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ON GRADED C^*-ALGEBRAS

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Abstract

We show that every topological grading of a C^*-algebra by a discrete abelian group is implemented by an action of the compact dual group.

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Suppose that A is an algebra over a field K and G is a group. We say that A is *G-graded* if there are linear subspaces $\{A_g : g \in G\}$ such that A is the direct sum of the A_g and $a \in A_g$, $b \in A_h$ imply $ab \in A_{gh}$. Then each element of A has a unique decomposition as a sum $a = \sum_{g\in G} a_g$ of homogeneous components $a_g \in A_g$ (and all but finitely many $a_g = 0$). We have known since the first paper on the subject that the Leavitt path algebras $L_K(E)$ of a directed graph E are $\mathbb{Z}$-graded [1, Lemma 1.7].

For graph C^*-algebras, the field K is always $\mathbb{C}$. The graph algebra $C^*(E)$ is not graded in the algebraic sense and the role of the grading in the general theory is played by a *gauge action* γ of the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ on $C^*(E)$. We can use this action to define homogeneous components of $a \in C^*(E)$ by

$$a_n := \int_0^1 \gamma_{e^{2\pi it}}(a)e^{-2\pi int}\,dt \quad \text{for } n \in \mathbb{Z}.$$

But $\{n : a_n \neq 0\}$ can be infinite and then the relationship between a and the sequence $\{a_n\}$ is well-known to be analytically subtle (see [12], for example).

In the recent book [2], the authors show that $C^*(E)$ is always graded in a weaker sense introduced by Exel [3]. He defined a C^*-algebra A to be *G-graded* if there is a family $\{A_g : g \in G\}$ of linearly independent closed subspaces such that $a \in A_g$ and $b \in A_h$ imply $ab \in A_{gh}$ and $a^* \in A_{g^{-1}}$, and such that A is the norm-closure of $\bigoplus_{g\in G} A_g$. It is proved in [2, Proposition 5.2.11] that every graph algebra $C^*(E)$ is $\mathbb{Z}$-graded in Exel's sense.

In fact, the result in [2] says rather more than this. Exel also introduced a stronger notion: a G-graded C^*-algebra A is *topologically graded* if there is a bounded linear map $F : A \to A$ which is the identity on A_e and vanishes on every A_g with $g \neq e$

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[4, Section 19]. (His original [3, Definition 3.4] looks a little stronger, since it asserts that F is a conditional expectation. But it follows from [3, Theorem 3.3] or [4, Theorem 19.1] that this extra requirement is automatic.) The extra information in [2, Proposition 5.2.11] implies that the $\mathbb{Z}$-grading of $C^*(E)$ is topological, and this information is obtained using the gauge action of $\mathbb{T}$.

In this paper, we revisit gradings of C^*-algebras. We work with topological gradings by an abelian group G, because that is enough to cover the $\mathbb{Z}^k$-graded graph algebras of higher-rank graphs and their twisted analogues. We show that every topological G-grading of a C^*-algebra A is implemented by a natural action of the Pontryagin dual $\widehat{G}$, which in the case of a graph algebra $C^*(E)$ is the usual gauge action of $\mathbb{T} = \widehat{\mathbb{Z}}$. We then use recent results on the C^*-algebras of Fell bundles [13] to reconstruct an arbitrary element of a topologically $\mathbb{Z}^k$-graded algebra from its graded components.

We begin by discussing a couple of illustrative examples from Exel's book [4].

Example 1. We consider the graph E with one vertex v and one loop e. The graph algebra $C^*(E)$ has identity P_v and is generated by the unitary element S_e. Because graph algebras are universal for Cuntz–Krieger families, this graph is universal for C^*-algebras generated by a unitary element, and hence $(C^*(E), S_e)$ is $(C(\mathbb{T}), z)$. The gauge action of $\mathbb{T}$ is implemented by rotations and, for $n \in \mathbb{Z}$, the graded components $C^*(\mathbb{T})_n$ are the scalar multiples of the polynomials z^n. Since these polynomials form an orthonormal basis for $L^2(\mathbb{T})$ and $C(\mathbb{T}) \subset L^2(\mathbb{T})$, the Fourier coefficients

$$\widehat{f}(n) = \int_{\mathbb{T}} f(z) z^{-n}\, dz := \int_0^1 f(e^{2\pi it}) e^{-2\pi int}\, dt$$

of $f \in C(\mathbb{T})$ determine f uniquely: $\widehat{f}(n) = \widehat{g}(n)$ for all n implies $f = g$ in $C(\mathbb{T})$. It has long been known that the Fourier series of f need not converge in the norm of the ambient C^*-algebra $C(\mathbb{T})$, but a classical theorem of Féjer (1900) tells us that the Césaro means of the partial sums of the Fourier series converge uniformly to f on $\mathbb{T}$. Thus we can recover f from its Fourier coefficients and, provided we remember that this recovery process is not the obvious one, we can view $C(\mathbb{T})$ as a $\mathbb{Z}$-graded algebra.

Example 2 (Motivated by the discussion following [4, Proposition 19.3]). We take a closed subset X of $\mathbb{T}$, which is infinite but not all of $\mathbb{T}$, and consider $C(X)$. We write e_n for the polynomial z^n, viewed as an element of $C(\mathbb{T})$. Then, as observed in [4], the subspaces

$$C(X)_n := \{c e_n|_X : c \in \mathbb{C}\}$$

are linearly independent (because X is infinite). Because the e_n span a dense subspace of $C(\mathbb{T})$ and $f \mapsto f|_X$ is a surjection of $C(\mathbb{T})$ onto $C(X)$, the direct sum $\bigoplus_{n\in\mathbb{Z}} C(X)_n$ is dense in $C(X)$. Thus the $C(X)_n$ give a $\mathbb{Z}$-grading of $C(X)$ in the sense of [3, 4].

Since X is a proper closed subset of $\mathbb{T}$, and the map $f \mapsto f|_X$ has infinite-dimensional kernel isomorphic to $C_0(\mathbb{T}\backslash X)$, each $f \in C(X)$ has many extensions g in $C(\mathbb{T})$. Each such extension g has a canonical sequence of homogeneous components $\widehat{g}(n)e_n$ and the Césaro means for this sequence converge uniformly in $C(\mathbb{T})$ to g. The restrictions of

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the Césaro means to X converge uniformly in $C(X)$ to $g|_X = f$. But different extensions of f have different Fourier coefficients, and hence there is no canonical choice of homogeneous components for f in $C(X)$.

Example 2 shows that a $\mathbb{Z}$-graded C^*-algebra need not have the properties one would expect of a grading. So Exel also considered his stronger notion of 'topological grading', in which the bounded linear map $F : A \to A_e$ gives a continuous choice of homogeneous component $a_e := F(a)$. In the discussion in [4, Section 19], he proves that the algebra $C(X)$ in Example 2 is not topologically graded. Our main result says that for a topologically G-graded C^*-algebra, the map F is implemented by integration of a continuous action of the compact dual group $\widehat{G}$ with respect to the normalised Haar measure.

Theorem 3. *Suppose that G is an abelian group and that A is a C^*-algebra which is topologically G-graded in Exel's sense. Then there is a strongly continuous action α of $\widehat{G}$ on A such that $\alpha_\gamma(a) = \gamma(g)a$ for $a \in A_g$, and then*

$$F(a) = \int_{\widehat{G}} \alpha_\gamma(a)\, d\gamma \quad \textit{for all } a \in A.$$

The subspaces $\{A_g : g \in G\}$ in the G-grading form a *Fell bundle B* over G. There is an extensive theory of Fell bundles, originally developed by Fell (he called them *C^*-algebraic bundles* [5]), and revisited by several authors in the 1990s. We shall lean heavily on results of Exel [3], as presented in his recent monograph [4].

Each Fell bundle B over a (discrete) group G has an enveloping C^*-algebra $C^*(B)$ that is universal for a class of Hilbert-space representations, consisting of linear maps $\pi_g : A_g \to B(H)$ such that $\pi_g(a)\pi_h(b) = \pi_{gh}(ab)$ and $\pi_g(a)^* = \pi_{g^{-1}}(a^*)$, and such that π_e is a nondegenerate representation of A_e. There is also a reduced C^*-algebra $C^*_{\mathrm{r}}(B)$ which is generated by a regular representation [4, Section 17]. Because we are interested in Fell bundles over abelian groups, all our Fell bundles are amenable in Exel's sense [4, Theorem 20.7] and $C^*(B) = C^*_{\mathrm{r}}(B)$.

Example 4. A G-graded algebra can be quite different from the C^*-algebra of its Fell bundle. To see this, consider the Fell bundles B_1 and B_2 over $\mathbb{Z}$ associated to the gradings of $C(\mathbb{T})$ in Example 1 and $C(X)$ in Example 2. The maps $ce_n \mapsto ce_n|_X$ are Banach-space isomorphisms of the fibres $B_{1,n}$ onto the fibres $B_{2,n}$ (both are one-dimensional) and respect the Fell-bundle structure. Since $C(\mathbb{T})$ is topologically graded (on any graph algebra there is a map $a \mapsto a_0$ defined by averaging over the gauge action), we have $C^*(B_1) = C(\mathbb{T})$. Thus we also have $C^*(B_2) = C(\mathbb{T})$.

Proof of Theorem 3. Because A is topologically graded there is a bounded linear map $F : A \to A$ such that $f(a) = a$ for $a \in A_e$ and $f(a) = 0$ for $a \in A_g$ with $g \neq e$. Let B be the corresponding Fell bundle over G with fibres A_g. From [4, Theorem 19.5], there are surjections ϕ of $C^*(B)$ onto A and ψ of A onto the reduced algebra $C^*_{\mathrm{r}}(B)$ such that $\psi \circ \phi$ is the regular representation of $C^*(B)$. Since the group G is abelian, the Fell bundle is amenable and the regular representation is an isomorphism. Hence so are ϕ

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and ψ. We deduce that A is generated by a representation ρ of B in A, and that (A, ρ) is universal for Hilbert-space representations of B.

We now fix $\gamma \in \widehat{G}$. For each $g \in G$, we define $\alpha_{\gamma,g} : A_g \to A$ by $\alpha_{\gamma,g}(a) = \gamma(g)a$. Since $|\gamma(g)| = 1$, $\alpha_{\gamma,g}$ is a linear and isometric embedding of the Banach space A_g in A. Since each A_g is a left Hilbert module over A_e, the action of A_e on A_g is nondegenerate [14, Corollary 2.7], and since $A = \overline{\bigoplus_g A_g}$, it follows that any approximate identity for A_e is also an approximate identity for A. Thus $\alpha_{\gamma,e}$ is nondegenerate. For $a \in A_g$ and $b \in A_h$,

$$\alpha_{\gamma,g}(a)\alpha_{\gamma,h}(b) = (\gamma(g)a)(\gamma(h)b) = \gamma(gh)ab = \alpha_{\gamma,gh}(ab)$$

and

$$u_{\gamma,g}(a)^* = (\gamma(g)a)^* = \overline{\gamma(g)}a^* = \gamma(g^{-1})a^* = \alpha_{\gamma,g^{-1}}(a^*).$$

Thus $\alpha_\gamma = \{\alpha_{\gamma,g}\}$ is a representation of the Fell bundle B, and the universal property of $A = C^*(B)$ gives a nondegenerate homomorphism $\alpha_\gamma : A \to A$ such that $\alpha_\gamma \circ \rho_g = \alpha_{\gamma,g}$ for $g \in G$.

For $\gamma, \chi \in \widehat{G}$, $\alpha_\gamma \alpha_\chi = \alpha_{\gamma\chi}$ on each A_g, and hence also on $A = \overline{\bigoplus_g A_g}$. Since α_1 is the identity on A, it follows that each α_γ is an isomorphism, and that $\gamma \mapsto \alpha_\gamma$ is a homomorphism of $\widehat{G}$ into the automorphism group $\operatorname{Aut} A$. Since convergence in the dual of a discrete abelian group is pointwise convergence, the map $\gamma \mapsto \alpha_\gamma(a)$ is continuous for each $a \in A_g$ and hence, by an $\epsilon/3$ argument, for all $a \in A = \overline{\bigoplus A_g}$. Thus α is a strongly continuous action of $\widehat{G}$ on A.

Now averaging with respect to the normalised Haar measure on $\widehat{G}$ gives a conditional expectation E of A onto the fixed-point algebra A^α such that

$$E(a) = \int_{\widehat{G}} \alpha_\gamma(a)\, d\gamma \quad \text{for all } a \in A$$

(following the discussion for $\widehat{G} = \mathbb{T}$ in the first few pages of [11, Ch. 3], for example). Since $\alpha_\gamma(a) = a$ for $a \in A_e$ and we are using the normalised Haar measure, $E(a) = a$ for $a \in A_e$. For $a \in A_g$ with $g \neq e$,

$$E(a) = \int_{\widehat{G}} \gamma(g)a\, d\gamma = \Big(\int_{\widehat{G}} \gamma(g)\, d\gamma \Big) a = 0.$$

Thus $E = F$ on $\bigoplus A_g$, and hence by continuity of E and F also on the closure A. □

Since E is a faithful conditional expectation, we deduce that F is too.

Corollary 5. *The bounded linear map $F : A \to A_e$ in Theorem 3 is a conditional expectation onto A_e, and is faithful in the sense that $F(a^*a) = 0$ implies $a = 0$.*

As we remarked earlier, Exel also proved directly in [3] that F is a conditional expectation.

REMARK 6. We have concentrated on Fell bundles over abelian groups because our motivation for looking at this material came from graph algebras, where the appropriate group is $G = \mathbb{Z}^k$. However, the first paragraph of the proof of Theorem 3 works for arbitrary amenable groups. Then we can use the universal property of $C^*(B)$ to construct a coaction $\delta : A \to A \otimes C^*(G)$ such that $\delta(a) = a \otimes u_g$ for $a \in A_g$ (see the preliminary material in [13, Appendix B]). The group algebra $C^*(G)$ has a trace τ characterised by $\tau(1) = 1$ and $\tau(u_g) = 0$ for $g \neq e$, and hence there is a slice map $\mathrm{id} \otimes \tau : A \otimes C^*(G) \to A$. Composing gives a contraction $E := (\mathrm{id} \otimes \tau) \circ \delta$ of A onto

$$A^\delta := \{a \in A : \delta(a) = a \otimes 1\}.$$

Again, $A^\delta = A_e$ and $E = F$.

When G is not amenable, Theorem 19.5 of [4] only tells us that A lies somewhere between $C^*(B)$ and $C^*_{\mathrm{r}}(B)$. For $A = C^*(B)$, we can use the coaction of the previous paragraph. If $A = C^*_{\mathrm{r}}(B)$, then we can use spatial arguments to construct a reduced coaction on A (see [9, Example 2.3(6)] and [10]). But in general, trying to construct suitable coactions on A seems likely to pose rather delicate problems in nonabelian duality.

We now return to the case of an abelian group G and the set-up of Theorem 3. The action $\alpha : \widehat{G} \to \operatorname{Aut} A$ allows us to construct homogeneous components

$$a_g := \int_{\widehat{G}} \alpha_\gamma(a)\overline{\gamma(g)}\, d\gamma \quad \text{for } a \in A \text{ and } g \in G.$$

For $a \in A_h$,

$$a_g = \int_{\widehat{G}} \alpha_h(a)\overline{\gamma(g)}\, d\gamma = \int_{\widehat{G}} \gamma(hg^{-1})a\, d\gamma = \begin{cases} a & \text{if } g = h \\ 0 & \text{if } g \neq h. \end{cases}$$

Comparing this with the formula in [4, Corollary 19.6], we see that a_g is the same as Exel's Fourier coefficient $F_g(a)$.

Since our motivation came from applications to graph algebras, we are particularly interested in $\mathbb{Z}^k$-graded C^*-algebras. Besides the usual graph algebras of directed graphs, for which $k = 1$, this includes the higher-rank graph algebras of [6] and the twisted higher-rank graph algebras of [7, 8] (which by [13, Corollary 4.9] can be realised as the C^*-algebras of Fell bundles over $\mathbb{Z}^k$). For all these graph algebras, the action of the dual $\mathbb{T}^k$ given by Theorem 3 is the usual gauge action.

When $G = \mathbb{Z}^k$, the dual is $\mathbb{T}^k$, and Theorem 3 gives us an action α of $\mathbb{T}^k$ on A. We then define the homogeneous components of $a \in A$ by

$$a_n = \int_{\mathbb{T}^k} \alpha_z(a)z^{-n}\, dz \quad \text{for } n \in \mathbb{Z}^k. \tag{1}$$

Now [13, Proposition B.1] tells us how to recover a from its homogeneous components a_n. More precisely, we have the following corollary.

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COROLLARY 7. *Suppose that a C^*-algebra A is $\mathbb{Z}^k$-graded in Exel's sense. Suppose also that there is a bounded linear map $F : A \to A_e$ such that $F|_{A_g} = 0$ for $g \neq e$ and $F|_{A_e}$ is the identity. For $a \in A$ and $n \in \mathbb{Z}^k$, define the homogeneous components a_n using (1). For $m, n \in \mathbb{Z}^k$, we write $m \leq n$ to mean $n - m \in \mathbb{N}^k$, and set*

$$s_n(a) := \sum_{-n \leq m \leq n} a_m \quad \text{for } n \in \mathbb{N}^k$$

and

$$\sigma_N(a) := \frac{1}{\prod_{j=1}^{k}(N_j + 1)} \sum_{0 \leq n \leq N} s_n(a) \quad \text{for } N \in \mathbb{N}^k.$$

Then $\|\sigma_N(a) - a\| \to 0$ as $N \to \infty$ in $\mathbb{N}^k$.

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