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PSEUDO HARMONIC MEASURES AND THE DIRICHLET PROBLEM

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1. Introduction. For the case of plane regions bounded by finitely many disjoint Jordan curves, the solution of the Dirichlet problem can be expressed in terms of the classical harmonic measure of boundary arcs. At an appropriate stage in the development it is, in fact, useful to observe that the existence of such harmonic measures is equivalent to solvability of the Dirichlet problem (although one subsequently proves that all such regions are Dirichlet regions). We propose here to carry over this order of ideas to a quite general setting, in which arbitrary regions and ideal boundary structures are allowed. The counterparts of the classical harmonic measures of arcs are then harmonic functions with analogous boundary properties, but they no longer appear as measures in the boundary sets, in general. We shall refer to them as "pseudo harmonic measures". Our main result shows how pseudo harmonic measures can be used to solve the Dirichlet problem.

Throughout the discussion, Ω will be taken as a region in Euclidean space or on a Riemann surface, and Γ will be assumed to be an ideal boundary which compactifies Ω. That is, Γ is a set disjoint from Ω such that $\Omega \cup \Gamma$, together with a specified topology, is a compact Hausdorff space in which Ω is dense and the induced topology on Ω is the same as its original topology. We shall be concerned with the Dirichlet problem for Ω relative to Γ, and a general discussion of this subject can be found in [**2**].

A function $\omega_K(z)$, defined for all points z on Ω and all compact subsets K of Γ, will be called a *strict pseudo harmonic measure* for Ω relative to Γ if it has the following properties:

(1) $\omega_K(z)$ is a bounded harmonic function of z over Ω (for fixed K) and an increasing function of compact subsets K of Γ (for fixed z),

(2) $\omega_K(z) \to 1$ as z tends to any interior point of K relative to Γ,

(3) $\omega_K(z) \to 0$ as z tends to any exterior point of K relative to Γ.

Strict pseudo harmonic measures are obviously direct analogues of the classical harmonic measures of arcs. Although the latter are finitely additive, they are not measures in the usual sense, and the same is true of strict pseudo harmonic measures. Let us look briefly at the relationship between strict pseudo harmonic measures and harmonic measures.

Suppose that Ω is a Dirichlet region relative to Γ, so that the strict Dirichlet problem on Ω can be solved for arbitrary continuous functions f on Γ. The

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solution $H_f(z)$ defines a positive linear functional in f, for fixed z, and the Riesz theorem therefore yields a unique Borel measure m_z on Γ such that

$$H_f(z) = \int_\Gamma f dm_z. \tag{4}$$

The resulting function $m_z(E)$ is a harmonic function of z over Ω (for fixed E) and a measure in E over the family of Borel subsets of Γ (for fixed z). This is the definition of *harmonic measure* (or, more properly, *strict harmonic measure*, since we are dealing with Dirichlet regions for the moment). It is inherent in the definition that harmonic measure yields an integral representation for the solution of the Dirichlet problem.

A strict harmonic measure $m_z(E)$ gives rise to a corresponding strict pseudo harmonic measure $\omega_K(z) = m_z(K)$ ($K \subset \Gamma$ compact). This has been noted in [**1**] for the case of a bounded plane region with Euclidean boundary, and it is easily proved in the general case. Indeed, property (1) follows in the usual way by using the fact that (4) defines H_f as a harmonic function for all measurable f and that it reduces to $m_z(K)$ when f is taken as the characteristic function of K. (See, e.g., [**2**, pp. 85–88].) We know also that Γ, as a compact Hausdorff space, must be completely regular (see [**3**, p. 238]). Thus, if ζ_0 is an interior point of K, then there exists a continuous function f on Γ such that $f(\zeta_0) = 1$ and f is dominated by the characteristic function of K. Property (2) follows easily from this, and property (3) is established by a dual argument.

Even in the setting of Dirichlet regions, however, there exist strict pseudo harmonic measures which are fundamentally different from harmonic measures. For example, fixing $0 < \alpha < 1$, we can define $\omega_K(z)$ as the strict pseudo harmonic measure given by

$$\omega_K(z) = m_z(K^\circ) + \alpha m_z(\partial K) \quad (z \in \Omega), \tag{5}$$

where K is any compact subset of Γ and the interior K° and boundary ∂K of K are taken relative to Γ. As evident here, it is readily seen in general that

$$m_z(K^\circ) \leqq \omega_K(z) \leqq m_z(K). \tag{6}$$

Thus, in particular,

$$0 \leqq \omega_K(z) \leqq 1, \tag{7}$$

although this property is obvious directly in our original setting.

2. Solution of the Dirichlet problem by means of strict pseudo harmonic measures. There is plainly no hope in trying to devise an analogue of the integral representation (4) in terms of strict pseudo harmonic measures. The measure properties of m_z are used in an essential way. Nevertheless, an integral representation can be formulated along different lines, as we proceed to show.

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THEOREM 1. *The existence of a strict pseudo harmonic measure $\omega_K(z)$ is necessary and sufficient for the region Ω with boundary Γ to be a Dirichlet region. The solution of the Dirichlet problem on Ω for a given nonnegative continuous function f on Γ is then given by*

$$H_f(z) = \int_0^\infty \omega_{[f \geqq \lambda]}(z) d\lambda \quad (z \in \Omega). \tag{8}$$

Proof. The infinite upper limit of integration in (8) can be replaced by $M = \max f$, since the integrand clearly vanishes for $\lambda > M$. Furthermore, the integrand is monotone decreasing, in view of the set-monotoneity hypothesis in (1). It follows that the right-hand member in (8) exists as a Riemann integral. Let us denote this integral by $U(z)$, i.e.

$$U(z) = \int_0^M \omega_{[f \geqq \lambda]}(z) d\lambda \quad (z \in \Omega).$$

The resulting function U is bounded and harmonic on Ω. Indeed, boundedness is clear from (7), and harmonicity can be deduced in the usual way, by using uniform boundedness of the Riemann sums to infer that these form a normal family of harmonic functions converging pointwise to U.

Our objective is to prove that U tends to f at all points of Γ, and this can be accomplished as follows. We fix ζ_0 as any point of Γ and put $\lambda_0 = f(\zeta_0)$. In terms of λ_0, we split up the integral for U as

$$U(z) = \int_0^{\lambda_0} \omega_{[f \geqq \lambda]}(z) d\lambda + \int_{\lambda_0}^M \omega_{[f \geqq \lambda]}(z) d\lambda. \tag{9}$$

Now, for $0 \leqq \lambda < \lambda_0$, the point ζ_0 lies interior to $[f \geqq \lambda]$ relative to Γ, so $\omega_{[f \geqq \lambda]}(z) \to 1$ as $z \to \zeta_0$, by (2). An application of the Lebesgue bounded convergence theorem then shows that the first integral in (9) tends to λ_0 as $z \to \zeta_0$. In the same way, for $\lambda_0 < \lambda \leqq M$, the point ζ_0 lies exterior to $[f \geqq \lambda]$ relative to Γ, forcing $\omega_{[f \geqq \lambda]}(z) \to 0$ as $z \to \zeta_0$, and we conclude that the second integral in (9) tends to zero. It follows that U tends to f on Γ, i.e., that the solution of the Dirichlet problem is given by formula (8).

Existence of a strict pseudo harmonic measure $\omega_K(z)$ is thus a sufficient condition for Ω to be a Dirichlet region relative to the boundary Γ. That this condition is also necessary has already been noted in § 1 (every Dirichlet region admits a strict harmonic measure), and the proof is complete.

3. Some extensions. At least as important in modern potential theory as the Dirichlet problem, is the Wiener-Dirichlet problem. For the latter, the solution is only required to be a bounded harmonic function which tends to the given continuous boundary function at the regular boundary points. The prevalence of this more general viewpoint accounts for the terminology "strict" pseudo harmonic measure when the limiting behavior in (2) and (3) is required to hold at all points of Γ. We show here how these concepts and results carry over to the setting of the Wiener-Dirichlet problem, and we indicate some extensions of the integral representation (8).

The ideal boundary Γ will now be assumed to be equipped with a set R of *regular* boundary points. Axiomatically, all that is required of R is that it is a subset of Γ with the property that the only bounded harmonic function on Ω which tends to zero on R is the function identically zero. We shall call Ω a *Wiener-Dirichlet region* relative to Γ and R provided that, for each continuous function f on Γ, there exists a bounded harmonic function H_f on Ω such that H_f tends to f at all regular points of Γ. (Uniqueness of H_f is a consequence of the hypothesis on the set R.) The resulting harmonic function H_f is referred to as the *Wiener function* for f relative to Ω, Γ, and R. By a *pseudo harmonic measure* for Ω relative to the boundary Γ and regular points R we mean a function $\omega_K(z)$ defined as before but with the limiting properties in (2) and (3) only required to hold at the regular boundary points.

Exactly the same argument as for Theorem 1 serves to prove the following counterpart for the Wiener-Dirichlet setting.

THEOREM 2. *The existence of a pseudo harmonic measure $\omega_K(z)$ is necessary and sufficient for the region Ω with boundary Γ and regular points R to be a Wiener-Dirichlet region. For any nonnegative continuous function f on Γ, the Wiener function H_f is then given in terms of pseudo harmonic measure by the integral formula* (8).

We remark that the strict Dirichlet setting is contained in the Wiener-Dirichlet setting, since one can always put $R = \Gamma$. It therefore suffices to deal with the Wiener-Dirichlet setting, and our concluding observations will apply to this case.

Let f be any nonnegative continuous function on Γ and λ_0 any positive real number. Then the least harmonic majorant of $(H_f - \lambda_0)^+$ has the integral representation

$$M(H_f - \lambda_0)^+(z) = \int_{\lambda_0}^{\infty} \omega_{[f\geqq\lambda]}(z)d\lambda \quad (z \in \Omega). \tag{10}$$

The proof is essentially contained in that of the original theorem, and, alternatively, (10) can be deduced as a corollary of (8). We omit the details.

Our final observation concerns the behavior of the function

$$U(z) = \int_0^{\infty} \omega_{[f\geqq\lambda]}(z)d\lambda \quad (z \in \Omega), \tag{11}$$

when the function f is required only to be upper semicontinuous and nonnegative on Γ. Clearly, formula (11) defines U as a nonnegative bounded harmonic function on Ω. Arguments of the sort used in proving Theorem 1 show easily that the inequality

$$\limsup_{z\to\zeta_0} U(z) \leqq f(\zeta_0) \tag{12}$$

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holds at all regular boundary points ζ_0 and that

$$\lim_{z \to \zeta_0} U(z) = f(\zeta_0) \tag{13}$$

holds at all regular boundary points ζ_0 which are points of continuity of f.

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