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Examples of discrete groups of hyperbolic motions conservative but not ergodic at infinity

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Abstract. For every $n \geq 2$, a discrete subgroup of isometries of the hyperbolic n-space, which is conservative but not ergodic on the sphere at infinity, is constructed.

1. *Introduction*

Let G be a discrete group of isometries of the hyperbolic n-space $\mathbb{H}^n$ with $n \geq 2$. When we take $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$ as the model space of $\mathbb{H}^n$, we can visualize the sphere at infinity of $\mathbb{H}^n$ as $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. And the horospherical limit set $L^h(G)$ of G is defined as follows;

$$L^h(G) = \{p \in S^{n-1}: \text{for every horosphere } S \text{ in } B^n \text{ based at } p, \text{ there is a } g \in G \text{ such that } g(0) \in S\},$$

where 0 is the origin of $\mathbb{R}^n$. Recently, D. Sullivan obtained several deep results on this horospherical limit set in [**5**]. In particular, he showed that $L^h(G)$ has full measure on S^{n-1} if and only if G is conservative on S^{n-1} ([**5**, IV. Theorem IV]). If G is ergodic on S^{n-1}, then G is conservative, as he noted in [**5**, IV. Note]. And one may think that ergodicity is equivalent to conservativity (cf. [**1**, Introduction]). But this is false. In fact, the main purpose of this note is to show the following

THEOREM. *For every* $n(\geq 2)$, *there is a discrete group* G *of isometries of* $\mathbb{H}^n$ *which is conservative but not ergodic on* S^{n-1}.

2. *Construction.*

First we note the following[1]

LEMMA 1 ([**4**], § 4). For every n (≥ 2), there is a torsion-free discrete group G_0 of isometries of $\mathbb{H}^n$ such that $M = \mathbb{H}^n / G_0$ is a compact manifold containing mutually disjoint compact (totally geodesic) submanifolds N_1, N_2 and N_3 of codimension one such that $M - (N_1 \cup N_2 \cup N_3)$ is connected.

Remark. When $n = 2$, we can take as M a compact surface of genus 3, and as $\{N_k\}_{k=1}^3$ mutually disjoint loops in a canonical homology base of M.

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Fix such G_0 as in Lemma 1, and let M' be the compact manifold with boundary obtained from $M-(N_1\cup N_2\cup N_3)$ by attaching six borders $\{N_k^+, N_k^-\}_{k=1}^3$ corresponding to $\{N_k\}_{k=1}^3$. Next, consider copies $\{M'(h,i,j)\}_{h,i,j\in\mathbb{Z}}$ of M', and gluing canonically the border N_k^+ of $M'(h,i,j)$ to that of N_k^- of $M'(h+\delta_{1k}, i+\delta_{2k}, j+\delta_{3k})$ for every h, i, j and k, where δ_{mk} is Kronecker's delta. Then we have a complete hyperbolic manifold M_1, which is a torsion-free abelian cover of M of rank 3. Equivalently, we have a normal subgroup G_1 of G_0 such that $M_1=\mathbb{H}^n/G_1$ and G_0/G_1 is a torsion-free abelian group of rank 3.

Next let N be the submanifold of M_1 of codimension one corresponding to N_1^+ of $M'(0,0,0)$, and fix a positive integer K greater than one. Using K copies of M_1-N, we can construct in the same way as above, an abelian cover M_2 of M_1 of order K, i.e. a normal subgroup G_2 of G_1 such that $M_2=\mathbb{H}^n/G_2$ and G_1/G_2 is isomorphic to $\mathbb{Z}/K\cdot\mathbb{Z}$, which is a desired group as is shown in the next section.

3. *Proof of theorem*

Let π_1 be the natural projection of M_2 to M_1 and T be an isometry of M_2 of order K such that $\pi_1\circ T=\pi_1$. Then attaching K points $\{e_k\}_{k=1}^K$ to M_2, we have the Kerékjártó-Stoilow's compactification $\bar{M}_2=M_2\cup(\bigcup_{k=1}^K\{e_k\})$ of M_2 such that T can be extended to an automorphism of $\bar{M}_2$ by setting $T(e_k)=e_{k+1}$ for every k (with $e_{K+1}=e_1$).

Let π_2 be the natural projection of $\mathbb{H}^n$ to $M_2=\mathbb{H}^n/G_2$. Fix $x\in\mathbb{H}^n$, and set

$$E_k(x)=\{p\in S^{n-1}\text{: letting } L(x,p) \text{ be the geodesic ray from } x \text{ tending to } p,\ \pi_2(L(x,p)) \text{ converges to } e_k \text{ in } \bar{M}_2\}$$

for every k. Then we have the following

LEMMA 2. *$E_k(x)$ is a G_2-invariant measurable set not depending on x.*

Proof. First we show that $E_k(x)$ does not depend on x. For any other $x'\in\mathbb{H}^n$, $\pi_2(L(x,p))$ and $\pi_2(L(x',p))$ are mutually asymptotic for every $p\in S^{n-1}$. Hence we can see that $p\in E_k(x)$ if and only if $p\in E_k(x')$.

In particular, $E_k(x)=E_k(g(x))$, or equivalently $E_k(x)=g^{-1}(E_k(x))$ for every $g\in G_2$; and since measurability of $E_k(x)$ is routine, we have the assertion. □

In the sequel, we will write simply E_k instead of $E_k(x)$.

LEMMA 3. *It holds that*

$$E_k\cap E_{k'}=\varnothing \quad \text{if } k\neq k',$$
$$m(E_k)>0 \quad \text{for every } k, \text{ and}$$
$$m\left(S^{n-1}-\bigcup_{k=1}^K E_k\right)=0,$$

where m is the canonical measure on S^{n-1}.

In particular, G_2 is not ergodic on S^{n-1}.

Proof. It is clear that $E_k \cap E_{k'} = \emptyset$ if $k \neq k'$. Let g^* be an element of G_1 such that the class of g^* in G_1/G_2 corresponds to T, then $g^*(E_k) = E_{k+1}$ for every k, hence $m(E_k) > 0$ if and only if it does so for some k.

Now for every $p \in S^{n-1} - \bigcup_{k=1}^{K} E_k$, we can see from the definition that there is a compact set A in M_2 and a sequence $\{x_n\}_{n=1}^{\infty}$ of points on $L(x, p)$ tending to p such that $\pi_2(x_n) \in A$ for every n, which in turn implies that there is a sequence $\{g_m\}_{m=1}^{\infty}$ of mutually distinct elements of G_2 such that the hyperbolic distances between $g_m(0)$ and $L(x, p)$ form a bounded sequence. Hence by definition, p belongs to the conical limit set $L^c(G_2)$ of G_2.

On the other hand, since M_1 admits Green's function by [**3**, Theorem 4], hence since M_2 does so, we can see that $m(L^c(G_2)) = 0$ by [**5**, Corollary III] (cf. [**2**, VII.7 and VII.8 Theorem 1]). Thus we have

$$m\left(S^{n-1} - \bigcup_{k=1}^{K} E_k\right) \leq m(L^c(G_2)) = 0,$$

hence $m(E_k) > 0$ for every k. Since every E_k is G_2-invariant by Lemma 2, G_2 is not ergodic on S^{n-1}. □

Now we say that a G_2-invariant set E on S^{n-1} is non-decomposable if either $m(E') = 0$ or $m(E - E') = 0$ for every G_2-invariant subset E' of E. Then we have the following

LEMMA 4. *Every E_k is non-decomposable.*

Proof. Assume that E_k is decomposable, and let E be a G_2-invariant subset of E_k such that $m(E) > 0$ and $m(E_k - E) > 0$. Set $F = \bigcup_{k=1}^{K} (g^*)^k(E)$, where g^* is as in the proof of Lemma 3, then F is G_1-invariant, $m(F) > 0$ and $m(S^{n-1} - F) > 0$. Then the Poisson's integral

$$f(t) = \int_F ((1 - |t|^2)/|t - y|^2)^{n-1}\, dm(y)$$

induces a non-constant bounded harmonic function on M_1 (cf. [**2**, Theorem V.9]), which contradicts [**3**, Theorem 1]. □

LEMMA 5. $m((S^{n-1} - L^h(G_2)) \cap E_k) = 0$ for every k. *Moreover G_2 is conservative on S^{n-1}.*

Proof. Suppose that $m((S^{n-1} - L^h(G_2)) \cap E_k) > 0$ for some k. Then by [**5**, IV. Theorem III], this set is contained in the dissipative part of G_2, hence is decomposable (which follows at once from the definition of the dissipative part). This contradicts Lemma 4, and we have the first assertion.

Moreover, Lemma 3 and the first assertion imply that $m(S^{n-1} - L(G_2)) = 0$, hence the second assertion follows by [**5**, IV. Theorem IV]. □

Remark. We have constructed a group G_2 such that S^{n-1} is divided into a set of measure zero and K G_2-invariant sets of positive measure. The author conjectures that a group G^* is conservative on S^{n-1} if and only if S^{n-1} is divided into a set of measure zero and (at most countable) non-decomposable G^*-invariant sets of positive measure.

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