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ALDO SCIMONE
Via Costantino Nigra 30,
90141 Palermo, Sicily
e-mail: *aldoscimone@outlook.it*

108.17 On a generalisation of the Lemoine axis

First let us briefly recall the definition of an *Apollonius* circle. Let A, B, P be three distinct points in the Euclidean plane. Then the ratio of the distances from P to A and B respectively is $\frac{|PA|}{|PB|}$. Then the locus of all points X in the plane with the property $\frac{|XA|}{|XB|} = \frac{|PA|}{|PB|}$ is a special *circle* if $\frac{|PA|}{|PB|} \neq 1$ (and, if $\frac{|PA|}{|PB|} = 1$, the perpendicular bisector of AB which can be viewed as a degenerated Apollonius circle with radius ∞). The diameter of this Apollonius circle is QR, where Q and R denote the intersection points of the angle bisector of $\angle APB$ with the straight line AB (Figure 1).

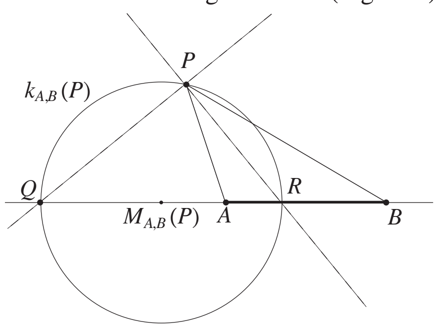

FIGURE 1: Construction of the Apollonius circle

In the following paragraphs we will denote this Apollonius circle by $k_{A,B}(P)$ and its centre by $M_{A,B}(P)$.

There is a well-known property of Apollonius circles, leading to the notion of the *Lemoine axis* (e.g. [1, pp. 294f]): Let ΔABC be a triangle and $k_{A,B}(C)$, $k_{B,C}(A)$ and $k_{C,A}(B)$ the three Apollonius circles through the vertices. Then the centres $M_{A,B}(C)$, $M_{B,C}(A)$ and $M_{C,A}(B)$ of the three circles are collinear (see Figure 2). The corresponding straight line L through these centres is called the *Lemoine axis* and the two intersection points of these circles are called the *isodynamic points* of the triangle ΔABC ($X(15)$ and $X(16)$ in Kimberling's *Encyclopedia of Triangle Centres* [2]).

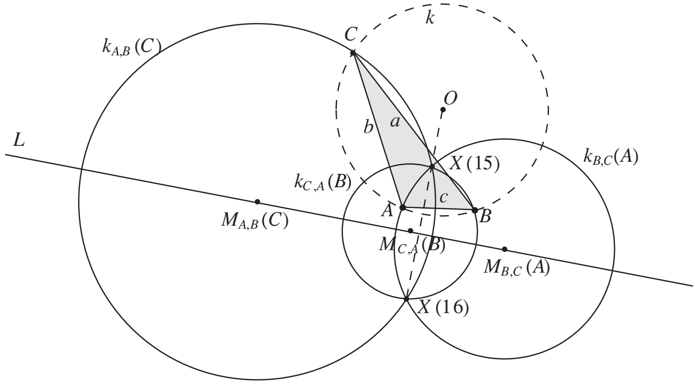

FIGURE 2: The Lemoine axis L of the triangle ABC

One can also see in Figure 2 that all three circles intersect the circumcircle k of ΔABC (dashed in Figure 2) orthogonally. According to that there is a different characterisation of the points $M_{A,B}(C)$, $M_{B,C}(A)$, $M_{C,A}(B)$ (and therefore of the Lemoine axis): for example, $M_{A,B}(C)$ is the intersection point of the straight line through A and B with the tangent of the circumcircle k at C. Furthermore, it is well-known that *every* Apollonius circle with respect to two fixed points intersects *every* other circle through these points orthogonally, not only the circumcircle ([3, p. 21], [4, p. 69]). Let us call this phenomenon "Lemma 1".

A short time ago we were interested in properties of the distances of a point P to the triangle vertices A, B, C as triangle coordinates, more exactly the ratio of these distances $\rho_A : \rho_B : \rho_C$ (see Figure 3).

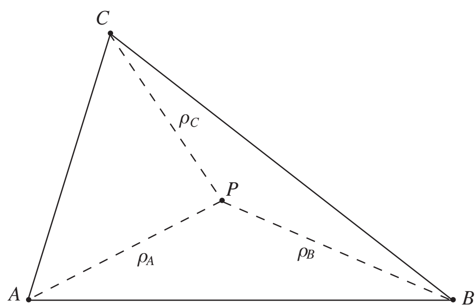

FIGURE 3: The distances ρ_A, ρ_B, ρ_C

It was therefore natural for us to construct the three corresponding Apollonius circles through P ($k_{A,B}(P)$, $k_{B,C}(P)$ and $k_{C,A}(P)$) and to look what happens.

In what follows we assume that P does not lie on a perpendicular bisector of a triangle side because otherwise the corresponding circle degenerates to a straight line (the perpendicular bisector, which can be viewed as a circle with infinite radius with its centre being a point at infinity).

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The result was the following (Figure 4) –

Theorem:

(1) The three Apollonius circles through P intersect each other exactly in two points: P, P'.

(2) The centres $M_{A,B}(P)$, $M_{B,C}(P)$, $M_{C,A}(P)$ of these three circles are collinear (like the Lemoine axis mentioned above). This straight line is the perpendicular bisector of PP'.

(3) For any of the three Apollonius circles, consider the straight line through its two intersection points with the circumcircle of ΔABC. These three straight lines intersect each other in one single point Q somewhere on the line segment PP'.

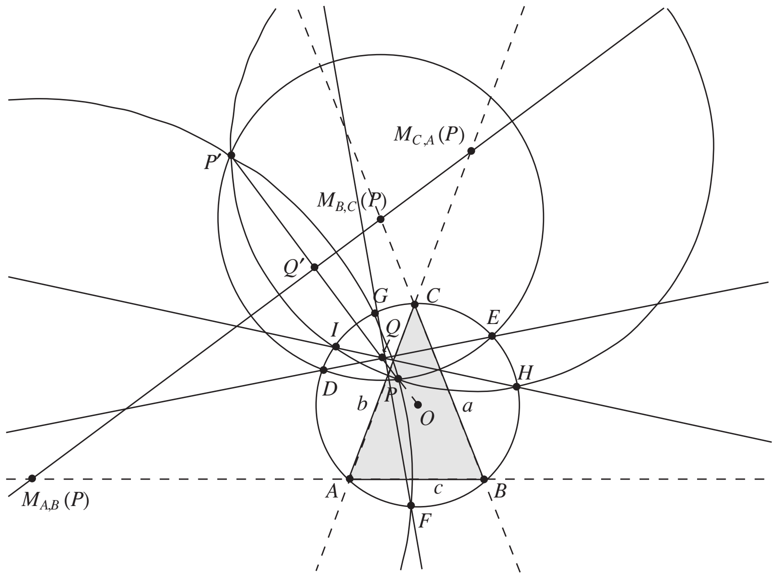

FIGURE 4: Sketch to the above Theorem

Since we initially could not find any observation of these facts in the literature we decided to write this short note. In the meantime, we know that the phenomena (1) and (2) have already been described by [5] (p. 9, in the chapter called *Olympiad Problems and More Applications*), but there other means are used (Menelaus' Theorem and its converse). If a reader knows further related references, please let the authors know.

Up to now we have not found any references for the point of view to consider the straight line through $M_{A,B}(P)$, $M_{B,C}(P)$, $M_{C,A}(P)$ as a generalisation of the Lemoine axis* (see below as well), so we decided to continue writing this short note.

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* Again: If a reader happens to know such references, please inform the authors.

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Proof:

(1) Let $P' \neq P$ be the other (than P) intersection point of the two Apollonius circles $k_{A,B}(P)$, $k_{B,C}(P)$ and ρ_A', ρ_B', ρ_C' the distances of P' to the points A, B, C. Then we have on the one hand $\frac{\rho_A}{\rho_B} = \frac{\rho_A'}{\rho_B'}$ and on the other hand $\frac{\rho_B}{\rho_C} = \frac{\rho_B'}{\rho_C'}$. This implies $\frac{\rho_A}{\rho_C} = \frac{\rho_A'}{\rho_C'}$ and thus $P' \in k_{C,A}(P)$.

(2) If three circles pass through two distinct points $P \neq P'$ then it is immediately clear that all three centres of the circles must lie on the perpendicular bisector of PP'.

(3) PP' is the common power line of the three Apollonius circles. DE (see Figure 4) is the *power line* of the circumcircle and the Apollonius circle centred at $M_{B,C}(P)$. Let Q be the intersection point of DE and PP'. Then Q has the same power with respect to all four circles: the circumcircle and the three Apollonius circles. Hence Q must also lie on FG (and HI, respectively), because FG contains all the points which have the same power with respect to the circumcircle and the Apollonius circle centred at $M_{A,B}(P)$ (and analogously with HI).

In fact, with the same argument as above, (3) holds not only for the circumcircle of ΔABC but also for any circle intersecting the three Apollonius circles. Q is simply the identical (!) *power centre* (*radical centre*) of any two of the three Apollonius circles and the fourth circle intersecting the three Apollonius circles.

Remarks:

- In case P coincides with the circumcentre, all three Apollonius circles degenerate to straight lines, all three "centres" are points at infinity, and also the "other" intersection point P' becomes a point at infinity. In case P lies on exactly one perpendicular bisector of a triangle side, only one Apollonius circle degenerates to a straight line with a point at infinity as its "centre", but in this case the other point P' is a real point in the Euclidean plane.
- Since the Apollonius circles intersect the circumcircle of ΔABC orthogonally it is clear that P' is the *inverse* point of P with respect to the circumcircle.

In order to establish a name for the straight line through $M_{A,B}(P)$, $M_{B,C}(P)$, $M_{C,A}(P)$ we suggest to call it the ***P*-Lemoine axis of the triangle** ΔABC.

Finally, we formulate a lemma concerning the aspect of **generalisation** mentioned in the title of this Note.

Lemma: The term ***P*-Lemoine axis of the triangle** ΔABC is a generalisation of the term **Lemoine axis**, because:

$$P\text{-Lemoine axis of } \Delta ABC = \text{Lemoine axis of } \Delta ABC \Leftrightarrow P \in \{X(15), X(16)\}.$$

Proof: The implication ⇐ is evident (Figures 2 and 4). In order to show the implication ⇒ we can argue as follows: Since the P-Lemoine axis is always orthogonal to the straight line PP' (Figure 4) passing through the circumcentre O, to satisfy the property on the left-hand side of the Lemma P must lie on the straight line $OX(15)$. Since P and P' are *inverse* points with respect to the circumcircle it is easy to see that the midpoint of PP' (i.e. Q' in Figure 4) is the nearer to the circumcircle the nearer P lies to it (in other words: on the line $OX(15)$ the distance of Q' to the circumcircle increases *monotonically* with the distance of P from the circumcircle, in both directions). Therefore, there are exactly two possible positions for P, namely $X(15)$ and $X(16)$.

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HANS HUMENBERGER, FRANZ EMBACHER

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University of Vienna,
Faculty of Mathematics,
Oskar-Morgenstern-Platz 1,
A – 1090 Vienna
e-mails: *hans.humenberger@univie.ac.at,*
franz.embacher@univie.ac.at

108.18 A one-line proof of the Finsler-Hadwiger inequality

Every proof is a one-line proof if you start sufficiently far to the left, [1].

The *Finsler-Hadwiger inequality* asserts that, in the triangle ABC with side-lengths a, b, c and area Δ,

$$\sum a^2 \geqslant \sum (b - c)^2 + 4\sqrt{3}\Delta, \tag{1}$$

with equality if, and only if, triangle ABC is equilateral.

In this short Note, we adapt an idea from [2] to give a very quick proof.