J. Austral. Math. Soc. (Series A) 44 (1988), 146–150

AN INEQUALITY IMPLICIT FUNCTION THEOREM

KUNG-FU NG

(Received 3 September 1986; revised 23 February 1987)

Communicated by B. Mond

Abstract

Let f be a continuous function, and u a continuous linear function, from a Banach space into an ordered Banach space, such that $f - u$ satisfies a Lipschitz condition and u satisfies an inequality implicit-function condition. Then f also satisfies an inequality implicit-function condition. This extends some results of Flett, Craven and S. M. Robinson.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 46 A 30, 26 B 10, 46 A 40, 49 B 27.

Following Rockafellar [13], by a convex process is meant a map T of points in a Banach space X into the subsets of another Banach space Y such that $0 \in T0$, $T(\lambda x) = \lambda Tx$ and $Tx_1 + Tx_2 \subseteq T(x_1 + x_2)$ for all $\lambda > 0$, x_1, x_2 and x in X. This is the case if and only if the graph $\mathcal{G}(T)$ of T is a convex cone in $X \times Y$. T is a *closed* convex process if $\mathcal{G}(T)$ is a closed convex cone. If T is also onto Y (in the sense that for each $y \in Y$ there exists $x \in X$ such that $y \in Tx$) then it is an open mapping (see [10, Theorem 2] and also [5, page 182], [8, Theorem 1]), that is, there exists a constant $k > 0$ with the following property: for each $y \in Y$ there is $x \in X$ with $||x|| \leq k||y||$ such that $y \in Tx$. (In this case we say that T is k-open.)

Suppose K is a closed convex cone in Y. Then, for any continuous linear map u from X into Y, we can associate a closed convex process U by putting

$$U(x) = u(x) + K \qquad (x \in X).$$

Thus, if U is onto Y, then U is k-open for some $k > 0$. The following Theorems 1 and 2 were proved by Flett [4, Lemmas 1 and 3] in the special case that $K = \{0\}$ (see also Craven [2], and [3, page 147]).

THEOREM 1. *Let U be k-open for some $k > 0$. Let f be a continuous (not necessarily linear) map from a subset of D of X containing 0 into Y such that $f(0) = 0$ and*

$$\|\{f(x_1) - u(x_1)\} - \{f(x_2) - u(x_2)\}\| \leq (\eta/k)\|x_1 - x_2\| \tag{1}$$

for some $\eta \in (0,1)$ and all $x_1, x_2 \in D$. If $z \in X$ and D contains the ball B with centre z and radius R with $R > (\eta/(1-\eta))\|z\|$, then there exists $x \in B$ such that $u(z) \in f(x) + K$.

The proof is based on the following contraction lemma, essentially due to Robinson [11] who considered Hausdorff distance ρ_H instead of unbalanced d (our proof is also simpler then that given in [11]). See also [7]. For subsets A, B of a metric space (Ω, ρ) and $x \in \Omega$, we define $d(x, B) := \inf\{\rho(x,b) \colon b \in B\}$, $d(A, B) := \sup\{d(a, B) \colon a \in A\}$, and $\rho_H(A, B) := \max\{d(A,B), d(B,A)\}$.

LEMMA 1. *Let (Ω, ρ) be a complete metric (or semi-metric) space, and let $T \colon \Omega \to 2^\Omega$ satisfy*

$$d(Tx_1, Tx_2) \leq \eta\rho(x_1, x_2) \tag{2}$$

for some $\eta \in (0,1)$ and all x_1, x_2 in a subset D of Ω. Suppose D contains a ball B with centre x_0 and radius $R > d(x_0, Tx_0)/(1-\eta)$. Then there exists $x \in B$ with $x \in \overline{Tx}$.

PROOF. Take $\varepsilon > 0$ such that $R > d(x_0, Tx_0)/(1-\eta) + \varepsilon$, and let $\sigma = d(x_0, Tx_0) + \varepsilon(1-\eta)$. Since $d(x_0, Tx_0) < \sigma$, there exists $x_1 \in Tx_0$ such that $\rho(x_0, x_1) < \sigma$. By (2),

$$d(x_1, Tx_1) \leq d(Tx_0, Tx_1) \leq \eta\rho(x_0, x_1) < \eta\sigma,$$

so there is $x_2 \in Tx_1$ such that $\rho(x_1, x_2) < \eta\sigma$. Suppose that $x_1, \dots, x_n$ from B have been selected respectively from $Tx_0, \dots, Tx_{n-1}$ such that $\rho(x_{k-1}, x_k) < \eta^{k-1}\sigma$ for all $k \leq n$. Then, since

$$d(x_n, Tx_n) \leq d(Tx_{n-1}, Tx_n) \leq \eta\rho(x_{n-1}, x_n) < \eta^n\sigma,$$

one can select $x_{n+1} \in Tx_n$ such that $\rho(x_n, x_{n+1}) < \eta^n\sigma$. Note that $\rho(x_0, x_{n+1}) < \sigma(1 + \eta + \cdots + \eta^n) < \sigma/(1-\eta) = d(x_0, Tx_0)/(1-\eta) + \varepsilon$; in particular $x_{n+1} \in B$. In this way, we have a Cauchy sequence, which converges, say to v. Then $d(x_0, v) \leq d(x_0, Tx_0)/(1-\eta) + \varepsilon$ so $v \in B$. The proof that $v \in \overline{Tv}$ is similar to [10]: take $\gamma > 0$ and a positive integer n. Then there is $y \in Tv$ such that $\rho(x_n, y) < d(x_n, Tv) + \gamma$ so

$$\rho(x_n, y) < d(Tx_{n-1}, Tv) + \gamma \leq \eta\rho(x_{n-1}, v) + \gamma$$

and

$$d(v, Tv) \leq \rho(v, y) \leq \rho(v, x_n) + \rho(x_n, y) \leq \rho(v, x_n) + \eta\rho(x_{n-1}, v) + \gamma.$$

Letting $n \to \infty$ and $\gamma \to 0$, we see that $v \in \overline{Tv}$.

We now turn to the proof of Theorem 1. We shall apply Lemma 1 to $\Omega = X$ with ρ the usual metric induced by the norm. The inverse U^{-1} of the multivalued function U is defined by

$$U^{-1}y = \{x \in X : y \in Ux\} \qquad (y \in Y).$$

By assumption each $U^{-1}y$ is non-empty. We will show that

$$(3) \qquad d(U^{-1}y_1, U^{-1}y_2) \leq k||y_1 - y_2|| \qquad (y_1, y_2 \in Y).$$

In fact, let $x_1 \in U^{-1}y_1$. Since U is k-open, there is $x \in X$ with $||x|| \leq k||y_2 - y_1||$ such that $y_2 - y_1 \in Ux$. Then

$$y_2 = (y_2 - y_1) + y_1 \in u(x) + K + u(x_1) + K = u(x + x_1) + K = U(x + x_1)$$

because K is a convex cone. Therefore $x + x_1 \in U^{-1}y_2$, and

$$d(x_1, U^{-1}y_2) \leq \rho(x_1, x + x_1) = ||x|| \leq k||y_2 - y_1||.$$

Since x_1 is arbitrary in $U^{-1}y_1$, (3) is proved.

Now define T on D by $Tw = U^{-1}(g(w))$ where $g(w) := u(z) - f(w) + u(w)$. By (1), we have, for all $w_1, w_2 \in D$, that

$$||g(w_1) - g(w_2)|| = ||\{f(w_2) - u(w_2)\} - \{f(w_1) - u(w_1)\}|| \leq \eta/k||w_1 - w_2||;$$

it follows from (3) that $d(Tw_1, Tw_2) \leq \eta||w_1 - w_2||$. Moreover, since $g(0) = u(z)$, $z \in U^{-1}(u(z)) = T0$, we have

$$d(z, Tz) \leq d(T0, Tz) \leq \eta||z - 0|| = \eta||z||.$$

By the Contraction Lemma, there exists $x \in B$ such that $x \in \overline{Tx}$. Take a sequence $\{x_n\}$ in Tx convergent to x. Then $g(x) \in U(x_n) = u(x_n) + K$, that is,

$$u(z) - f(x) + u(x) \in u(x_n) + K.$$

Since K is closed it follows that $u(z) \in f(x) + K$.

THEOREM 2. *Let C be a closed convex cone in Y, and Q a subset of Y such that $Q + C \subseteq Q$ and $\lambda Q \subseteq Q$ for all $\lambda \in [0, 1]$. Let f be a C^1-function at 0 from an open set in X containing 0 into Y, with $f(0) = 0$ and $f'(0) = u$. Define U by $U(x) = u(x) - C$ for all $x \in X$. If U is onto Y, then $U^{-1}(Q)$ is contained in the tangent cone of $f^{-1}(Q)$ at 0.*

PROOF. It is known that U is k-open for some $k > 0$ as noted before. Let $h \in U^{-1}(q)$ with $||h|| = 1$ and $q \in Q$. Then $q \in U(h) = u(h) - C$ so $u(h) \in C + Q \subseteq Q$

https://doi.org/10.1017/S1446788700029736 Published online by Cambridge University Press

and consequently $u(\lambda h) \in Q$ for all $\lambda \in [0,1]$. Take $\eta \in (0,1)$; then there exists $\xi > 0$ such that $||f'(x) - u|| \leq \eta/k$ for all x in ξB_X the ξ-ball with centre 0 in X. By the Mean Value Theorem, (1) of Theorem 1 holds with $D := \xi B_X$. Take $\lambda > 0$, small enough that D contains the open ball with centre λh and radius $2\eta\lambda/(1-\eta)$. Applying Theorem 1 there is $x \in X$ with $||x-\lambda h|| \leq 2\eta||\lambda h||/(1-\eta)$ such that $u(\lambda h) \in f(x) - C$, that is, $f(x) \in u(\lambda h) + C \subseteq Q + C \subseteq Q$.

Do the above for all $\eta = 1/n$ with integers $n > 3$ and choose $\lambda = \lambda_n > 0$ such that $\lambda_n \to 0$ as $n \to \infty$; we write x_n for x accordingly constructed above. Note that $x_n \neq 0$, $x_n \in f^{-1}(Q)$, $x_n \to 0$ and

$$\begin{aligned} ||x_n||x_n||^{-1} - \lambda_n h||\lambda_n h||^{-1}|| &\leq 2||x_n - \lambda_n h||\,||\lambda_n h||^{-1} \\ &\leq 4\eta/(1-\eta) \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

where we have used the elementary inequality $||a||a||^{-1} - b||b||^{-1}|| \leq 2||a-b||\,||b||^{-1}$ for non-zero elements in a normed space, which is true because

$$||(a||b|| - a||a|| - b||a|| + a||a||)(||a||\,||b||)^{-1}|| \leq 2||a||\,||b-a||(||a||\,||b||)^{-1}.$$

Therefore h is in the tangent cone of $f^{-1}(Q)$ at 0.

REMARK. A related result has been given by Robinson [12, Corollary 2] where he considered the case $Q = C$. Applications of results of this type to Optimization Theory, have been given in [1], [2], [3], [4], [6], [9], [12] and [14].

References

[1] P. J. Bender, 'Nonlinear programming in normed linear spaces', *J. Optimization Theory Appl.* **24** (1978), 263–285.

[2] B. D. Craven, 'A generalization of Lagrange multipliers', *Bull. Austral. Math. Soc.* **3** (1970), 353–362.

[3] B. D. Craven, *Mathematical programming and control theory*, (Chapman and Hall, 1978).

[4] T. M. Flett, 'On differentiation in normed vector spaces', *J. London Math. Soc.* **42** (1967), 523–533.

[5] G. Jameson, *Ordered linear spaces*, (Springer-Verlag, 1970).

[6] D. Luenberger, *Optimization by vector space method*, (John Wiley, 1968).

[7] S. B. Nadler, Jr., 'Multi-valued contraction mappings', *Pacific J. Math.* **30** (1969), 475–488.

[8] K. F. Ng, 'An open mapping theorem', *Math. Proc. Cambridge Philos. Soc.* **74** (1973), 61–66.

[9] K. F. Ng and D. Yost, 'Quasi-regularity in optimization', *J. Austral. Math. Soc. Ser. A* **41** (1986), 188–192.

[10] S. M. Robinson, 'Normed convex processes', *Trans. Amer. Math. Soc.* **174** (1972), 127–140.

[11] S. M. Robinson, 'An inverse-function theorem for a class of multivalued functions', *Proc. Amer. Math. Soc.* **41** (1973), 211–218.

[12] S. M. Robinson, 'Stability theory for systems of inequalities, Part II: differentiable nonlinear systems', *SIAM J. Numer. Anal.* **13** (1976), 497–513.

[13] R. T. Rockafellar, 'Monotone processes of convex and concave type', *Mem. Amer. Math. Soc.* **77** (1967).
[14] J. Zowe and S. Kurcyusz, 'Regularity and stability for the mathematical programming problem in Banach spaces', *Appl. Math. Optim.* **5** (1979), 49–62.

Department of Mathematics
The Chinese University of Hong Kong
Hong Kong

https://doi.org/10.1017/S1446788700029736 Published online by Cambridge University Press