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ON NAGUMO'S CONDITION

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1. **Background.** The classical uniqueness theorem of Nagumo [1] for ordinary differential equations is as follows.

THEOREM. *If $f(t, y)$ is continuous on $0 \leq t \leq 1$, $-\infty < y < \infty$ and if*

$$|f(t, y)-f(t, x)| \leq \frac{1}{t}|x-y|,$$

then there is at most one solution to the initial value problem $y'=f(t, y)$, $y(0)=0$.

In this paper we attempt to establish uniqueness criteria when $1/t$ is replaced by $1/t^2$ in Nagumo's theorem. Following the approach favoured by Hille [2, Ch. 1], we shall approach the problem from the standpoint of integral inequalities.

2. **A uniqueness theorem.** We require the following lemma.

LEMMA. *If* (i) $f(t)$ *is continuous and nonnegative in* $[0, 1]$,

(ii) $f(t) \leq \int_0^t \frac{1}{s^2} f(s)\, ds$,

(iii) $f(t) = o(e^{-1/t})$, *as* $t \to 0$,

then $f(t) \equiv 0$.

Proof. Set $F(t)=\int_0^t 1/s^2 f(s)\, ds$. Differentiating and using (ii) we obtain for $t>0$,

$$F'(t) = \frac{f(t)}{t^2} \leq \frac{F(t)}{t^2},$$

$$F'(t) - \frac{F(t)}{t^2} \leq 0,$$

$$\frac{d}{dt}(e^{1/t}F(t)) \leq 0,$$

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so that $e^{1/t}F(t)$ is nonincreasing. Choose $\varepsilon>0$. Then from (iii), we have for small t,

$$e^{1/t}F(t) = e^{1/t}\int_0^t \frac{1}{s^2} f(s)\,ds$$
$$\leq e^{1/t}\int_0^t \frac{\varepsilon}{s^2} e^{-(1/s)}\,ds = \varepsilon.$$

Therefore $\lim_{t\to 0+} e^{1/t}F(t)=0$ and so $e^{1/t}F(t)\leq 0$ for $t>0$. The result now follows from (i) and the definition of $F(t)$.

Suppose now that $f(t, y)$ is defined and continuous on $0\leq t\leq 1$, $-\infty<y<\infty$ and satisfies $|f(t, x)-f(t, y)|\leq(1/t^2)\,|x-y|$. Then if x and y solve $x'=f(t, x)$, $x(0)=0$, we have

$$|x(t)-y(t)| \leq \int_0^t \frac{1}{s^2}\,|x(s)-y(s)|\,ds.$$

From the lemma it follows that for that class of solutions of the initial value problem for which differences of solutions x, y satisfy

$$x(t)-y(t) = o(e^{-(1/t)}), \quad \text{as } t\to 0$$

there is uniqueness.

THEOREM. *Let $f(t, y)$ be continuous on $0\leq t\leq 1$, $-\infty<y<\infty$, and satisfy the conditions*

$$|f(t, y)-f(t, x)| \leq \frac{1}{t^2}\,|x-y|,$$
$$f(t, y) = o(e^{-(1/t)}t^{-2}), \quad \text{as } t\to 0$$

uniformly for $0\leq y\leq\delta$, $\delta>0$ arbitrary. Then $y'=f(t, y)$, $y(0)=0$ has at most one solution.

Proof. If x, y satisfy the initial value problem, then from the first condition

$$|x(t)-y(t)| \leq \int_0^t \frac{1}{s^2}\,|x(s)-y(s)|\,ds.$$

Choose $\varepsilon>0$. Then from the second condition, we have for small t,

$$|x(t)-y(t)| \leq \int_0^t |f(s, x(s))-f(s, y(s))|\,ds$$
$$\leq \varepsilon\int_0^t e^{-(1/s)}s^{-2}\,ds = \varepsilon e^{-(1/t)}.$$

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From the remark preceding the theorem, $x(t)=y(t)$ for all t.

3. **Examples.** As an example of this result consider the following function.

$$f(t, y) = \begin{cases} \dfrac{e^{-(1/t)}}{t} + e^{-(1/t)}, & y \geq te^{-(1/t)} \\ \dfrac{y}{t^2} + e^{-(1/t)}, & 0 \leq y \leq te^{-(1/t)} \\ e^{-(1/t)}, & y \leq 0 \end{cases}$$

$f(t, y)$ is continuous on $0 \leq t \leq 1$, and it is easily checked to satisfy the growth conditions of the theorem. The unique solution of $y' = f(t, y)$, $y(0)=0$, is $y(t) = te^{-1/t}$.

The assumption $f(t, x) = o(e^{-1/t}\, t^{-2})$ is necessary in the theorem, as the following example shows. Each of $y(t) = ce^{-1/t}$, $0 \leq c \leq 1$ solves

$$y' = f(t, y) = \begin{cases} 0, & y \leq 0 \\ \dfrac{y}{t^2}, & 0 \leq y \leq e^{-(1/t)} \\ \dfrac{e^{-(1/t)}}{t^2}, & y \geq e^{-(1/t)}. \end{cases}$$

But for $y \geq e^{-1/t}$, $f(t, y)e^{1/t}t^2 = 1$.

Finally we remark that although the function y/t is admissible in the general Kamke uniqueness theorem, so that Nagumo's theorem is implied by Kamke's theorem, the function y/t^2 is not admissible.

References

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