $f(u)$ with respect to the affine plane containing the points $f(v), f(w), f(z), f(t)$, that is, it is given by

$$\widetilde{f(u)} + f(u) = f(v) + f(z) = f(w) + f(t). \tag{22.1}$$

Clearly, $\mathrm{conv}(\widetilde{f(u)}, f(u), f(v), f(w), f(z), f(t))$ is an octahedron, which is the union of two square pyramids. The following lemma follows from the definitions.

Lemma 22.3. *Let $M = (G, f)$ be a membrane, and let $G^* \to \tilde{G}^*$ be any local move of the plabic graph G^* of type (I), (II), or (III). Let $\tilde{G}$ be the plane dual of the plabic graph $\tilde{G}^*$. The vertex set* $\tilde{\mathrm{Vert}}$ *of $\tilde{G}$ can be naturally identified with the vertex set* Vert *of G. Let $\tilde{f} : \tilde{\mathrm{Vert}} \to \mathbb{Z}^n$ be the map defined, as follows.*

- *For a move of type* (I) *or* (III)*, assume that*

$$f(u) \neq f(w), \tag{22.2}$$

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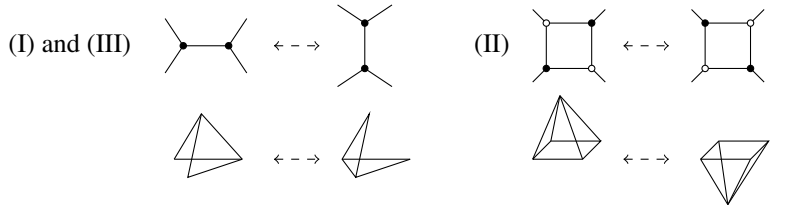

Figure 4. *Moves of plabic graphs (top) and membranes (bottom): tetrahedron moves (left), and octahedron move (right).*

where F_u and F_w are two of the four faces F_u, F_v, F_w, F_z of G^* involved in the move, which are not adjacent faces of G^*. Then define $\tilde{f} := f$.

- *For a square move (II), let F_u, F_v, F_w, F_z, F_t be the faces of G^* involved in the move, labelled as in Lemma 22.2, then set $\tilde{f}(u) := \widetilde{f(u)}$, given by (22.1). For all other $x \neq u$, set $\tilde{f}(x) := f(x)$.*

Then $\tilde{M} := (\tilde{G}, \tilde{f})$ is a valid membrane with the same boundary loop L as M.

Definition 22.4. Local moves of membranes of types (I), (II), or (III) are the moves $M \to \tilde{M}$ in Lemma 22.3.

Moves of types (I) and (III) correspond to *tetrahedron moves* of membranes, where we replace two triangles on the surface of a tetrahedron by the other two triangles. Moves of type (II) correspond to *octahedron moves* of membranes, where we replace four triangles forming a half of the surface of an octahedron by the four triangles on the other half of the surface of the octahedron, as shown in Figure 4 (see also [FP]).

Remark 22.5. There is only one situation when there is a valid move $G^* \to \tilde{G}^*$ of plabic graphs, but there is no corresponding move of membranes $M \to \tilde{M}$. This happens if the move of plabic graphs is of type (I) or (III) and condition (22.2) fails. In this case, in the picture shown on the bottom left of Figure 4, the two triangles before the move coincide with each other. The move would transform them into two "degenerate triangles", that is, line segments, which we do not allow in a membrane (recall that we require that $f(u) \neq f(v)$ for any edge $\{u, v\}$ of G).

However, for any membrane $M = (G, f)$ and any square move (II) of the plabic graph G^*, there is always the associated valid octahedron move of the membrane M.

Remark 22.6. According to [Po18], the three types of moves of plabic graphs correspond to the three 3-dimensional hypersimplices: Δ_{14} (tetrahedron), Δ_{24} (octahedron), and Δ_{34} (upside down tetrahedron). Moves of certain 3-dimensional membranes were used in [FP, Section 12.6] to graphically describe "chain reactions" on plabic graphs.

Clearly, $\mathrm{Area}(M)$ is preserved under the local moves of membranes, and the boundary loop L does not change. So the class of minimal membranes with a given boundary loop L is invariant under these three types of moves of membranes.

Remark 22.7. The exceptional case when condition (22.2) fails and the move of membranes is not defined (see Remark 22.5) can never happen in a minimal membrane. Indeed, in this case, the two coinciding triangles in the picture on the bottom left of Figure 4 can be removed from the membrane M, so that we get a membrane with a smaller area but with the same boundary loop L.

23. Minimal membranes and reduced plabic graphs

We recall the notion of reducedness for plabic graphs. The following definition is equivalent to [Po06, Definition 12.5] by [Po06, Theorem 13.2].

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Definition 23.1. A plabic graph G^* is *reduced* if it satisfies the conditions:

1. G^* has no self-intersecting strands.
2. G^* has no closed strands.
3. G^* has no pair of strands S, T with a *bad double crossing,* which means, that S and T intersect at two edges $\{a, b\}$ and $\{c, d\}$ and both strands are directed from $\{a, b\}$ to $\{c, d\}$.

Theorem 23.2 [Po06, Theorem 13.4]. *For any two reduced plabic graphs with the same number of marked points on the disk, the graphs have the same strand permutations if and only if they can be obtained from each other by a sequence of local moves of types (I), (II), and (III).*

Let us discuss a relationship between minimal membranes and reduced plabic graphs.

Theorem 23.3. *If $M = (G, f)$ is a minimal membrane, then G^* is a reduced plabic graph. Moreover, for a minimal membrane $M = (G, f)$ with boundary loop $L = (\lambda^{(1)}, \dots, \lambda^{(m)})$ (where $\lambda^{(a+1)} - \lambda^{(a)} = e_{i_a} - e_{j_a}$, for $a \in \mathbb{Z}/m\mathbb{Z}$), one can recover the strand labelling g (and thus the face labelling f) from the plabic graph G^* and loop L as follows: If $S \in \mathrm{Strand}(G^*)$ is a strand in G^* connecting two marked points labelled s (the source of S) and t (the target of S) on the boundary of the disk, then we have*

$$g(S) = i_s = j_t\,.$$

Proof. Suppose that $M = (G, f)$ is a minimal membrane, but G^* is not a reduced plabic graph. According to [Po06, Sections 12, 13], one can apply a sequence of moves (I), (II), and (III) to G^* and obtain a plabic graph with a double edge. By Remark 22.7, if $M = (G, f)$ is a *minimal* membrane, then, for any sequence of local moves of plabic graphs starting with G^*, there is an associated sequence of local moves of membranes. So we get some minimal membrane whose plabic graph has a double edge. However, we have:

1. There is no membrane (minimal or not) whose plabic graph has a double edge with vertices of different colors. Indeed, such a plabic graph would have a self-intersecting strand, which contradicts Theorem 21.5.
2. If the plabic graph of a membrane has a double edge with vertices of the same color, then the membrane is not minimal, because we can remove two triangles from it (corresponding to the two vertices of the plabic graph connected by the double edge), and get a membrane of smaller area with the same boundary loop.

In both cases, we get a contradiction, which proves the first part of the theorem.

The second part follows from the fact that a reduced plabic graph G^* cannot contain a closed strand. So any strand S of G^* connects two marked points s (source) and t (target) on the boundary of the disk. Applying Theorem 21.5 (that relates the strand labelling g to the face labelling of f of a plabic graph) to the first and the last edges of the strand S, that is, the edges of S connected to the marked points s and t on the boundary of the disk, we get exactly the needed equality $g(S) = i_s = j_t$. □

In general, a nonminimal membrane $M = (G, f)$ can give rise to a reduced plabic graph G^*. However, for a special class of loops, minimality of M is equivalent to reducedness of G^*.

Recall that a sequence $(c_1, \dots, c_m)$ of real numbers is *unimodal* if $c_1 \le \cdots \le c_k \ge c_{k+1} \ge \cdots \ge c_m$, for some $k \in [m]$. Let us say that a sequence $(c_1, \dots, c_m)$ is *cyclically unimodal* if it is unimodal up to a possible cyclic shift, that is, if $(c_r, c_{r+1}, \dots, c_m, c_1, \dots, c_{r-1})$ is unimodal for some $r \in [m]$.

Definition 23.4. We say that a loop $L = (\lambda^{(1)}, \dots, \lambda^{(m)})$ is *unimodal* if each of its coordinate sequences $(\lambda_i^{(1)}, \dots, \lambda_i^{(m)})$, for $i \in [n]$, is a cyclically unimodal sequence.

Equivalently, a loop L (with $\lambda^{(a+1)} - \lambda^{(a)} = e_{i_a} - e_{j_a}$, for $a \in \mathbb{Z}/m\mathbb{Z}$) is unimodal if there is no 4-tuple of indices $a < b < c < d$ in $[m]$, such that $i_a = j_b = i_c = j_d$ or $j_a = i_b = j_c = i_d$.

Consider a disk with m marked points on its boundary labelled $1, \dots, m$ in the clockwise order. For $s, t \in [m]$, let $|s, t|$ denote the chord in the disk that connects two marked boundary points labelled s and t. We say that two chords $|s, t|$ and $|s', t'|$ are *noncrossing* if they do not intersect each other in the disk.

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Definition 23.5. For a unimodal loop $L = (\lambda^{(1)}, \dots, \lambda^{(m)})$ (with $\lambda^{(a+1)} - \lambda^{(a)} = e_{i_a} - e_{j_a}$ for $a \in \mathbb{Z}/m\mathbb{Z}$), define $\pi = \pi_L$ as the unique permutation $\pi : [m] \to [m]$, such that

- For any $s \in [m]$, we have $i_s = j_{\pi(s)}$.
- If $i_s = i_{s'}$, for some $s \neq s' \in [m]$, then the two chords $|s, \pi(s)|$ and $|s', \pi(s')|$ are noncrossing.

Explicitly, the permutation $\pi = \pi_L$ is given, as follows. For any $i \in [n]$, let $s_1, \dots, s_p$ be all indices, such that $i_{s_1} = i_{s_2} = \cdots = i_{s_p} = i$; and let $t_1, \dots, t_p$ be all indices, such that $j_{t_1} = j_{t_2} = \cdots = j_{t_p} = i$. Since L is unimodal, we may assume that these indices are arranged as $s_1 < s_2 < \cdots < s_p < t_1 < t_2 < \cdots < t_p$ (up to a cyclic shift). Then we have $\pi(s_1) = t_p$, $\pi(s_2) = t_{p-1}, \dots, \pi(s_p) = t_1$.

Theorem 23.6. *Fix a unimodal loop L. The map $M = (G, f) \mapsto G^*$ gives a bijection between the following two sets:*

- *The set of minimal membranes M with boundary loop L.*
- *The set of reduced plabic graphs G^* with strand permutation π_L.*

All minimal membranes with boundary loop L are connected by local moves of membranes of types (I)*,* (II)*, and* (III)*.*

Proof. Let $M = (G, f)$ be a minimal membrane with boundary loop L. According to Theorem 21.5, two strands S and T of G^* that have the same strand label $g(S) = g(T)$ cannot intersect each other. For $i \in [n]$, let $s_1, \dots, s_p, t_1, \dots, t_p$ be sequence of sources and targets as in Definition 23.5. These sources should be connected with the targets by strands $S_1, \dots, S_p$ that all have the same label $g(S_1) = \cdots = g(S_p) = i$. Since the sources and the targets are separated from each other on the boundary of the disk, there exists a unique matching between them, given by a collection of pairwise noncrossing strands. Namely, the strand S_a starting at the source s_a should end at the target t_{p+1-a}, for $a = 1, \dots, p$. Thus, the strand permutation of G^* is exactly the permutation π_L given by Definition 23.5. Since we know that all reduced plabic graphs with the same strand permutation are connected with each other by the local moves, and there are corresponding local moves of membranes, we deduce all the claims of the theorem. □

Let us give explicit expressions for the surface area and the number of lattice points of a minimal membrane using the results of [Po06].

Definition 23.7 [Po06, Section 17]. Let $\pi : [m] \to [m]$ be a derangement, that is, a permutation, such that $\pi(a) \neq a$, for any a. Define:

- The *number of antiexceedances* in π

$$k(\pi) := \#\{a \in [m] \mid \pi(a) < a\}.$$

- The *number of alignments* in π

$$A(\pi) := \left\{ (a, b) \in [m]^2 \mid \begin{array}{l} a < b < \pi(b) < \pi(a), \text{ or } \pi(a) < a < b < \pi(b), \text{ or} \\ \pi(b) < \pi(a) < a < b, \text{ or } b < \pi(b) < \pi(a) < a \end{array} \right\}.$$

Recall that the surface area $\mathrm{Area}(M)$ of a membrane $M = (G, f)$ is the number of faces (triangles) Δ of the cactus G, or, equivalently, the number of vertices of the plabic graph G^*. Also denote the number of *lattice points* of M by

$$\mathrm{LatticePoints}(M) := \#(\langle M \rangle \cap \mathbb{Z}^n) = \#\{\text{vertices of } G\} = \#\{\text{faces of } G^*\}.$$

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Proposition 23.8 (cf. [Po06, Proposition 17.10]). *Let L be a unimodal loop, and let $\pi = \pi_L : [m] \to [m]$ be the associated permutation (see Definition 23.5). The number of lattice points and the surface area of a minimal membrane M with boundary loop L are equal to*

$$\begin{aligned}\mathrm{LatticePoints}(M) &= k(\pi)(m - k(\pi)) - A(\pi) + 1,\\ \mathrm{Area}(M) &= 2(k(\pi)(m - k(\pi)) - A(\pi)) - m.\end{aligned}$$

Proof. For a membrane $M = (G, f)$, the number $\mathrm{LatticePoints}(M)$ equals the number of faces of the plabic graph G^*. The needed expression for the number of faces of a reduced plabic graph was given in [Po06, Proposition 17.10]. $\mathrm{Area}(M)$ equals the number of vertices of the plabic graph G^*. Using Euler's formula together with the fact that G^* is a 3-valent graph, we deduce

$$\mathrm{Area}(M) = 2(\mathrm{LatticePoints}\, M - 1) - m,$$

which gives the stated expression for $\mathrm{Area}(M)$. □

In the following sections, we will discuss several special classes of unimodal loops and membranes that have special properties:

$$\left\{\begin{matrix}\text{positroid}\\ \text{loops}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{polypositroid}\\ \text{loops}\end{matrix}\right\} \subset \left\{\begin{matrix}j\text{-increasing}\\ \text{loops}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{unimodal}\\ \text{loops}\end{matrix}\right\}.$$

They are described in terms of the associated sequences of roots $e_{i_1} - e_{j_1}, \ldots, e_{i_m} - e_{j_m} \in \mathbb{Z}^n$, by the following conditions:

- Positroid loops: $m = n$; $i_1, \ldots, i_n$ is a permutation of $1, \ldots, n$; and $j_a = a$, for $a = 1, \ldots, n$.
- Polypositroid loops: $j_1 \le \cdots \le j_m$; and if $j_a = j_{a+1}$, then $i_{a+1} \in \{i_a - 1, i_a - 2, \ldots, j_a + 1\}$ (a cyclic interval in $[n]$).
- j-increasing loops: $j_1 \le \cdots \le j_m$.

For example, we'll see that positroid loops (considered up to affine translations) are in bijection with positroids, and polypositroid loops are in bijection with integer polypositroids.

24. Positroid membranes

We now discuss the distinguished class of loops and membranes related to positroids. Assume (in the notations of Section 21), that L is a loop, such that $m = n$ and $\{i_1, \ldots, i_n\} = \{j_1, \ldots, j_n\} = [n]$ are usual sets. Moreover, by permuting the coordinates in $\mathbb{R}^n$, we assume that $j_a = a$, for $a = 1, \ldots, n$. Such loops L (up to affine translations) correspond to Grassmann necklaces associated with positroids.

Let $\mathcal{M} \subset \binom{[n]}{k}$ be a positroid and $\mathcal{I} = (I_1, \ldots, I_n)$ be the associated Grassmann necklace (see Section 2). To simplify the presentation, we shall assume that $\mathcal{M}$ has no loops or coloops, and thus, $I_{a+1} \neq I_a$ for any a. We have $I_{a+1} = (I_a \setminus \{a\}) \cup \{i_a\}$, for $a \in \mathbb{Z}/n\mathbb{Z}$, where $\pi = \pi_{\mathcal{M}} : a \to i_a$ is a certain permutation (derangement) of size n.

Consider the loop $L = L_{\mathcal{M}} := (e_{I_1}, \ldots, e_{I_n})$ (recall that $e_I := \sum_{i \in I} e_i$). It corresponds to the sequence of roots $e_{i_1} - e_1, \ldots, e_{i_n} - e_n$.

Theorem 24.1. *Minimal membranes M with boundary loop $L_{\mathcal{M}}$ are in bijection with reduced plabic graphs with strand permutations $\pi_{\mathcal{M}}$. The bijection is given by $M = (G, f) \mapsto G^*$.*

Moreover, for any such minimal membrane M, its embedding $\langle M \rangle \subset \mathbb{R}^n$ is contained in the positroid polytope $P_{\mathcal{M}} := \mathrm{conv}(e_I \mid I \in \mathcal{M})$.

Proof. Clearly, the loop $L_{\mathcal{M}}$ is unimodal. The first part of the above theorem follows from Theorem 23.6.

For such a minimal positroid membrane $M = (G, f)$, the vectors $f(v) \in \mathbb{Z}^n$ are related to the *face labels* $I(F) \in \binom{[n]}{k}$ of the corresponding plabic graph G^*, which were studied in [OPS], as follows:

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$f(v) = e_{I(F_v)}$, for any face F_v of G^*. Indeed, in the case of positroid membranes, the relationship between f and the strand labelling g (given in Theorem 21.5) specializes to the definition of face labels of reduced plabic graphs.

The second claim of the above theorem now follows from the result proved in [OPS] that any face label $I(F)$ of a reduced plabic graph G^* associated with a positroid $\mathcal{M}$ belongs to the positroid: $I(F) \in \mathcal{M}$. □

25. Polypositroid membranes

We now discuss the class of loops and membranes related to integer polypositroids, which includes positroid loops and membranes from Section 24. Recall that in Section 6, we gave bijections between polypositroids $P \subset \mathbb{R}^n$, Coxeter necklaces $\mathbf{v} = (v^{(1)}, \dots, v^{(n)})$, and balanced digraphs. If P is an integer polypositroid, then the $v^{(i)}$ are integer vectors, and the edge weights m_{ij} of the balanced digraph are nonnegative integers.

Define the *perimeter*[7] of a membrane M with boundary loop $L = (\lambda^{(1)}, \dots, \lambda^{(m)})$ by $\operatorname{Perim} M = m$.

Definition 25.1. Let $P \subset \mathbb{R}^n$ be an integer polypositroid, and let $\mathbf{v} = (v^{(1)}, \dots, v^{(n)})$ be its Coxeter necklace. We say that a membrane M is a *minimal P-membrane* if

1. The boundary loop $L = (\lambda^{(1)}, \dots, \lambda^{(m)})$ of M contains the points $v^{(1)}, \dots, v^{(n)}$ in this particular cyclic order.
2. The membrane M has minimal possible perimeter $\operatorname{Perim} M$ among all membranes satisfying condition (1).
3. The membrane M has minimal possible surface area $\operatorname{Area} M$ among all membranes satisfying conditions (1) and (2).

Remark 25.2. In the above definition, it is important to first minimize the perimeter of M, and only after that minimize the surface area. If we skip condition (2), we can always find a membrane M satisfying (1), whose surface area $\operatorname{Area} M$ equals zero.

Define the *standard root order* as the total order $<$ on all roots $e_i - e_j$, $i, j \in [n]$, $i \neq j$:

$$\begin{aligned}
&(e_n - e_1) < (e_{n-1} - e_1) < (e_{n-2} - e_1) < \cdots < (e_2 - e_1) < \\
&< (e_1 - e_2) < (e_n - e_2) < (e_{n-1} - e_2) < \cdots < (e_3 - e_2) < \\
&< (e_2 - e_3) < (e_1 - e_3) < (e_n - e_3) < \cdots < (e_4 - e_3) < \\
&\qquad\qquad \cdots\cdots\cdots\cdots \\
&< (e_{n-1} - e_n) < (e_{n-2} - e_n) < (e_{n-3} - e_n) < \cdots < (e_1 - e_n).
\end{aligned} \tag{25.1}$$

In other words, we have $e_i - e_j < e_{i'} - e_{j'}$ whenever $j < j'$, or ($j = j'$ and $i' \in \{i-1, i-2, \dots, j+1\}$), where elements of the interval are considered modulo n.

Remark 25.3. The total order (25.1) differs from the one in Example 10.8: instead, it arises from the simple system $\{e_n - e_1, e_{n-1} - e_n, \dots, e_2 - e_3\}$ and Coxeter element $c = s_0 s_{n-1} s_{n-2} \cdots s_2$, where $s_0 := (1n)$ is the reflection associated to the root $e_n - e_1$.

Definition 25.4. A *polypositroid loop* is a loop $L = (\lambda^{(1)}, \dots, \lambda^{(m)})$, with $\lambda^{(a)} \in \mathbb{Z}^n$, such that $\lambda^{(a+1)} - \lambda^{(a)}$, $a = 1, \dots, m$, is a weakly increasing sequence of roots in the standard root order.

The following two lemmas easily follow from the definitions.

Lemma 25.5. *Any polypositroid loop is unimodal.*

Let $P \subset \mathbb{R}^n$ be an integer polypositroid with Coxeter necklace $\mathbf{v} = (v^{(1)}, \dots, v^{(n)})$, and let m_{ij} be the edge multiplicities of the associated balanced digraph. Define the loop $L_P = (\lambda^{(1)}, \dots, \lambda^{(m)})$, such that

[7] $\operatorname{Perim} M$ equals the usual Euclidean length of the perimeter of $\langle M \rangle$ divided by $\sqrt{2}$.

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$\lambda^{(1)} = v^{(1)}$, and the sequence of roots $\lambda^{(a+1)} - \lambda^{(a)}$, $a = 1, \dots, m$, is the weakly increasing sequence of roots (in the standard root order), such that each root $e_j - e_i$ is repeated m_{ij} times. Note that m equals the total number of edges of the balanced bigraph.

Lemma 25.6. (1) *The map $P \to L_P$ is a bijection between integer polypositroids $P \subset \mathbb{R}^n$ and polypositroid loops in $\mathbb{R}^n$.*

(2) *The Coxeter necklace* $\mathbf{v}$ *is a subsequence of the associated polypositroid loop L_P. Explictly, $v^{(i+1)} = \lambda^{(1+d_1+d_2+\cdots+d_i)}$, for $i = 0, \dots, n$; here, $d_i = -(v^{(i+1)} - v^{(i)})_i = \sum_{j \neq i} m_{ij}$, is the outdegree (or the indegree) of vertex i of the associated balanced digraph.*

Recall that, for a unimodal loop L, we defined the permutation $\pi_L : [m] \to [m]$ (see Definition 23.5).

Theorem 25.7. *Let $P \subset \mathbb{R}^n$ be an integer polypositroid, and let $L = L_P$ be the corresponding polypositroid loop. The following two sets coincide:*

- *the set of minimal P-membranes,*
- *the set of minimal membranes with boundary loop L.*

Each of these sets is in bijection (via $M = (G, f) \mapsto G^*$) with

- *the set of reduced plabic graphs with strand permutation π_L.*

All minimal P-membranes are connected with each other by a sequence of local moves of types (I), (II), *and* (III).

Proof. Let us show that the boundary loop L of any minimal P-membrane is exactly the polypositroid loop L_P. Indeed, by Definition 25.1, for each $i = 1, \dots, n$, the loop L contains the points $v^{(i)}$ and $v^{(i+1)}$ connected by a shortest possible piecewise-linear curve with line segments given some roots. Thus, this portion of the loop L between the points $v^{(i)}$ and $v^{(i+1)}$ should contain exactly m_{ij} copies of the root $e_j - e_i$, for all $j \neq i$, cf. formulas (6.1) and (6.3) in Section 6. Note that any way to arrange these roots (in each portion of L between $v^{(i)}$ and $v^{(i+1)}$) would produce a unimodal loop. So the surface area of the membrane M is given by Proposition 23.8 in terms of the number of alignments and the number of antiexceedances of the permutation π associated with L (see Definitions 23.5 and 23.7). The number of antiexceedances of π equals $k(\pi) = \sum_{i>j} m_{ij}$. In order to minimize the surface area of M, we need to maximize the number of alignments $A(\pi)$ of π. This maximum is achieved if and only if $L = L_P$. The theorem now follows from Theorem 23.6. □

For a balanced digraph on the vertex set $[n]$ with edge multiplicities m_{ij}, $i, j \in [n]$ define the number m of edges, the number k of antiexceedances, and the number A of alignments, as follows:

$$m := \sum_{i \neq j} m_{ij}, \qquad k := \sum_{i>j} m_{ij},$$

$$A := \sum_{\substack{i < i' < j' < j, \text{ or} \\ i' < j' < j < i, \text{ or} \\ j' < j < i < i', \text{ or} \\ j < i < i' < j'}} m_{ij}\, m_{i'j'} + \sum_{i,\, j<j'} (m_{ij} m_{ij'} + m_{ji} m_{j'i}) + \sum_{i \neq j} \binom{m_{ij}}{2}$$

(notice that we regard a pair of edges of the digraph with the same initial points and/or the same endpoints as an alignment). Proposition 23.8 implies the following formulae.

Corollary 25.8. *The number of lattice points and the surface area of any minimal P-membrane M are equal to*

$$\mathrm{LatticePoints}(M) = k(m-k) - A + 1,$$
$$\mathrm{Area}(M) = 2(k(m-k) - A) - m,$$

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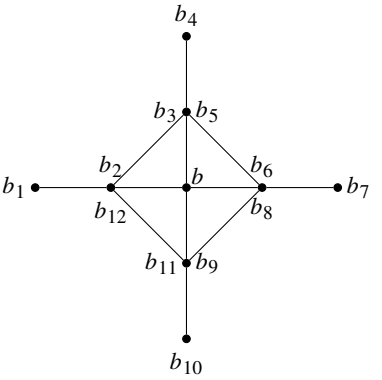

Figure 5. *The cactus graph G.*

where m is the number of edges, k is the number of antiexceedances, and A is the number of alignments of the balanced digraph associated with P.

Example 25.9. Let $n = 4$. Then $L =$

$$((3,2,1,0),(2,2,1,1),(1,2,2,1),(0,3,2,1),(1,2,2,1),(1,1,2,2),$$
$$(1,0,3,2),(1,1,2,2),(2,1,1,2),(2,1,0,3),(2,1,1,2),(2,2,1,1))$$

is a polypositroid loop, associated to the balanced digraph, where $m_{ij} = 1$ for all $i \neq j$, studied in Example 6.11. The permutation $\pi_L : [12] \to [12]$ is given by

$$(12,8,4,3,11,7,6,2,10,9,5,1),$$

and we have $k(\pi_L) = 6$. We have $A = 28$, and thus for a minimal membrane M, we have LatticePoints$(M) = 9$ and Area$(M) = 4$. Let us give an example of a minimal membrane $M = (G, f)$. Let G (shown in Figure 5) have boundary vertices $\{b_1, \ldots, b_{12}\}$ arranged clockwise, with repetitions $b_2 = b_{12}, b_3 = b_5, b_6 = b_8, b_9 = b_{11}$, and a single nonboundary vertex b. The edges of G are the boundary edges $\{b_i, b_{i+1}\}$ and the edges of the four faces $(b, b_2, b_3), (b, b_5, b_6), (b, b_8, b_9), (b, b_{11}, b_{12})$. We define $f(b_i) = \lambda^{(i)}$ and $f = (2, 1, 2, 1)$.

26. Positroid lifts

For a membrane $M = (G, f)$ and a vertex v of G, let $f(v)_i$ denote the i-th coordinate of the vector $f(v) \in \mathbb{Z}^n$.

Lemma 26.1. *Let $M = (G, f)$ be a minimal membrane, and let $i \in [n]$. The minimal/maximal value of the i-th coordinate $f(v)_i$ over all vertices v of G is achieved on some boundary vertex b_j of G.*

Proof. Suppose that this is not true, and the minimal value of the i-th coordinate is achieved on some internal vertex v of G, and it is strictly less than $f(b_j)_i$ for all boundary vertices b_j. Let F_v be the face of the plabic graph G^* that corresponds to the vertex v of G. The i-th coordinate might take the same minimal value on some other vertices of G that correspond to other faces of G^* adjacent to F_v. Let R be the maximal connected region formed by such faces of the plabic G^*. By our assumption, the region R does not include any boundary regions of G^*, thus the region R is bounded by a closed curve C formed by some edges of G^*. Assume that C is oriented clockwise. For any other face of G^* adjacent to R, the i-th coordinate is strictly greater. This mean that, for any edge $a \to b$ of G^* on the curve C (oriented in same the clockwise direction), the strand S that passes through the edge $a \to b$ has label $g(S) = i$; see

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Theorem 21.5. Since any two intersecting strands cannot have the same label, we conclude that all edges on the closed curve C belong to the same strand S. This means that the strand S is either self-intersecting or closed. Since the membrane M is minimal, the plabic graph G^* is reduced (see Theorem 23.3). However, by Definition 23.1, a reduced plabic graph cannot contain self-intersecting or closed strands. We obtain a contradiction. The proof of the claim about the maximal value of $f(v)_i$ is analogous. □

Let $\mathbf{d} = (d_1, \dots, d_n)$ be a nonnegative integer vector. We say that a loop $L = (\lambda^{(1)}, \dots, \lambda^{(m)}) \in (\mathbb{Z}^n)^m$ is $\mathbf{d}$-*boxed* if $\min_{j\in[m]} \lambda_i^{(j)} = 0$ and $\max_{j\in[m]} \lambda_i^{(j)} = d_i$, for all $i \in [n]$. In other words, the curve $\langle L\rangle$ lies in the box $[0, d_1] \times \cdots \times [0, d_n] \subset \mathbb{R}^n$ and has points on each facet of the box.

According to Lemma 26.1, for any minimal membrane M with a $\mathbf{d}$-boxed boundary loop L, we have $\langle M\rangle \subset [0, d_1] \times \cdots \times [0, d_n]$. Moreover, both $\langle L\rangle$ and $\langle M\rangle$ belong to the intersection of the box $[0, d_1] \times \cdots \times [0, d_n]$ with some affine hyperplane $x_1 + \cdots + x_n = k$.

Definition 26.2. Let $d = d_1 + \cdots + d_n$. For an integer vector $\lambda = (\lambda_1, \dots, \lambda_n) \in [0, d_1] \times \cdots \times [0, d_n]$, let $\mathrm{lift}(\lambda)$ be the 0-1-vector in $\mathbb{Z}^d$ given by

$$\mathrm{lift}(\lambda) := (0^{d_1-\lambda_1}, 1^{\lambda_1}, 0^{d_2-\lambda_2}, 1^{\lambda_2}, \dots, 0^{d_n-\lambda_n}, 1^{\lambda_n}) \in \{0, 1\}^d,$$

where a^r denotes a repeated r times.

Let $L = (\lambda^{(1)}, \dots, \lambda^{(m)}) \in (\mathbb{Z}^n)^m$ be a $\mathbf{d}$-boxed loop. The *lift* of L is the loop $\mathrm{lift}(L) := (\mathrm{lift}(\lambda^{(1)}), \dots, \mathrm{lift}(\lambda^{(m)})) \in (\mathbb{Z}^d)^m$.

For a minimal membrane $M = (G, f)$ with boundary loop L, the *lift* of M is the membrane $\mathrm{lift}(M) := (G, \mathrm{lift}(f))$, where $\mathrm{lift}(f) : v \mapsto \mathrm{lift}(f(v)) \in \mathbb{Z}^d$, for a vertex v of G.

Let $\mathrm{proj} : \mathbb{R}^d \to \mathbb{R}^n$ be the map[8] given by

$$\mathrm{proj} : (x_1, \dots, x_d) \mapsto (x_1 + x_2 + \cdots + x_{d_1}, x_{d_1+1} + \cdots + x_{d_1+d_2}, \cdots, x_{d_1+\cdots+d_{n-1}+1} + \cdots + x_d).$$

For a membrane $\tilde{M} = (G, \tilde{f})$, where $f : \mathrm{Vert} \to \mathbb{Z}^d$, define $\mathrm{proj}(\tilde{M}) := (G, \mathrm{proj}(\tilde{f}))$.

Proposition 26.3. *Let $L = (\lambda^{(1)}, \dots, \lambda^{(m)}) \in (\mathbb{Z}^n)^m$ be a $\mathbf{d}$-boxed unimodal loop. Then $m = d := d_1 + \cdots + d_n$. The sequence of roots $e_{i_a} - e_{j_a} = \mathrm{lift}(\lambda^{(a+1)}) - \mathrm{lift}(\lambda^{(a)}) \in \mathbb{Z}^m$, $a = 1, \dots, m$, associated with the loop $\mathrm{lift}(L)$ satisfies the condition: both sequences $i_1, \dots, i_m$ and $j_1, \dots, j_m$ are permutations of $1, \dots, m$.*

The following three sets are in bijection with each other:

- *Minimal membranes M with boundary loop L.*
- *Minimal membranes $\tilde{M}$ with boundary loop $\mathrm{lift}(L)$.*
- *Reduced plabic graphs G^* with strand permutation π_L (see Definition 23.5).*

Explicitly, the bijections are given by the maps (which form a commutative diagram): $M \mapsto \tilde{M} = \mathrm{lift}(M)$, $\tilde{M} \mapsto M = \mathrm{proj}(\tilde{M})$, $M = (G, f) \mapsto G^*$, $\tilde{M} = (G, \tilde{f}) \mapsto G^*$.

The loop $\mathrm{lift}(L)$ is obtained from a positroid loop in $\mathbb{Z}^m$ by a permutation of coordinates in $\mathbb{Z}^m$. This is exactly the positroid whose permutation is equal to π_L. Thus, the lifted membranes $\tilde{M}$ are permuted positroid membranes.

Proof. The first claim easily follows from the definitions. The claim about bijections between the sets of membranes M, $\tilde{M}$, and reduced plabic graphs G^* follows from Theorem 23.6. Indeed, both loops L and $\mathrm{lift}(L)$ correspond to the same permutation $\pi_L = \pi_{\mathrm{lift}(L)}$ (see Definition 23.5). □

Let us specialize this construction to polypositroid loops. As in the previous section, let $P \subset \mathbb{R}^n$ be an integer polypositroid. Assume that P belongs to the positive orthant $\mathbb{R}^n_{\geq 0}$ and has points on each coordinate plane in $\mathbb{R}^n$. Let m_{ij} be the edge multiplicities of the balanced digraph associated with P, let $m := \sum_{i,j} m_{ij}$ be the total number of edges of the digraph, let $k = \sum_{i>j} m_{ij}$ be its number of

[8]The map proj is a projection if all d_i's are positive.

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antiexceedances, and let $d_i := \sum_j m_{ij}$, $i \in [n]$, be the outdegrees (or the indegrees) of the balanced digraph. Clearly, we have $m = d_1 + \cdots + d_n$. Let $\mathbf{d} = (d_1, \ldots, d_n)$.

Corollary 26.4. *For a polypositroid P as above, the polypositroid loop $L = L_P$ is $\mathbf{d}$-boxed. Its lift* $\mathrm{lift}(L)$ *is a positroid loop in $\mathbb{Z}^m$.*

The maps lift : $M \mapsto \tilde{M}$ *and* proj : $\tilde{M} \mapsto M$ *give a bijection between minimal membranes M with boundary loop L minimal positroid membranes $\tilde{M}$ with boundary loop* $\mathrm{lift}(L)$.

The first claims in this statement are straightforward from the definitions, the last claim is a special case of Proposition 26.3.

Example 26.5. Consider the polypositroid loop L in Example 25.9. Then L is $\mathbf{d} = (3, 3, 3, 3)$-boxed, and

$$\begin{aligned}\mathrm{lift}(L) =&((1,1,1,0,1,1,0,0,1,0,0,0),(0,1,1,0,1,1,0,0,1,0,0,1),\\ &(0,0,1,0,1,1,0,1,1,0,0,1),(0,0,0,1,1,1,0,1,1,0,0,1),\ldots).\end{aligned}$$

Let $\mathcal{M} \subset \binom{[m]}{k}$ be the positroid associated with the positroid loop $\mathrm{lift}(L)$, and let $P_{\mathcal{M}} := \mathrm{conv}(e_I \mid I \in \mathcal{M}) \subset \mathbb{R}^m$ be the positroid polytope of the positroid $\mathcal{M}$.

Lemma 26.6. *Under the map* proj : $\mathbb{R}^m \to \mathbb{R}^n$, *we have* $\mathrm{proj}(P_{\mathcal{M}}) \subseteq P$.

Proof. The positroid polytope $P_{\mathcal{M}}$, being an alcoved polytope in $\mathbb{R}^m$, is given by inequalities of the form $x_i + x_{i+1} + \cdots + x_j \le c_{ij}$ for all cyclic intervals $[i, j]$ in $[m]$. Similarly, the polytope P, being an alcoved polytope in $\mathbb{R}^n$, is given by inequalities of the form $y_a + y_{a+1} + \cdots + y_b \le d_{ab}$ for all cyclic intervals $[a, b]$ in $[n]$. One can check from the definitions that the inequalities for $P_{\mathcal{M}}$ corresponding to cyclic intervals $[i, j]$ that consist of unions of blocks $\{1, \ldots, d_1\}$, $\{d_1 + 1, \ldots, d_1 + d_2\}$, etc., project exactly to the inequalities defining the polytope P. □

Theorem 24.1 now implies the following claim.

Corollary 26.7. *For any minimal P-membrane M, the embedding $\langle M \rangle \subset \mathbb{R}^n$ belongs to the polypositroid: $\langle M \rangle \subset P$.*

Remark 26.8. The same plabic graph G^* can appear in different membranes $M = (G, f)$ of different dimensions. For any reduced plabic graph G^* with m boundary edges, there is always the associated positroid membrane of dimension $m - 1$ that lies in a hyperplane $\{x_1 + \cdots + x_m = k\} \subset \mathbb{R}^m$. But there might also be other lower dimensional membranes with the same plabic graph, which are obtained by projections of this positroid membrane.

Define the *essential dimension* of a reduced plabic graph G^* as the minimal dimension of a minimal membrane whose plabic graph is equal to G^*.

Proposition 26.9. *For a reduced plabic graph G^* with m boundary edges, the essential dimension equals $m - 1$ if and only if G^* is a plabic graph for the top positroid cell in $\mathrm{Gr}(k, m)_{\ge 0}$ for some $k \in [m - 1]$, that is, if its strand permutation is $\pi : i \mapsto i + k \pmod m$.*

Proof. If a graph G^* has a maximal possible essential dimension $m - 1$, then its strand permutation π does not have alignments. Indeed, if π has an alignment (i, j), then we can project the positroid membrane for G^* to a lower-dimensional membrane by mapping $(x_i, x_j) \mapsto x_i + x_j$ and leaving all other coordinates. The only permutations with no alignments are permutations given by $\pi : i \mapsto i + k \pmod m$, for some k.

Assume now that G^* is a plabic graph for the top positroid cell in $\mathrm{Gr}(k, m)_{\ge 0}$ and $M = (G, f)$ is a minimal membrane with $f : \mathrm{Vert} \to \mathbb{Z}^n$. If $n < m$, then we can find two different strands S and T in G^*, with the same label $g(S) = g(T)$ (see Theorem 21.5). According to Theorem 21.5, the strands S and T can not intersect in the plabic graph G^*. It is not hard to show, using the techniques of [Po06], that for any i and j, there is *some* plabic graph for the top cell in $\mathrm{Gr}(k, m)_{\ge 0}$ whose i-th and j-th strands intersect. Also, according to Theorem 23.2 ([Po06, Theorem 13.4]), all plabic graphs for the top cell are

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connected with each other by local moves. This means that even if the strands S and T do not intersect in G^*, one can always find a sequence of local moves that result in a plabic graph, where the pair strands with the same sources and targets as S and T intersect each other. Since local moves preserve minimal membranes, we deduce that for two different strands, we cannot have $g(S) = g(T)$. Thus, $n = m$. □

Reduced plabic graphs of essential dimension 2 are the bipartite plabic graphs that can be drawn on the plane as subgraphs of the regular hexagonal lattice.

27. Semisimple membranes

Recall (e.g., see [Hum]), that for a Coxeter element $c \in W$ in the Weyl group associated with root system $R \subset V \simeq \mathbb{R}^r$, there exists a unique 2-dimensional plane $P \subset V$, $P \simeq \mathbb{R}^2$, called the *Coxeter plane,* such that P is c-invariant, and the Coxeter element c acts on P by rotations by $2\pi/h$. Note that the Coxeter element c defines an orientation on the Coxeter plane P, assuming that c acts on P by a clockwise rotation. Let $p : V \to P$ be the orthogonal projection onto the Coxeter plane.

Definition 27.1. Fix a Coxeter element $c \in W$. We say that an R-membrane M is *semisimple* if the projection $p : \langle M \rangle \to p(\langle M \rangle)$ onto the Coxeter plane is a bijective map between $\langle M \rangle$ and $p(\langle M \rangle)$ and each component of the projection $p(\langle L \rangle)$ of the boundary loop L of M is oriented clockwise in the Coxeter plane.

Equivalently, an R-membrane $M = (G, f)$ is semisimple if the orientation of any triangle Δ in the cactus G agrees with the orientation of the projection $p(\langle \Delta \rangle)$ in the Coxeter plane.

Example 27.2. Consider the B_4-membrane in Example 20.4. This membrane is not semisimple: the two faces $(b, b_2, b_3), (b, b_3, b_6)$ are oppositely oriented in the Coxeter plane. Here, the Coxeter plane can be taken to be the span of the real and imaginary parts of $v = (1, \eta, \eta^2, \eta^3)$, where η is a primitive 8-th root of unity.

Remark 27.3. A *simple* membrane M is a semisimple membrane, such that $\langle M \rangle \simeq p(\langle M \rangle)$ is homeomorphic to a disk (or to a line segment when $m = 2$). A semisimple membrane is simple if and only if the graph G^* is connected. Any semisimple membrane is obtained by taking wedges of simple membranes along their boundary vertices.

Let us now discuss the type A case. Assume that $c = (12 \cdots n) \in S_n$ is the standard long cycle in S_n, which is a Coxeter element in type A case. We can identify the corresponding Coxeter plane with $\mathbb{R}^2$ and assume that

$$p : \mathbb{R}^n \to \mathbb{R}^2, \qquad p : e_i \mapsto u_i, \text{ for } i = 1, \dots, n,$$

is the projection that sends the coordinate vectors $e_1, \dots, e_n$ in $\mathbb{R}^n$ to the vertices $u_1, \dots, u_n$ of a regular n-gon in $\mathbb{R}^2$ centered at the origin 0 arranged in the clockwise order.

Recall that a loop $L = (\lambda^{(1)}, \dots, \lambda^{(m)})$, with $\lambda^{(a+1)} - \lambda^{(a)} = e_{i_a} - e_{j_a}$, for $a \in \mathbb{Z}/m\mathbb{Z}$, is called *j-increasing* if $j_1 \le j_2 \le \cdots \le j_m$. In particular, any polypositroid loop is j-increasing.

Theorem 27.4. *Let L be any j-increasing polypositroid loop. A membrane M with boundary loop L is minimal if and only if M is semisimple.*

Proof. Let $M = (G, f)$ be a minimal membrane with boundary loop L. So the plabic graph G^* is reduced. Let Δ be any triangle in the cactus G, let d be the corresponding vertex in the plabic graph G^*, and let S_1, S_2, S_3 be the three strands in G^* that pass through the three edges of G^* adjacent to the vertex d in the directions away from d arranged, respectively, in the clockwise order. Since G^* is a reduced plabic graph, the segments of these three strands between the vertex d and their target points t_1, t_2, t_3 on the boundary of the disk cannot intersect each other. So the three target points t_1, t_2, t_3 of the strands S_1, S_2, S_3, respectively, are arranged in the clockwise order on the boundary of the disk. Thus, the labels

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$g(S_1), g(S_2), g(S_3) \in [n]$ of these three strands (which are $j_{t_1}, j_{t_2}, j_{t_3}$, respectively; see Theorem 21.5) are ordered as $g(S_1) < g(S_2) < g(S_3)$ (up to a cyclic shift).

The triangle Δ is embedded into $\mathbb{R}^n$ either as $\mathrm{conv}(-e_{g(S_1)}, -e_{g(S_2)}, -e_{g(S_3)})$ or as $\mathrm{conv}(e_{g(S_1)}, e_{g(S_2)}, e_{g(S_3)})$ (up to a parallel translation) depending on the color of the triangle. In both cases, the rules of the road imply that the orientation of the triangle Δ in the cactus G agrees with the orientation of the projection $p(\langle\Delta\rangle)$ onto the Coxeter plane. This implies that the membrane M is semisimple.

On the other hand, let us now assume that M is a semisimple membrane and deduce that it should be a minimal membrane. Whenever we apply a local move to M, it remains semisimple. Indeed, for a square move (II), that is, an octahedron move shown on Figure 4, if the upper half of the surface of the octahedron projects bijectively onto the Coxeter plane, then the lower part of the surface projects bijectively onto the Coxeter plane. For a tetrahedron move, that is, a move of type (I) or (III) shown on Figure 4, observe that the four vertices of the two triangles involved in the move project onto the Coxeter plane as some points $u_i + v, u_j + v, u_k + v, u_l + v$. Since $u_1, \ldots, u_n$ are vertices of a convex n-gon, it is impossible that one of these four points lies in the convex hull of the three other points. Thus, if the union of two triangles involved in a move of type (I) or (III) projects bijectively onto the Coxeter plane, then the same remains true after the move.

If M is not minimal, then we can find a sequence of local moves that results in a plabic graph with a pair of parallel edges and a membrane with two coinciding triangles $\langle\Delta\rangle = \langle\Delta'\rangle$. However, in a semisimple membrane, two triangles cannot coincide. Thus, M should be a minimal membrane. □

Remark 27.5. For a positroid $\mathcal{M}$, projections of minimal membranes with boundary loop $L_{\mathcal{M}}$ onto the Coxeter plane are related to the *plabic tilings* of Oh et al. [OPS]. Plabic tilings are certain subdivisions (or tilings) of a polygon on the plane into smaller polygons (or tiles) colored in two colors. The tiles are not necessarily triangles. A plabic tiling corresponds to an equivalence class under moves of types (I) and (III) of projections of minimal membranes with boundary loop $L_{\mathcal{M}}$ onto the Coxeter plane. In other words, a plabic tiling is obtained from a projection of a membrane by combining its adjacent triangles colored in the same color into tiles.

28. Higher octahedron recurrence and cluster algebras

Consider the collection of variables x_λ, labelled by integer vectors $\lambda \in \mathbb{Z}^n$, that satisfy the following *higher octahedron relations*:

$$x_{e_i+e_k+\lambda} \cdot x_{e_j+e_l+\lambda} = x_{e_i+e_j+\lambda} \cdot x_{e_k+e_l+\lambda} + x_{e_i+e_l+\lambda} \cdot x_{e_j+e_k+\lambda}\,, \tag{28.1}$$

for any $i < j < k < l$ in $[n]$, and any $\lambda \in \mathbb{Z}^n$. We will call the recurrence (28.1) the *higher octahedron recurrence*. The polytope $\mathrm{conv}(e_i + e_k, e_j + e_l, e_i + e_j, e_k + e_l, e_i + e_l, e_j + e_k)$ is an octahedron in $\mathbb{R}^n$, which explains the name of the above relations.

Clearly, each relation (28.1) involves only the variables x_λ, for λ in an affine hyperplane $\{\lambda_1 + \cdots + \lambda_n = \mathrm{Const}\}$. So, essentially, (28.1) is a recurrence relation on variables corresponding to points of the $(n-1)$-dimensional integer lattice. For $n = 4$, this recurrence on $\mathbb{Z}^3$ is equivalent to the *octahedron recurrence* (see, for example [Spe]).

Let us now define algebras generated by certain finite subsets of variables x_λ satisfying the octahedron relations.

Definition 28.1. For a loop $L \subset \mathbb{Z}^n$, define its *cloud* as the union of integer lattice points of $\langle M\rangle$ over all minimal membranes M with boundary loop L:

$$\mathrm{cloud}(L) := \bigcup_{M=(G,f)\ \text{min. membr. with bound. } L} \{f(v) \mid v \text{ is vertex of } G\}.$$

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Remark 28.2. According to Corollary 26.7, for a polypositroid loop $L = L_P$, $\mathrm{cloud}(L_P)$ belongs to the set $P \cap \mathbb{Z}^n$ of lattice points of the polypositroid P. For some (poly)positroids, $\mathrm{cloud}(L_P) = P \cap \mathbb{Z}^n$. For example, the equality holds if $P = P_{\mathcal{M}}$, where $\mathcal{M} = \binom{[n]}{k}$ is the uniform matroid. In this case, P is the hypersimplex $\Delta(k,n)$ and $\mathrm{cloud}(L_P) = \{e_I \mid I \in \binom{[n]}{k}\}$ is the set of all lattice points of $\Delta(k,n)$.

However, in general, $\mathrm{cloud}(L_P)$ is not equal to $P \cap \mathbb{Z}^n$. For example, if the loop L is a wedge of line segments, then any minimal membrane with boundary L has no triangles. In this case, $\mathrm{cloud}(L) = L$. Apart from some trivial cases, this set cannot be equal to the set of lattice points of the polypositroid P, which is a convex polytope.

For the case $n = 3$, a generic polypositroid is a hexagon, and the associated cloud is a triangle with line segments attached to its vertices.

Definition 28.3. For a loop L, let $\mathrm{Octa}_L := \mathbb{C}[x_\lambda, x_\mu^{-1}]_{\lambda \in \mathrm{cloud}(L), \mu \in L}$ be the commutative algebra over $\mathbb{C}$ generated by the variables x_λ, for $\lambda \in \mathrm{cloud}(L)$, and x_μ^{-1} for $\mu \in L$, modulo the octahedron relations (28.1). We call Octa_L the *octahedron algebra* of the loop L.

Recall that, for any finite quiver (i.e., a directed graph) Q with a chosen subset of vertices B, there is a cluster algebra, whose initial cluster variables correspond to vertices of Q and frozen cluster variables correspond to the subset of vertices B (see [FZ02]). By convention, we will assume that the inverses of frozen cluster variables belong to the cluster algebra.

Definition 28.4. For a membrane $M = (G, f)$, define the *quiver* of M as the directed graph $Q(M)$ on the same set of vertices Vert as the cactus G, whose edges $u \to v$ are the edges of G that separate triangles of different colors, directed so that the adjacent black triangle is on the right of the edge $u \to v$ and the adjacent white triangle is on the left of the edge $u \to v$.

Let $\mathcal{A}_M$ denote the cluster algebra over $\mathbb{C}$ given by the quiver $Q(M)$ of the membrane M with frozen variables corresponding to the boundary vertices $b_i \in \mathrm{Vert}$ of G.

Theorem 28.5. *Let L be any j-increasing loop (in particular, L can be any polypositroid loop). Let $M = (G, f)$ be any minimal membrane with boundary loop L, and let* Vert *be the vertex set G.*

For any other minimal membrane M' with the same boundary loop L, the quivers $Q(M)$ and $Q(M')$ are mutation equivalent, and we have a canonical isomorphism $\mathcal{A}_M \simeq \mathcal{A}_{M'}$. The octahedron algebra Octa_L *is a (finitely generated) subalgebra of the cluster algebra $\mathcal{A}_M$.*

More explicitly, let us identify the collection of variables $\{x_{f(v)} \mid v \in \mathrm{Vert}\}$ with the initial cluster of $\mathcal{A}_M$. We have

1. *The collection of variables $\{x_{f(v)} \mid v \in \mathrm{Vert}\}$ is an algebraically independent set in* Octa_L*. Any other x_λ, for $\lambda \in \mathrm{cloud}(L)$ is expressed in terms of these variables by Laurent polynomials with positive integer coefficients.*
2. *Local moves of membranes of types (I) and (III) (tetrahedron moves) do not change the set of variables $\{x_{f(v)} \mid v \in \mathrm{Vert}\}$, and they do not change the quiver $Q(M)$ of M and the cluster algebra $\mathcal{A}_M$.*
3. *Local moves of membranes of type (II) (octahedron moves) change exactly one element in the set $\{x_{f(v)} \mid v \in \mathrm{Vert}\}$. They correspond to (a certain class of) mutations of the cluster algebra $\mathcal{A}_M$.*
4. *Any minimal membrane $M' = (G', f')$ with the same boundary loop L is obtained from M by a sequence of local moves. The collection of variables $\{x_{f'(v')} \mid v' \in \mathrm{Vert}'\}$ (where* Vert' *is the set of vertices of G') is a cluster of the cluster algebra $\mathcal{A}_M$.*
5. *The isomorphism $\mu_{M,M'} : \mathcal{A}_M \to \mathcal{A}'_M$ given by the composition of mutations coming from a sequence of local moves connecting the membranes M and M' depends only on the membranes M and M', and it does not depend on a choice of a sequence of local moves connecting the membranes.*
6. *The octahedron algebra* Octa_L *is the subalgebra of the cluster algebra $\mathcal{A}_M$ generated by all cluster variables from all clusters of $\mathcal{A}_M$ (and inverses of frozen variables) that correspond to minimal membranes M' with the same boundary loop L.*

Proof. Part (2) is clear, because tetrahedron moves of membranes do not change the set of points $\{f(v) \mid v \in \mathrm{Vert}\}$, and they do not change the quiver Q of a membrane.

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Observe that, if we apply a move $M \to \tilde{M}$ of type (II) (an octahedron move), that is, apply a square move of the associated plabic graph G^*, then the labels i, j, k, l of the four strands (arranged clockwise) going out of the vertices of the square are ordered as $i < j < k < l$ (up to a cyclic shift) (see the argument in the proof of Theorem 27.4). One easily checks that the quiver $Q(\tilde{M})$ is a mutation of the quiver $Q(M)$. This move results in replacing one element $x_{e_i+e_k+\lambda}$ of the set $\{x_{f(v)} \mid v \in \mathrm{Vert}\}$ by $x_{e_j+e_l+\lambda}$. One checks from the definitions that the transformation:

$$x_{e_i+e_k+\lambda} \to x_{e_j+e_l+\lambda} = (x_{e_i+e_j+\lambda} \cdot x_{e_k+e_l+\lambda} + x_{e_i+e_l+\lambda} \cdot x_{e_j+e_k+\lambda})/x_{e_i+e_k+\lambda},$$

is exactly the cluster mutation of the associated variable in the initial cluster of $\mathcal{A}_M$. So we get part (3).

Part (4) follows from Theorem 23.6 and the fact the that loop L is unimodal.

Part (1) now follows from general results on cluster algebras, namely, Fomin-Zelevinsky's Laurent phenomenon [FZ03] and the positivity result of Lee-Schiffler [LS].

Part (5) follows from the observation that each element of the initial seed of the cluster algebra $\mathcal{A}_{M'}$ corresponds to some variable $x_{f'(v')}$, $v' \in \mathrm{Vert}'$. By part (1), this element is expressed by a Laurent polynomial in terms of the variables $x_{f(v)}$, $v \in \mathrm{Vert}$, corresponding to the initial seed of $\mathcal{A}_M$. This Laurent expression depends only on the membrane M and the integer vector $f'(v') \in \mathbb{Z}^n$, and it does not depend on a choice of a sequence of local moves connecting the membranes M and M'.

Part (6) is clear from the above discussion. □

For a j-increasing loop L, we denote by $\mathcal{A}_L$ the cluster algebra $\mathcal{A}_M$ for a minimal membrane M with boundary loop L. The quiver $Q(M)$, and thus the cluster algebra $\mathcal{A}_M$, of a minimal membrane $M = (G, f)$ depends only on the reduced plabic graph G^*. Since every reduced plabic graph appears in a minimal *positroid* membrane, the class of cluster algebras $\mathcal{A}_L$ is as general as its subclass corresponding to positroid loops. However, the description of these cluster algebras in terms of the higher octahedron recurrence allows us to associate some cluster variables with points of the integer lattice $\mathbb{Z}^n$ and some clusters with membranes, which provides an additional geometrical intuition into the structure of these cluster algebras.

Let $\mathcal{M}$ be a positroid, and let $\mathcal{A}_{L_\mathcal{M}}$ be the cluster algebra for the positroid loop $L_\mathcal{M}$ (see Section 24). The cluster algebra $\mathcal{A}_{L_\mathcal{M}}$ implicitly appeared in [Po06]. It was shown in [GL] that the cluster algebra $\mathcal{A}_{L_\mathcal{M}}$ is isomorphic to the coordinate ring $\mathbb{C}[\mathring{\Pi}_\mathcal{M}]$ of an *open positroid variety* $\mathring{\Pi}_\mathcal{M}$ [KLS], confirming conjectures of Muller–Speyer [MS] and Leclerc [Lec].

Remark 28.6. Suppose that L_1 and L_2 are two j-increasing loops, and $M_1 = (G_1, f_1)$ and $M_2 = (G_2, f_2)$ are minimal membranes with boundary loops L_1 and L_2, respectively. If the reduced plabic graphs G_1^* and G_2^* are connected by the local moves (I), (II), and (III), then it follows from the above remarks that $\mathrm{cloud}(L_1)$ and $\mathrm{cloud}(L_2)$ are naturally in bijection.

Remark 28.7. The cluster algebra $\mathcal{A}_L$ is typically a cluster algebra of infinite type, with infinitely many cluster variables. On the other hand, $\mathrm{cloud}(L)$ is a finite set, corresponding to a finite subset of the cluster variables of $\mathcal{A}_L$, and thus the octahedron algebra Octa_L is a finitely generated subalgebra of the cluster algebra $\mathcal{A}_L$. However, even when $\mathcal{A}_L$ is of infinite type, we may have $\mathrm{Octa}_L = \mathcal{A}_L$. For example, this holds when the reduced plabic graph G^* corresponds to the top positroid cell of $\mathrm{Gr}(k, n)$. In this case, $\mathcal{A}_L$ is isomorphic to the homogeneous coordinate ring of the Grassmannian with the cyclic minors $\Delta_{12\cdots k}, \Delta_{23\cdots(k+1)}, \ldots$ inverted. The equality $\mathrm{Octa}_L = \mathcal{A}_L$ follows from the fact that the homogeneous coordinate ring of the Grassmannian is generated by Plücker coordinates Δ_I.

29. Asymptotic cluster algebra

As we discussed in Section 28, membranes are closely related to a class of cluster algebras generated by some collections of variables satisfying the higher octahedron recurrence. Minimal membranes correspond to certain clusters in these algebras, and local moves of membranes correspond to cluster

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mutations. It would be interesting to investigate the asymptotic behavior of these structures under dilations of the boundary loop.

We can call this area of research the "Asymptotic Cluster Algebra." We anticipate that many results from statistical physics and from asymptotic representation theory (e.g., the study of asymptotics properties of representations of symmetric groups), will have their analogs in the asymptotic cluster algebra.

Under dilations of the boundary loop L, minimal membranes might approach a certain limit surface S.

Let $\mathrm{Memb}(L)$ be the set of all minimal membranes with boundary loop L. For $t \in \mathbb{Z}_{>0}$, let tL denote the loop L dilated t times.

Conjecture 29.1. *Let L be a unimodal loop. For a positive integer t, consider the uniform distribution on the set* $\mathrm{Memb}(tL)$.

There exists a unique surface $S \subset \mathbb{R}^n$ (with boundary $\langle L \rangle$), such that, for any $\epsilon > 0$, there exists $N > 0$, such that, for any $t \geq N$, the probability that $\frac{1}{t}\langle M \rangle$, for $M \in \mathrm{Memb}(tL)$, belongs to the ϵ-neighborhood of S is greater than $1 - \epsilon$.

A related conjecture can be formulated in terms of measures. For a loop L, consider the measure μ_L on the set $\mathrm{cloud}(L) \subset \mathbb{Z}^n$ given by

$$\mu_L(a) = \frac{\#\{M \in \mathrm{Memb}(L) \text{ such that } a \in \langle M \rangle\}}{\#\mathrm{Memb}(L) \times \#(\text{lattice points in any } M \in \mathrm{Memb}(L))},$$

for $a \in \mathbb{Z}^n$. Clearly, $\sum_{a \in \mathbb{Z}^n} \mu_L(a) = 1$, so μ_L is a probability distribution.

Equivalently, $\mu_L(a)$ is the probability that a random (uniformly chosen) minimal membrane with boundary L contains a lattice point a. In other words, μ_L is the density of a random membrane.

For a positive integer t, let $\mu_{L,t}(a) := \mu_{tL}(ta)$. The measure (probability distribution) $\mu_{L,t}$ is supported on a certain subset of the lattice $(\frac{1}{t}\mathbb{Z})^n$.

Conjecture 29.2. *Let L be a unimodular loop. As $t \to \infty$, the measures $\mu_{L,t}$ converge to a certain limit measure*

$$\mu_{L,\infty} := \lim_{t \to \infty} \mu_{L,t}$$

supported on a certain limit surface $S \subset \mathbb{R}^n$.

Remark 29.3. For $n = 4$, semisimple membranes are related to the 6-vertex and 8-vertex models [Bax], whose asymptotic properties have been extensively studied.

Remark 29.4. It would be interesting to investigate a relationship between the "limit membrane" S and Plateau's problem (see Remark 20.7).

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