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To the memory of André Haefliger

Abstract

The Haefliger–Thurston conjecture predicts that Haefliger's classifying space for C^r-foliations of codimension n whose normal bundles are trivial is $2n$-connected. In this paper, we confirm this conjecture for piecewise linear (PL) foliations of codimension 2. Using this, we use a version of the Mather–Thurston theorem for PL homeomorphisms due to the author to derive new homological properties for PL surface homeomorphisms. In particular, we answer the question of Epstein in dimension 2 and prove the simplicity of the identity component of PL surface homeomorphisms.

1. Introduction

Haefliger defined the notion of C^r-Haefliger structures on manifolds which are more flexible than C^r-foliations to be able to construct a classifying space $B\Gamma_n^r$ for them [Hae71, Bot72]. This space is the classifying space of the étale groupoid Γ_n of germs of local C^r-diffeomorphisms of $\mathbb{R}^n$. For $r > 0$, there is a map

$$\nu \colon B\Gamma_n^r \to \mathrm{BGL}_n(\mathbb{R})$$

which classifies the normal bundle to the C^r-Haefliger structures and the homotopy fiber of ν is denoted by $\overline{B\Gamma}_n^r$.

The work of Haefliger [Hae71] and Thurston's h-principle theorems [Thu74b, Thu76] say that if the normal bundle of a C^r-Haefliger structure γ on a manifold M can be embedded into the tangent bundle TM, then there is a genuine foliation in the homotopy class of γ. Hence, in principle, the classification of foliations on M is translated into the homotopy type of the mysterious space $\overline{B\Gamma}_n^r$. Haefliger proved that $\overline{B\Gamma}_n^r$ is n-connected and Thurston proved [Thu74a] that the identity component of the smooth diffeomorphism group of any compact manifold is a simple group and used it to show that $\overline{B\Gamma}_n^\infty$ is $(n+1)$-connected; shortly after, Mather [Mat74, § 7] proved the same statement for $\overline{B\Gamma}_n^r$ for all regularities r except $r = n + 1$.

The theory of differentiable cohomology for groupoids developed by Haefliger made him speculate that $\overline{B\Gamma}_n^r$ might be $2n$-connected, and Thurston also stated [Thu74a] this range of connectivity for $\overline{B\Gamma}_n^r$ as a conjecture.

CONJECTURE 1.1 (Haefliger–Thurston). *The space $\overline{\mathrm{B\Gamma}}_n^r$ is $2n$-connected.*

A geometric consequence of this conjecture [Thu74b, Thu76] is that any subbundle of the tangent bundle of a smooth M whose dimension is at most $(\dim(M) + 1)/2$ is C^r-integrable up to homotopy, that is, one can change it up to homotopy to become the tangent field of a C^r-foliation on M. As a consequence of Mather's acyclicity result [Mat71] and McDuff's theorem [McD80], we know that $\overline{\mathrm{B\Gamma}}_n^0$ is contractible, and it is a consequence of the remarkable theorem of Tsuboi [Tsu89] that $\overline{\mathrm{B\Gamma}}_n^1$ is also contractible. But for regularity $r > 1$, because of the existence and nontriviality of Godbillon–Vey invariants, it is known that $\overline{\mathrm{B\Gamma}}_n^r$ is not $(2n + 1)$-connected.

In this paper, we consider the piecewise linear (PL) category instead and we prove the analog of this conjecture for PL foliations of codimension 2. To formulate the conjecture in this category, we shall first define $\overline{\mathrm{B\Gamma}}_n^{\mathrm{PL}}$. Let Γ_n^{PL} be the étale groupoid of germs of local orientation-preserving PL homeomorphisms of $\mathbb{R}^n$. The classifying space $\mathrm{B\Gamma}_n^{\mathrm{PL}}$ classifies codimension n PL Haefliger structures that are co-oriented up to concordance [Hae71, § 2]. Another perspective is that it classifies foliated PL microbundles of dimension n [Hae70, p. 188, Proposition] or [Tsu09, § 4]. On the other hand, the Kuiper–Lashof theorem implies that oriented PL microbundles of dimension n are classified by $\mathrm{BPL}^+(\mathbb{R}^n)$, where $\mathrm{PL}^+(\mathbb{R}^n)$ is the realization of the simplicial group of orientation-preserving PL homeomorphisms of $\mathbb{R}^n$ [KL66]. So forgetting the germ of the foliation near the zero section of microbundles induces a map

$$\nu \colon \mathrm{B\Gamma}_n^{\mathrm{PL}} \to \mathrm{BPL}^+(\mathbb{R}^n).$$

This map classifies the normal microbundle of the PL Haefliger structures. Let $\overline{\mathrm{B\Gamma}}_n^{\mathrm{PL}}$ be the homotopy fiber of the above map ν. Haefliger's argument [Hae71, § 6] implies that $\overline{\mathrm{B\Gamma}}_n^{\mathrm{PL}}$ is $(n - 1)$-connected. He proved in [Hae70, Theorem 3] that Phillips' submersion theorem in the smooth category implies that $\overline{\mathrm{B\Gamma}}_n^r$ is n-connected for $r > 0$. Given that Phillips' submersion theorem also holds in the PL category [HP64], one could argue similarly to the smooth case to show that $\overline{\mathrm{B\Gamma}}_n^{\mathrm{PL}}$ is in fact n-connected. The analog of the Haefliger–Thurston conjecture in the PL category is that the classifying space $\overline{\mathrm{B\Gamma}}_n^{\mathrm{PL}}$ is $2n$-connected. Unlike the smooth case, it is not even known whether $\overline{\mathrm{B\Gamma}}_n^{\mathrm{PL}}$ is $(n + 1)$-connected for all n. Our main theorem is about the connectivity of this space for $n = 2$.

THEOREM 1.2. *The space $\overline{\mathrm{B\Gamma}}_2^{\mathrm{PL}}$ is 4-connected.*

As a consequence of this theorem, we prove new homological properties of PL surface homeomorphisms.

1.1 Applications
Let M be a compact connected PL n-manifold possibly with nonempty boundary. Let $\mathrm{PL}(M, \mathrm{rel}\, \partial)$ be the group of PL homeomorphisms of M which agree with the identity on an open neighborhood of the boundary and let $\mathrm{PL}_0(M, \mathrm{rel}\, \partial)$ be the identity component of $\mathrm{PL}(M, \mathrm{rel}\, \partial)$. Epstein [Eps70, Theorem 3.1] considered the abstract group $G = \mathrm{PL}_0^{\delta}(M, \mathrm{rel}\, \partial)$ and showed that the commutator subgroup $[G, G]$ is a simple group. Hence, to prove that G is simple, it is enough to show that it is perfect. He then proved that $\mathrm{PL}([0, 1], \mathrm{rel}\, \partial)$ and $\mathrm{PL}_0(S^1)$ are perfect by observing that in dimension 1, PL homeomorphisms are generated by certain 'typical elements' and those typical elements can be easily written as commutators. To generalize his argument to higher dimensions, he suggested the following approach [Eps70, p. 173].

DEFINITION 1.3. Let B be a ball in $\mathbb{R}^n$. It is PL homeomorphic to $S^{n-2} \star [0,1]$, the join of S^{n-2} with $[0,1]$. Note that for PL manifolds M and N, a PL homeomorphism of N extends naturally to a PL homeomorphism of the join $M \star N$. A *glide* homeomorphism of the ball B is a PL homeomorphism that is induced by the extension of a compactly supported PL homeomorphism of $(0,1)$ to a PL homeomorphism of $S^{n-2} \star [0,1]$. For a PL n-manifold M, a glide homeomorphism $h \colon M \to M$ is the extension by the identity of a glide homeomorphism supported in a PL embedded ball $B \hookrightarrow M$.

Question (Epstein). Is $\mathrm{PL}_0^\delta(M, \mathrm{rel}\,\partial)$ generated by glide homeomorphisms?

The affirmative answer to this question implies that $\mathrm{PL}_0^\delta(M, \mathrm{rel}\,\partial)$ is simple, and conversely, since the group generated by glide homeomorphisms is a normal subgroup of $\mathrm{PL}_0^\delta(M, \mathrm{rel}\,\partial)$, if the group $\mathrm{PL}_0^\delta(M, \mathrm{rel}\,\partial)$ is simple, it is generated by glide homeomorphisms.

THEOREM 1.4. *Let Σ be an oriented compact surface possibly with a boundary. Then the group $\mathrm{PL}_0^\delta(\Sigma, \mathrm{rel}\,\partial)$ is simple.*

The *simplicity* of $\mathrm{PL}_0^\delta(M, \mathrm{rel}\,\partial)$ in all dimensions is still open. We use Theorem 1.2 and the version of the *Mather–Thurston theorem* that the author proved in [Nar23, §5] for PL homeomorphisms to prove the perfectness of this group in dimension 2, and as a consequence, we answer affirmatively Epstein's question in dimension 2.

Note that this line of argument is the opposite of Thurston's point of view in the smooth category where he first proved the perfectness of the identity component of smooth diffeomorphism groups to improve the connectivity of the space $\overline{\mathrm{B\Gamma}}_n^\infty$. Our argument gives a homotopy-theoretic proof of the perfectness of $\mathrm{PL}_0^\delta(M, \mathrm{rel}\,\partial)$ for a compact surface M. It would be still interesting to find a direct algebraic proof and study the commutator length of PL surface homeomorphisms.

Recall that the perfectness of a group is equivalent to the vanishing of its first group homology. We in fact determine the group homology of PL surface homeomorphisms up to degree 2. By Hauptvermutung in dimension 2, any two PL structures on a surface Σ are PL homeomorphic [Moi77, Chapter 8]. Hence, the homotopy type of $\mathrm{PL}(\Sigma, \mathrm{rel}\,\partial)$ does not depend on the choice of the PL structure on the surface Σ. Moreover, we have the weak equivalence $\mathrm{PL}(\Sigma, \mathrm{rel}\,\partial) \simeq \mathrm{Homeo}(\Sigma, \mathrm{rel}\,\partial)$ [BH81, p. 8].

THEOREM 1.5. *Let Σ be a compact orientable surface. Then the natural map*

$$\mathrm{BPL}_0^\delta(\Sigma, \mathrm{rel}\,\partial) \to \mathrm{BPL}_0(\Sigma, \mathrm{rel}\,\partial)$$

induces an isomorphism on $H_*(-;\mathbb{Z})$ for $* \leq 2$ and surjection on $* = 3$.

Since $\mathrm{BPL}_0(\Sigma, \mathrm{rel}\,\partial)$ is simply connected, this theorem implies that $\mathrm{PL}_0^\delta(\Sigma, \mathrm{rel}\,\partial)$ is a perfect group. Therefore, Theorem 1.5 proves the perfectness of PL homeomorphisms of surfaces without following Epstein's strategy through glide homeomorphisms. On the other hand, by Epstein [Eps70, Theorem 3.1], the perfectness for the group $\mathrm{PL}_0^\delta(\Sigma, \mathrm{rel}\,\partial)$ implies that it is also *simple*. Given that the group generated by glides is a normal subgroup, we can also answer Epstein's question about glide homeomorphisms.

In fact, the homotopy type of the topological group $\mathrm{PL}_0(\Sigma, \mathrm{rel}\,\partial)$ is completely determined, so the second group homology of $\mathrm{PL}_0^\delta(\Sigma, \mathrm{rel}\,\partial)$ can also be determined.

Our second application is about the invariants of flat surface bundles with transverse PL structures. We first show that the powers of the universal Euler class are all nontrivial in $H^*(\mathrm{B\Gamma}_2^{\mathrm{PL}}; \mathbb{Z})$ and we use it to prove the following result.

THEOREM 1.6. *Let Σ be a compact orientable surface. Then the map*

$$H^*(\mathrm{BPL}(\Sigma, \mathrm{rel}\,\partial); \mathbb{Q}) \to H^*(\mathrm{BPL}^\delta(\Sigma, \mathrm{rel}\,\partial); \mathbb{Q})$$

induces an injection when $* \le (2g(\Sigma) - 2)/3$ where $g(\Sigma)$ is the genus of the surface Σ.

As a consequence of the Madsen–Weiss theorem [MW07], $H^*(\mathrm{BPL}(\Sigma, \mathrm{rel}\,\partial); \mathbb{Q})$ is isomorphic to the polynomial ring $\mathbb{Q}[\kappa_1, \kappa_2, \dots]$ in the stable range, $* \le (2g(\Sigma) - 2)/3$. Here κ_i are certain characteristic classes of surface bundles known as ith Miller–Morita–Mumford (MMM) classes whose degree is $2i$.

COROLLARY 1.7. *The κ_i are all nontrivial in $H^*(\mathrm{BPL}^\delta(\Sigma, \mathrm{rel}\,\partial); \mathbb{Q})$ as long as $i \le (2g(\Sigma) - 2)/6$.*

This is in contrast to the case of smooth diffeomorphisms. It is known by the Bott vanishing theorem [Mor87, Theorem 8.1] that κ_i vanishes in $H^*(\mathrm{BDiff}^\delta(\Sigma, \mathrm{rel}\,\partial); \mathbb{Q})$ for all $i > 2$, and Kotschick and Morita [KM05] proved that κ_1 is nontrivial in $H^2(\mathrm{BDiff}^\delta(\Sigma, \mathrm{rel}\,\partial); \mathbb{Q})$ as long as $g(\Sigma) \ge 3$. However, it is still open [KM05] whether κ_2 is nontrivial in $H^4(\mathrm{BDiff}^\delta(\Sigma, \mathrm{rel}\,\partial); \mathbb{Q})$.

2. The curious case of PL foliations

In this section, we are mainly concerned with codimension 2 PL foliations that are co-oriented (i.e., their normal bundles are oriented). The Haefliger classifying space for these structures $\mathrm{B}\Gamma_2^{\mathrm{PL}}$ is the (fat) geometric realization of the nerve of the étale groupoid Γ_2^{PL} whose space of objects is $\mathbb{R}^2$ with the usual topology and the space of morphisms are germs of orientation-preserving PL homeomorphisms of $\mathbb{R}^2$ with the sheaf topology [Mat11, §1]. The main inputs to prove Theorem 1.2 are Greenberg's inductive model for the classifying space PL foliations [Gre92] and Suslin's work [Sus91] on low-degree K-groups of real numbers. And then we use our variant of the Mather–Thurston theorem [Nar23] for PL homeomorphisms to relate the connectivity of $\overline{\mathrm{B}\Gamma}_2^{\mathrm{PL}}$ to the homology of PL surface homeomorphisms.

We first recall Greenberg's recursive construction for such classifying spaces [Gre92, Theorem 3.2(c)] in the case that we are interested in.

2.1 Greenberg's construction and the connectivity of $\overline{\mathrm{B}\Gamma}_2^{\mathrm{PL}}$

Let A be the subgroup of $\mathrm{GL}_2^+(\mathbb{R})$ consisting of matrices of the form

$$M = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

where a and d are positive reals. Let $\epsilon \colon A \to \mathbb{R}^+$ be the homomorphism $\epsilon(M) = a$. Let R_A be the following homotopy pushout:

$$\begin{array}{ccc}
\mathrm{B}(A \times_{\mathbb{R}^+} A)^\delta & \xrightarrow{\;\;p_1\;\;} & \mathrm{B}A^\delta \\
\Big\downarrow{\scriptstyle p_2} & & \Big\downarrow \\
\mathrm{B}A^\delta & \xrightarrow{\hspace{2cm}} & R_A
\end{array} \tag{2.1}$$

where the p_i are induced by the projection to the ith factor and $A \times_{\mathbb{R}^+} A$ is the fiber products of A over $\mathbb{R}^+$ using the map ϵ. Let $\widetilde{\mathrm{GL}_2^+}(\mathbb{R})$ be the universal cover of $\mathrm{GL}_2^+(\mathbb{R})$. Note that the inclusion of A into $\mathrm{GL}_2^+(\mathbb{R})$ lifts to the universal cover $\widetilde{\mathrm{GL}_2^+}(\mathbb{R})$. Let the map $\alpha \colon \mathrm{B}A^\delta \to R_A$ be induced by the diagonal embedding $A \to A \times_{\mathbb{R}^+} A$ and then composing with $\mathrm{B}(A \times_{\mathbb{R}^+} A)^\delta \to R_A$. We let X

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be the homotopy pushout of the following diagram:

$$
\begin{array}{ccc}
\mathrm{B}A^\delta & \longrightarrow & R_A \\
\downarrow & & \downarrow \\
\widetilde{\mathrm{BGL}_2^+(\mathbb{R})}^\delta & \longrightarrow & X
\end{array}
\tag{2.2}
$$

Finally, let LX be the space of continuous free loops in X and let $LX /\!\!/ S^1$ be the homotopy quotient of the circle action on LX. We define rX to be the homotopy pushout

$$
\begin{array}{ccc}
LX & \xrightarrow{\ \mathrm{ev}\ } & X \\
{\scriptstyle j}\downarrow & & \downarrow \\
LX /\!\!/ S^1 & \xrightarrow[\ q\ ]{} & rX
\end{array}
\tag{2.3}
$$

where $\mathrm{ev}\colon LX \to X$ is the evaluation of the loops at their base point and j is the inclusion of the fiber in the Borel fibration $LX \to LX /\!\!/ S^1 \to \mathrm{B}S^1$. Greenberg's theorem [Gre92, Theorem 3.2(c)] says that $rX \simeq \mathrm{B}\Gamma_2^{\mathrm{PL}}$.[1] Recall that $\overline{\mathrm{B}\Gamma}_2^{\mathrm{PL}}$ is the homotopy fiber of the map

$$
\nu\colon \mathrm{B}\Gamma_2^{\mathrm{PL}} \to \mathrm{BPL}^+(\mathbb{R}^2) \simeq \mathrm{B}S^1,
$$

and it was already known as we mentioned in the introduction that $\overline{\mathrm{B}\Gamma}_2^{\mathrm{PL}}$ is at least 2-connected. So to prove Theorem 1.2, it is enough to show that the map ν induces a homology isomorphism up to and including degree 4. To do this, we shall calculate the homology of rX using the Mayer–Vietoris sequence for the homotopy pushout square (2.3). But we first need to prove the following key lemma about the homotopy type of X.

THEOREM 2.4. *The space X is 2-connected.*

The fact that X is simply connected was already observed by Greenberg [Gre92, Proof of Corollary 2.6]. This can also be seen using Van Kampen's theorem to compute the fundamental group. The map $\mathrm{B}\epsilon\colon \mathrm{B}A^\delta \to \mathrm{B}\mathbb{R}^{+,\delta}$ induces a map $h\colon R_A \to \mathrm{B}\mathbb{R}^{+,\delta}$. Using Van Kampen's theorem, one can easily see that h induces an isomorphism $\pi_1(R_A) \xrightarrow{\cong} \mathbb{R}^+$. So $\pi_1(X)$ is isomorphic to the quotient of $\widetilde{\mathrm{GL}_2^+(\mathbb{R})}^\delta$ by the smallest normal subgroup generated by the image of $\ker(\epsilon)$. A priori, $\ker(\epsilon)$ lies in $\mathrm{GL}_2^+(\mathbb{R})^\delta$ and it is easy to see that it normally generates the entire group $\mathrm{GL}_2^+(\mathbb{R})^\delta$. Therefore, its lift also normally generates $\widetilde{\mathrm{GL}_2^+(\mathbb{R})}^\delta$. Hence, $\pi_1(X)$ is trivial.

One can use the Milnor–Friedlander conjecture for solvable Lie groups which was already proved in Milnor's original paper [Mil83] on this topic and Suslin's stability theorem [Sus84] to show that $\pi_2(X) \otimes \mathbb{F}_p = 0$ for all prime p. But to prove the integral result that $\pi_2(X) = 0$ we need to work a bit harder.

Remark 2.5. To see that $\pi_2(X) \otimes \mathbb{F}_p = 0$, by the Hurewicz theorem, it is enough to show that the group $H_2(X; \mathbb{F}_p)$ vanishes for all prime p. We shall first observe that R_A is an $\mathbb{F}_p$-acyclic space, that is, $H_*(R_A; \mathbb{F}_p) = 0$ for all $* > 0$. We have the short exact sequence of groups

$$
\mathrm{Aff}^+(\mathbb{R})^\delta \to A^\delta \to \mathbb{R}^+.
$$

---

[1] He also explains his statement in his introduction but the statement in the introduction is missing a diagram and only describes the space X instead of rX.

Therefore, the group A^δ is solvable. Similarly, $(A \times_{\mathbb{R}^+} A)^\delta$ is solvable. On the other hand, as topological groups, both A and $A \times_{\mathbb{R}^+} A$ are contractible. Hence, by Milnor's theorem [Mil83, Lemma 3], the groups A and $A \times_{\mathbb{R}^+} A$ are $\mathbb{F}_p$-acyclic for all prime p, and by applying the Mayer–Vietoris sequence to the pushout (2.1) we deduce that R_A is also $\mathbb{F}_p$-acyclic for all prime p. Now using the Mayer–Vietoris sequence with $\mathbb{F}_p$ coefficients for the pushout (2.2), it is enough to show that $H_2(\widetilde{\mathrm{BGL}_2^+}(\mathbb{R})^\delta; \mathbb{F}_p) = 0$. Recall that we know by Suslin's theorem [Sus84] that in general the map

$$\mathrm{BGL}_n^+(\mathbb{R})^\delta \to \mathrm{BGL}_n^+(\mathbb{R})$$

induces an isomorphism on $H_*(-; \mathbb{F}_p)$ for $* \leq n$. On the other hand, we have a short exact sequence

$$\mathbb{Z} \to \widetilde{\mathrm{GL}_2^+}(\mathbb{R})^\delta \to \mathrm{GL}_2^+(\mathbb{R})^\delta.$$

Therefore, by a spectral sequence argument, we deduce that the map

$$\widetilde{\mathrm{BGL}_2^+}(\mathbb{R})^\delta \to \widetilde{\mathrm{BGL}_2^+}(\mathbb{R})$$

induces an isomorphism on $H_*(-; \mathbb{F}_p)$ for $* \leq 2$. But $\widetilde{\mathrm{GL}_2^+}(\mathbb{R})$ is contractible which implies that $H_2(\widetilde{\mathrm{BGL}_2^+}(\mathbb{R})^\delta; \mathbb{F}_p) = 0$.

To prove Theorem 2.4, we need some preliminary lemmas to do the calculations integrally.

LEMMA 2.6. *Let D be the subgroup of diagonal matrices in A. Let $\iota \colon D \hookrightarrow A$ be the inclusion map of the subgroup of diagonal matrices. The map ι has a left inverse $r \colon A \to D$,*

$$r\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}.$$

The maps $\tilde{\iota}$ and $\tilde{r}$,

$$\mathrm{B}D^\delta \xrightarrow{\tilde{\iota}} \mathrm{B}A^\delta \xrightarrow{\tilde{r}} \mathrm{B}D^\delta,$$

induce homology isomorphisms.

Proof. There is a trick that apparently goes back to Quillen over rational coefficients [dLHM83, Lemma 4] and to Nesterenko and Suslin [NS90, Theorem 1.11] over integer coefficients that the map $\mathrm{BGL}_n^+(\mathbb{R})^\delta \to \mathrm{BAff}^+(\mathbb{R}^n)^\delta$, which also has a left inverse, induces a homology isomorphism. Taking $n = 1$, we have a map between fibrations

$$
\begin{array}{ccccc}
\mathrm{B}\mathbb{R}^{+,\delta} & \longrightarrow & \mathrm{BAff}^+(\mathbb{R}^1)^\delta & \longrightarrow & \mathrm{B}\mathbb{R}^{+,\delta} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{B}D^\delta & \xrightarrow{\tilde{\iota}} & \mathrm{B}A^\delta & \xrightarrow{\tilde{r}} & \mathrm{B}D^\delta \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{B}\mathbb{R}^{+,\delta} & \xrightarrow{\cong} & \mathrm{B}\mathbb{R}^{+,\delta} & \xrightarrow{\cong} & \mathrm{B}\mathbb{R}^{+,\delta}
\end{array}
\qquad (2.7)
$$

where the top horizontal maps induce homology isomorphisms and the bottom maps are the identity. Therefore, by the comparison of Serre spectral sequences, $\tilde{\iota}$ and $\tilde{r}$ also induce homology isomorphisms. $\qquad \square$

Let Y be the homotopy pushout of the diagram

$$
\begin{array}{ccc}
\mathrm{B}(D \times_{\mathbb{R}^+} D)^\delta & \xrightarrow{\ p_1\ } & \mathrm{B}D^\delta \\
\downarrow{\scriptstyle p_2} & & \downarrow \\
\mathrm{B}D^\delta & \xrightarrow{\hspace{2cm}} & Y
\end{array}
\tag{2.8}
$$

Given that we have the inclusion $D \xrightarrow{\iota} A$ and its left inverse r, we have a natural map $\theta \colon Y \to R_A$.

LEMMA 2.9. *The map $\theta \colon Y \to R_A$ admits a left inverse and it induces a homology isomorphism. The space Y is homotopy equivalent to $\mathrm{B}\mathbb{R}^{+,\delta} \times (\mathrm{B}\mathbb{R}^{+,\delta} \star \mathrm{B}\mathbb{R}^{+,\delta}) \to R_A$ where $\star$ means the join of topological spaces.*

Proof. The inclusion $D \xrightarrow{\iota} A$ induces a map of homotopy pushout diagrams from (2.8) to (2.1) and the left inverse r induces a map of diagrams from (2.1) to (2.8) which gives the left inverse to θ. Since the maps between corresponding terms induce homology isomorphisms by Lemma 2.6, the Mayer–Vietoris sequence implies that θ induces a homology isomorphism.

To see that Y is homotopy equivalent to $\mathrm{B}\mathbb{R}^{+,\delta} \times (\mathrm{B}\mathbb{R}^{+,\delta} \star \mathrm{B}\mathbb{R}^{+,\delta})$, note that D is isomorphic to $\mathbb{R}^+ \times \mathbb{R}^+$. So Y is homotopy equivalent to the homotopy pushout of

$$
\begin{array}{ccc}
\mathrm{B}\mathbb{R}^{+,\delta} \times \mathrm{B}\mathbb{R}^{+,\delta} \times \mathrm{B}\mathbb{R}^{+,\delta} & \xrightarrow{\ p_{1,2}\ } & \mathrm{B}\mathbb{R}^{+,\delta} \times \mathrm{B}\mathbb{R}^{+,\delta} \\
\downarrow{\scriptstyle p_{1,3}} & & \downarrow \\
\mathrm{B}\mathbb{R}^{+,\delta} \times \mathrm{B}\mathbb{R}^{+,\delta} & \xrightarrow{\hspace{2cm}} & Y
\end{array}
\tag{2.10}
$$

where $p_{1,i}$ is the projection to the first and the ith factor. Therefore, Y is homotopy equivalent to $\mathrm{B}\mathbb{R}^{+,\delta} \times (\mathrm{B}\mathbb{R}^{+,\delta} \star \mathrm{B}\mathbb{R}^{+,\delta})$. □

Recall that the map $\alpha \colon \mathrm{B}A^\delta \to R_A$ is defined to be the composition of the diagonal embedding $\mathrm{B}A^\delta \to \mathrm{B}(A \times_{\mathbb{R}^+} A)^\delta$ and $\mathrm{B}(A \times_{\mathbb{R}^+} A)^\delta \to R_A$. Similarly, we obtain a map $\beta \colon \mathrm{B}D^\delta \to Y$. So we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{B}D^\delta & \xrightarrow{\ \beta\ } & Y \\
\downarrow & & \downarrow \\
\mathrm{B}A^\delta & \xrightarrow{\ \alpha\ } & R_A
\end{array}
\tag{2.11}
$$

where the vertical maps induce homology isomorphisms. Note that the join $\mathrm{B}\mathbb{R}^{+,\delta} \star \mathrm{B}\mathbb{R}^{+,\delta}$ is 2-connected and there is an isomorphism from $H_*(\mathrm{B}\mathbb{R}^{+,\delta}; \mathbb{Z})$ to $\bigwedge^* \mathbb{R}^+$ which is an exterior product of $\mathbb{R}^+$ over $\mathbb{Z}$. By considering the commutative diagram

$$
\begin{array}{ccc}
H_*(\mathrm{B}D^\delta; \mathbb{Z}) & \xrightarrow{\ \beta_*\ } & H_*(Y; \mathbb{Z}) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
H_*(\mathrm{B}\mathbb{R}^{+,\delta} \times \mathrm{B}\mathbb{R}^{+,\delta}; \mathbb{Z}) & \longrightarrow & H_*(\mathrm{B}\mathbb{R}^{+,\delta} \times (\mathrm{B}\mathbb{R}^{+,\delta} \star \mathrm{B}\mathbb{R}^{+,\delta}); \mathbb{Z})
\end{array}
\tag{2.12}
$$

and using Künneth's formula, it is easy to determine the kernel of the map β_* in low homological degrees. So we record the following corollary about $\ker \alpha_*$ in low homological degrees where

$$\alpha_* \colon H_*(BA^\delta; \mathbb{Z}) \to H_*(R_A; \mathbb{Z}).$$

COROLLARY 2.13. *Let* $t \colon \mathbb{R}^+ \to A$ *be the map* $t(a) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right]$ *and let* $s \colon \mathbb{R}^+ \to A$ *be the map* $s(a) = \left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

(1) *The map* $\iota \colon \mathbb{R}^+ \times \mathbb{R}^+ \cong D \to A$ *induces a split surjection*

$$\textstyle\bigwedge^2 \mathbb{R}^+ \oplus (\mathbb{R}^+ \otimes \mathbb{R}^+) \oplus \bigwedge^2 \mathbb{R}^+ \cong H_2(BD^\delta; \mathbb{Z}) \xrightarrow{\cong} H_2(BA^\delta; \mathbb{Z}) \to H_2(R_A; \mathbb{Z}) \cong \textstyle\bigwedge^2 \mathbb{R}^+,$$

which maps the first summand $\bigwedge^2 \mathbb{R}^+$ *isomorphically to* $H_2(R_A; \mathbb{Z})$. *So* $\ker \alpha_2$ *is isomorphic to* $(\mathbb{R}^+ \otimes \mathbb{R}^+) \oplus \bigwedge^2 \mathbb{R}^+$.

(2) *The map* t *induces an isomorphism* $\mathbb{R}^+ \cong H_1(B\mathbb{R}^{+,\delta}; \mathbb{Z}) \to \ker \alpha_1$ *and an injective map* $\bigwedge^2 \mathbb{R}^+ \cong H_2(B\mathbb{R}^{+,\delta}; \mathbb{Z}) \to \ker \alpha_2$ *which is an isomorphism to the* $\bigwedge^2 \mathbb{R}^+$ *summand of* $\ker \alpha_2$ *in the identification in item (1) above.*

(3) *The composition*

$$\textstyle\bigwedge^2 \mathbb{R}^+ \cong H_2(B\mathbb{R}^{+,\delta}; \mathbb{Z}) \xrightarrow{s_2} H_2(BA^\delta; \mathbb{Z}) \to H_2(R_A; \mathbb{Z}) \cong \textstyle\bigwedge^2 \mathbb{R}^+,$$

where the first map is induced by s, *is an isomorphism.*

We also need part of Suslin's calculation [Sus91, Theorem 2.1] and [Wei13, Chapter 6, §5, Proof of Theorem 5.7] of $H_2(BGL_2(\mathbb{R})^\delta; \mathbb{Z})$ to determine the image of

$$H_2(BA^\delta; \mathbb{Z}) \to H_2(\widetilde{BGL_2^+}(\mathbb{R})^\delta; \mathbb{Z}).$$

To find $H_2(\widetilde{BGL_2^+}(\mathbb{R})^\delta; \mathbb{Z})$, first note that there is an isomorphism $f \colon SL_2(\mathbb{R}) \times \mathbb{R}^+ \xrightarrow{\cong} GL_2^+(\mathbb{R})$ where

$$f(A, a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \cdot A.$$

This isomorphism can be lifted to give an isomorphism $\tilde{f} \colon \widetilde{SL_2}(\mathbb{R}) \times \mathbb{R}^+ \xrightarrow{\cong} \widetilde{GL_2^+}(\mathbb{R})$. On the other hand, the groups $SL_2(\mathbb{R})^\delta$ and $\widetilde{SL_2}(\mathbb{R})^\delta$ are perfect and it is known [PS83, pp. 190–191] that $H_2(\widetilde{BSL_2}(\mathbb{R})^\delta; \mathbb{Z}) \cong K_2(\mathbb{R})$, and we have a short exact sequence

$$0 \to H_2(\widetilde{BSL_2}(\mathbb{R})^\delta; \mathbb{Z}) \to H_2(BSL_2(\mathbb{R})^\delta; \mathbb{Z}) \to \mathbb{Z} \to 0.$$

Therefore, the Künneth formula implies that we have the isomorphism $H_2(\widetilde{BGL_2^+}(\mathbb{R})^\delta; \mathbb{Z}) \cong K_2(\mathbb{R}) \oplus \bigwedge^2 \mathbb{R}^+$ where $K_2(\mathbb{R})$ summand comes from the image of $H_2(\widetilde{BSL_2}(\mathbb{R})^\delta; \mathbb{Z}) \to H_2(\widetilde{BGL_2^+}(\mathbb{R})^\delta; \mathbb{Z})$. Also the map

$$u \colon H_2(\widetilde{BGL_2^+}(\mathbb{R})^\delta; \mathbb{Z}) \to H_2(BGL_2^+(\mathbb{R})^\delta; \mathbb{Z})$$

is split injective with a co-kernel which is isomorphic to $\mathbb{Z}$.

The map $\mathbb{R}^+ \times \mathbb{R}^+ \cong D \to A \to GL_2^+(\mathbb{R})$ induces the map

$$\textstyle\bigwedge^2 \mathbb{R}^+ \oplus (\mathbb{R}^+ \otimes \mathbb{R}^+) \oplus \bigwedge^2 \mathbb{R}^+ \cong H_2(B\mathbb{R}^{+,\delta} \times B\mathbb{R}^{+,\delta}; \mathbb{Z}) \to H_2(BGL_2^+(\mathbb{R})^\delta; \mathbb{Z}).$$

Let σ be the involution of diagonal entries of $D \cong \mathbb{R}^+ \times \mathbb{R}^+$. The spectral sequence in the proof of [Wei13, Chapter 6, Theorem 5.7] implies that this map factors through the co-invariants

$$H_2(B\mathbb{R}^{+,\delta} \times B\mathbb{R}^{+,\delta}; \mathbb{Z})_\sigma.$$

The group $H_2(\mathrm{B}\mathbb{R}^{+,\delta} \times \mathrm{B}\mathbb{R}^{+,\delta}; \mathbb{Z})_\sigma$ is isomorphic to $\bigwedge^2 \mathbb{R}^+ \oplus \tilde{\bigwedge}^2 \mathbb{R}^+$, where $\tilde{\bigwedge}^2 \mathbb{R}^+$ denotes the quotient of the group $\mathbb{R}^+ \otimes \mathbb{R}^+$ by the subgroup generated by all $a \otimes b + b \otimes a$. The proof of [Wei13, Chapter 6, Theorem 5.7] also implies that the restriction of the map

$$\bigwedge\nolimits^2 \mathbb{R}^+ \oplus \tilde{\bigwedge}\nolimits^2 \mathbb{R}^+ \to H_2(\mathrm{BGL}_2^+(\mathbb{R})^\delta; \mathbb{Z})$$

on the summand $\bigwedge^2 \mathbb{R}^+$ is injective and maps $\tilde{\bigwedge}^2 \mathbb{R}^+$ surjectively to the summand $K_2(\mathbb{R})$ in $H_2(\mathrm{BGL}_2^+(\mathbb{R})^\delta; \mathbb{Z})$. So we summarize what we need from Suslin's calculation in the following lemma.

LEMMA 2.14. *The map*

$$\mathbb{R}^+ \otimes \mathbb{R}^+ \to H_2(\mathrm{B}\mathbb{R}^{+,\delta} \times \mathrm{B}\mathbb{R}^{+,\delta}; \mathbb{Z}) \to H_2(\mathrm{BGL}_2^+(\mathbb{R})^\delta; \mathbb{Z})$$

surjects to the image of $K_2(\mathbb{R}) \cong H_2(\widetilde{\mathrm{BSL}_2}(\mathbb{R})^\delta; \mathbb{Z}) \to H_2(\mathrm{BGL}_2^+(\mathbb{R})^\delta; \mathbb{Z})$.

Proof of Theorem 2.4. Since X is simply connected, to prove that it is 2-connected, we need to show that $H_2(X; \mathbb{Z}) = 0$. The homotopy pushout in diagram (2.2) gives the Mayer–Vietoris sequence

$$
\begin{aligned}
H_2(\mathrm{B}A^\delta; \mathbb{Z}) &\xrightarrow{i_2} H_2(R_A; \mathbb{Z}) \oplus H_2(\widetilde{\mathrm{BGL}_2^+}(\mathbb{R})^\delta; \mathbb{Z}) \longrightarrow H_2(X; \mathbb{Z}) \\
&\xrightarrow{\delta} \\
&H_1(\mathrm{B}A^\delta; \mathbb{Z}) \xrightarrow{i_1} H_1(R_A; \mathbb{Z}) \oplus H_1(\widetilde{\mathrm{BGL}_2^+}(\mathbb{R})^\delta; \mathbb{Z}) \longrightarrow H_1(X; \mathbb{Z}) = 0
\end{aligned}
\tag{2.15}
$$

First, we observe that i_1 is an isomorphism. From Corollary 2.13, we know that the kernel of the map

$$\alpha_1 \colon H_1(\mathrm{B}A^\delta; \mathbb{Z}) \to H_1(R_A; \mathbb{Z})$$

is given by the image of $H_1(\mathrm{BAff}^+(\mathbb{R})^\delta; \mathbb{Z}) \to H_1(\mathrm{B}A^\delta; \mathbb{Z})$. So to prove that i_1 is an isomorphism, it is enough to show the composition

$$\mathrm{BAff}^+(\mathbb{R})^\delta \to \mathrm{B}A^\delta \to \widetilde{\mathrm{BGL}_2^+}(\mathbb{R})^\delta$$

induces an isomorphism on the first homology. On the other hand, using the isomorphism $f \colon \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}^+ \xrightarrow{\cong} \mathrm{GL}_2^+(\mathbb{R})$, we know that

$$H_1(\mathrm{BAff}^+(\mathbb{R})^\delta; \mathbb{Z}) \to H_1(\mathrm{BGL}_2^+(\mathbb{R})^\delta; \mathbb{Z})$$

is an isomorphism. Hence, to prove that i_1 is an isomorphism, it is enough to show that

$$H_1(\widetilde{\mathrm{BGL}_2^+}(\mathbb{R})^\delta; \mathbb{Z}) \to H_1(\mathrm{BGL}_2^+(\mathbb{R})^\delta; \mathbb{Z})$$

is an isomorphism. The Serre spectral sequence for the fibration

$$\mathbb{S}^1 \to \widetilde{\mathrm{BGL}_2^+}(\mathbb{R})^\delta \to \mathrm{BGL}_2^+(\mathbb{R})^\delta \tag{2.16}$$

gives the long exact sequence

$$
\begin{aligned}
H_2(\widetilde{\mathrm{BGL}_2^+}(\mathbb{R})^\delta; \mathbb{Z}) &\xrightarrow{u} H_2(\mathrm{BGL}_2^+(\mathbb{R})^\delta; \mathbb{Z}) \xrightarrow{e} \mathbb{Z} \\
&\xrightarrow{} \\
&H_1(\widetilde{\mathrm{BGL}_2^+}(\mathbb{R})^\delta; \mathbb{Z}) \xrightarrow{v} H_1(\mathrm{BGL}_2^+(\mathbb{R})^\delta; \mathbb{Z}) \longrightarrow 0.
\end{aligned}
\tag{2.17}
$$

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The map e is the Euler class for flat $\mathrm{GL}_2^+(\mathbb{R})$-bundles over surfaces. By Milnor's theorem [Mil58, Theorem 2], the Euler number of flat $\mathrm{GL}_2^+(\mathbb{R})$-bundles over a surface of genus g can take any value between $-g+1$ and $g-1$. So by varying g, we conclude that e is surjective (it is in fact split surjective). Therefore, the map v is an isomorphism.

So to prove that $H_2(X;\mathbb{Z}) = 0$, it is enough to show that

$$i_2\colon H_2(\mathrm{B}A^\delta;\mathbb{Z}) \to H_2(R_A;\mathbb{Z}) \oplus H_2(\widetilde{\mathrm{BGL}_2^+}(\mathbb{R})^\delta;\mathbb{Z}) \tag{2.18}$$

is a surjection. In Corollary 2.13, we determined the kernel of the split surjective map

$$\alpha_2\colon H_2(\mathrm{B}A^\delta;\mathbb{Z}) \to H_2(R_A;\mathbb{Z}).$$

Hence, it is enough to show that $\ker(\alpha_2) \to H_2(\widetilde{\mathrm{BGL}_2^+}(\mathbb{R})^\delta;\mathbb{Z})$ is surjective.

Recall that

$$u\colon H_2(\widetilde{\mathrm{BGL}_2^+}(\mathbb{R})^\delta;\mathbb{Z}) \to H_2(\mathrm{BGL}_2^+(\mathbb{R})^\delta;\mathbb{Z})$$

is split injective where $\mathrm{Im}(u) \cong K_2(\mathbb{R}) \oplus \bigwedge^2\mathbb{R}^+$ and by the above discussion the co-kernel is isomorphic to $\mathbb{Z}$ via the map e in the long exact sequence (2.17). So we need to show that $\ker(\alpha_2) \cong (\mathbb{R}^+ \otimes \mathbb{R}^+) \oplus \bigwedge^2\mathbb{R}^+$ maps surjectively to the summand $\mathrm{Im}(p)$.

Recall from Corollary 2.13 that the map $t\colon \mathbb{R}^+ \to A$ given by $t(a) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right]$ induces a map on the second homology groups

$$\bigwedge^2\mathbb{R}^+ \to \ker(\alpha_2),$$

which is isomorphic to the summand $\bigwedge^2\mathbb{R}^+$ in $\ker(\alpha_2)$ in Corollary 2.13. On other hand, under the isomorphism $f\colon \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}^+ \xrightarrow{\cong} \mathrm{GL}_2^+(\mathbb{R})$, the matrix $t(a)$ comes from $(\mathrm{Id}, \sqrt{a})$. Given that the square root is an isomorphism of $\mathbb{R}^+$, we obtain that the composition of maps

$$H_2(\mathrm{B}\mathbb{R}^{+,\delta};\mathbb{Z}) \xrightarrow{t_*} H_2(\mathrm{B}A^\delta;\mathbb{Z}) \to H_2(\mathrm{BGL}_2^+(\mathbb{R})^\delta;\mathbb{Z}) \cong H_2(\mathrm{BSL}_2(\mathbb{R})^\delta;\mathbb{Z}) \oplus \bigwedge^2\mathbb{R}^+$$

is injective and isomorphic to the summand $\bigwedge^2\mathbb{R}^+$. Hence, to finish the proof of surjectivity of

$$(\mathbb{R}^+ \otimes \mathbb{R}^+) \oplus \bigwedge^2\mathbb{R}^+ \cong \ker(\alpha_2) \to \mathrm{Im}(p) \cong K_2(\mathbb{R}) \oplus \bigwedge^2\mathbb{R}^+,$$

it is enough to prove that the summand $\mathbb{R}^+ \otimes \mathbb{R}^+$ in $\ker(\alpha_2)$ maps surjectively to $K_2(\mathbb{R})$. Recall that the summand $\mathbb{R}^+ \otimes \mathbb{R}^+$ in Corollary 2.13 is induced by embedding of diagonal matrices and using the Kenneth formula

$$\mathbb{R}^+ \otimes \mathbb{R}^+ \to H_2(\mathrm{B}\mathbb{R}^{+,\delta} \times \mathrm{B}\mathbb{R}^{+,\delta};\mathbb{Z}) \to H_2(\mathrm{B}A^\delta;\mathbb{Z}).$$

So from Lemma 2.14, it follows that the summand $\mathbb{R}^+ \otimes \mathbb{R}^+$ in $\ker(\alpha_2)$ maps surjectively to $K_2(\mathbb{R})$. $\qquad\square$

Recall that we have natural maps $\nu\colon \mathrm{B}\Gamma_2^{\mathrm{PL}} \to \mathrm{BPL}^+(\mathbb{R}^2)$ and $\zeta\colon \mathrm{BPL}^+(\mathbb{R}^2) \xrightarrow{\simeq} \mathrm{BHomeo}^+(\mathbb{R}^2) \simeq \mathrm{B}S^1$. So they induce a map

$$\psi\colon rX \to \mathrm{BHomeo}^+(\mathbb{R}^2).$$

We think of the map ψ as the map that classifies the normal bundle to codimension 2 PL Haefliger structures as $\mathbb{R}^2$-bundles. Therefore, to prove Theorem 1.2 which says that $\overline{\mathrm{B}\Gamma}_2^{\mathrm{PL}}$ is 4-connected, it is enough to prove the following result.

THEOREM 2.19. *The map ψ induces an isomorphism on $H_*(-;\mathbb{Z})$ for $* \leq 4$.*

We need another preliminary lemma. In Greenberg's homotopy pushout diagram (2.3), there is a map $q\colon LX /\!\!/ S^1 \to rX$ and also there is a natural map $p\colon LX /\!\!/ S^1 \to \mathrm{B}S^1$ that classifies the

universal circle bundle over the homotopy quotient $LX /\!/ S^1$. Since the map $S^1 \hookrightarrow \mathrm{Homeo}_0(S^1)$ is a homotopy equivalence, we shall consider the following equivalent models for these maps:

$$q\colon LX /\!/ \mathrm{Homeo}_0(S^1) \to rX \simeq \mathrm{B}\Gamma_2^{\mathrm{PL}},$$

$$p\colon LX /\!/ \mathrm{Homeo}_0(S^1) \to \mathrm{BHomeo}_0(S^1).$$

There is also the composition $\mathrm{Homeo}_0(S^1) \to \mathrm{Homeo}_0(D^2) \to \mathrm{Homeo}^+(\mathbb{R}^2)$ where the first map is the Alexander cone construction and the second map is the restriction to the identity. This inclusion induces a weak homotopy equivalence

$$\iota\colon \mathrm{BHomeo}_0(S^1) \xrightarrow{\simeq} \mathrm{BHomeo}^+(\mathbb{R}^2).$$

LEMMA 2.20. *The maps $\iota \circ p$ and $\psi \circ q$ induce the same map on homology.*

Proof. This is already implicit in Greenberg's paper [Gre92], but for the convenience of the reader we shall first recall the relevant object for this proof. As in Greenberg's paper [Gre92, p. 188], let P_0 be the group of germs of orientation-preserving PL homeomorphisms of $\mathbb{R}^2$ that fix the origin. Ghys and Sergiescu [GS87, § 2] and Greenberg [Gre92, Theorem 2.25 and Corollary 2.26] proved a general version of the Mather–Thurston homology isomorphism theorem for certain groupoids on the circle. As a result, there is a map

$$f\colon \mathrm{B}P_0 \to LX /\!/ \mathrm{Homeo}_0(S^1)$$

that induces a homology isomorphism. To prove the lemma, we use Greenberg's description of the Ghys–Sergiescu theorem to show that the two maps

$$\iota \circ p \circ f\colon \mathrm{B}P_0 \to \mathrm{BHomeo}^+(\mathbb{R}^2),$$

$$\psi \circ q \circ f\colon \mathrm{B}P_0 \to \mathrm{BHomeo}^+(\mathbb{R}^2)$$

induce isomorphic $\mathbb{R}^2$-bundles over $\mathrm{B}P_0$.

By Greenberg's description [Gre92, § 2.22], the composition $q \circ f\colon \mathrm{B}P_0 \to \mathrm{B}\Gamma_2^{\mathrm{PL}}$ is induced by the inclusion of P_0 as the group of germs into the groupoid Γ_2^{PL}. One can canonically extend each germ in P_0 to a PL homeomorphism of $\mathbb{R}^2$. So there is a natural action of P_0 on $\mathbb{R}^2$. Therefore, the map $\psi \circ q \circ f$,

$$\mathrm{B}P_0 \to \mathrm{B}\Gamma_2^{\mathrm{PL}} \xrightarrow{\nu} \mathrm{BPL}^+(\mathbb{R}^2) \to \mathrm{BHomeo}^+(\mathbb{R}^2),$$

classifies the $\mathbb{R}^2$-bundle on $\mathrm{B}P_0$ induced by the action of P_0 on $\mathbb{R}^2$.

On the other hand, P_0 acts on rays out of the origin. So P_0 also maps into $\mathrm{PP}(S^1)$ the group of orientation-preserving piecewise projective homeomorphisms of S^1. In particular, it is a subgroup of orientation-preserving homeomorphisms of the circle. The map $p \circ f$,

$$\mathrm{B}P_0 \to \mathrm{BHomeo}_0(S^1),$$

classifies the natural circle bundle over $\mathrm{B}P_0$ induced by the action of P_0 on S^1. Therefore, the map $\iota \circ p \circ f$ classifies the Euclidean $\mathbb{R}^2$-bundle induced by the natural action of P_0 on $\mathbb{R}^2$. $\square$

Let $\mathrm{ev}\colon LX \to X$ be the map induced by evaluating loops at the base point 1 of the unit circle in the complex plane. The circle action $\eta\colon S^1 \times LX \to LX$ sends the pair $(s, \gamma(t))$ where $\gamma(t)$ is a free loop in X to the loop $\gamma(st)$. The map η induces the map

$$\Delta_*\colon H_*(LX; \mathbb{Z}) \to H_{*+1}(LX; \mathbb{Z}).$$

For each positive integer k, let $h_k\colon \pi_k(\Omega X) \to H_{k+1}(X; \mathbb{Z})$ be the map that sends the homotopy class of $f\colon S^k \to \Omega X$ to $F_*([S^1 \times S^k])$, where $F\colon S^1 \times S^k \to X$ is the map induced by the adjoint of f (it is the adjoint of f composed with swapping S^1 and S^k factors).

LEMMA 2.21. *We have a commutative diagram*

$$
\begin{array}{ccc}
\pi_*(\Omega X) & \xrightarrow{\quad h_* \quad} & H_{*+1}(X;\mathbb{Z}) \\
\downarrow{\scriptstyle H} & & \uparrow{\scriptstyle ev_*} \\
H_*(\Omega X;\mathbb{Z}) & \xrightarrow{\;G\;} H_*(LX;\mathbb{Z}) \xrightarrow{\;\Delta_*\;} & H_{*+1}(LX;\mathbb{Z})
\end{array}
$$

where the map H is the Hurewicz map and the map G is induced by the inclusion $\Omega X \to LX$.

Proof. Let $f\colon S^k \to \Omega X$ be an element in $\pi_k(\Omega X)$ and let $\tilde{f}$ be $G \circ H(f) \in H_k(LX;\mathbb{Z})$. Then $\Delta_k(\tilde{f})$ is defined to be the map

$$\Delta_k(\tilde{f})\colon S^1 \times S^k \to LX$$

which sends the pair (s,x) to the action of s on the loop $f(x)(t)$ which is $f(x)(st)$. The evaluation map evaluates this loop at $t=1$ which gives the same map as the adjoint $F\colon S^1 \times S^k \to X$. Hence, we have $ev_k \circ \Delta_k(\tilde{f}) = h_k(f)$. $\qquad\square$

COROLLARY 2.22. *Let X be a 2-connected space.*

- *The map Δ_2 is injective and $\Delta_2(H_2(LX;\mathbb{Z}))$ maps isomorphically to $H_3(X;\mathbb{Z})$ via the evaluation map* ev.
- *$\Delta_3(H_3(LX;\mathbb{Z}))$ maps surjectively to $H_4(X;\mathbb{Z})$ via the evaluation map* ev.

Proof. Since X is 2-connected, the Hurewicz map $\pi_3(X) \to H_3(X;\mathbb{Z})$ is an isomorphism and also $\pi_4(X) \to H_4(X;\mathbb{Z})$ is surjective. On the other hand, LX is also simply connected, therefore we have the isomorphisms

$$\pi_2(\Omega X) \xrightarrow{\cong} H_2(\Omega X;\mathbb{Z}) \xrightarrow{\cong} H_2(LX;\mathbb{Z}),$$

where ΩX is the based loop space on X. Also, note that the map

$$h_2\colon \pi_2(\Omega X) \to H_3(X;\mathbb{Z})$$

is an isomorphism. From Lemma 2.21, we know that $ev_2 \circ \Delta_2\colon \pi_2(\Omega X) \to H_3(X;\mathbb{Z})$ is the same map as h_2, which proves the first statement.

Since ΩX is simply connected, the Hurewicz map

$$\pi_3(\Omega X) \to H_3(\Omega X;\mathbb{Z})$$

is surjective. So to prove the second statement, it is enough to show that the composition

$$\pi_3(\Omega X) \xrightarrow{\cong} H_3(\Omega X;\mathbb{Z}) \to H_3(LX;\mathbb{Z}) \xrightarrow{\Delta_3} H_4(LX;\mathbb{Z}) \xrightarrow{ev} H_4(X;\mathbb{Z}) \qquad (2.23)$$

is surjective. But again by Lemma 2.21 the above composition is the same as the natural map

$$h_3\colon \pi_3(\Omega X) \to H_4(X;\mathbb{Z})$$

that sends the homotopy class of $f\colon S^3 \to \Omega X$ to $F_*([S^1 \times S^3])$, where $F\colon S^1 \times S^3 \to X$ is the map induced by the adjoint of f. Now since X is 2-connected, the map h_3 is surjective. Therefore, the composition (2.23) is also surjective. $\qquad\square$

Proof of Theorem 2.19. Recall from the introduction that the space $\overline{\mathrm{B}\Gamma}_2^{\mathrm{PL}}$, which is weakly equivalent to the homotopy fiber of the map

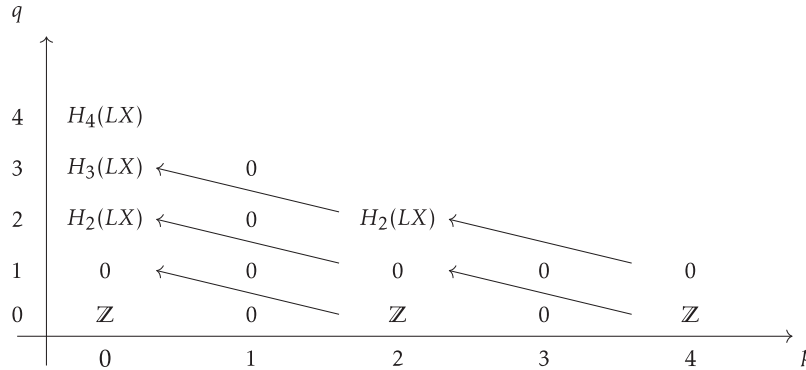$$\psi\colon rX \to \mathrm{BHomeo}^+(\mathbb{R}^2),$$

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FIGURE 1. The second page of the Serre spectral sequence for the fibration $LX \to LX/\!\!/S^1 \to \mathrm{B}S^1$.

is known to be at least 2-connected. So the map ψ induces isomorphisms on $H_*(-;\mathbb{Z})$ for $* \leq 2$. Hence, we need to show two things; one is that $H_3(rX;\mathbb{Z}) = 0$ and the other is that

$$\psi_4 \colon H_4(rX;\mathbb{Z}) \to H_4(\mathrm{BHomeo}^+(\mathbb{R}^2);\mathbb{Z}) \cong \mathbb{Z}$$

is an isomorphism. First, note that the Mayer–Vietoris sequence for the pushout (2.3) gives

$$H_i(LX) \to H_i(X) \oplus H_i(LX/\!\!/S^1) \to H_i(rX) \to H_{i-1}(LX) \to H_{i-1}(X) \oplus H_{i-1}(LX/\!\!/S^1).$$

To compute $H_*(rX;\mathbb{Z})$ for $* \leq 4$, we use that X is 2-connected and the fact that fibrations

$$\Omega X \to LX \xrightarrow{\mathrm{ev}} X,$$
$$LX \to LX/\!\!/S^1 \xrightarrow{p} \mathrm{B}S^1 \tag{2.24}$$

have sections. The first fibration has a section by considering constant loops and the second fibration has a section because the action of S^1 has fixed points, that is, the constant loops. Therefore, $H_*(LX) \xrightarrow{\mathrm{ev}_*} H_*(X)$ is surjective and so is $H_*(LX/\!\!/S^1) \to H_*(\mathrm{B}S^1)$, and since they have sections, $H_*(X)$ and $H_*(\mathrm{B}S^1)$ split off as summands from $H_*(LX)$ and $H_*(LX/\!\!/S^1)$, respectively.

From the Serre spectral sequence for the fibration (2.24), we see that $H_2(LX;\mathbb{Z}) \to H_2(LX/\!\!/S^1;\mathbb{Z})$ is injective. So to show that $H_3(rX;\mathbb{Z}) = 0$, it is enough to prove that

$$H_3(LX) \to H_3(X) \oplus H_3(LX/\!\!/S^1) \tag{2.25}$$

is surjective.

Note that the differentials of the spectral sequence out of the terms isomorphic to $\mathbb{Z}$s in the 0th row are trivial because of the existence of the section for the map p in fibration (2.24). And it is standard that the differentials

$$d_2 \colon H_i(LX) \to H_{i+1}(LX)$$

are the same as the map Δ_i in Corollary 2.22 [BO99, Proposition 3.3].

From the first part of Corollary 2.22, we know that the map d_2 in

$$H_2(LX;\mathbb{Z}) \xrightarrow{d_2} H_3(LX;\mathbb{Z}) \xrightarrow{\mathrm{ev}_*} H_3(X;\mathbb{Z})$$

is injective and the natural map $\mathrm{ev}_* \colon d_2(H_2(LX;\mathbb{Z})) \to H_3(X;\mathbb{Z})$ is an isomorphism. Given that $d_2(H_2(LX;\mathbb{Z}))$ is the kernel of the surjection $H_3(LX) \twoheadrightarrow H_3(LX/\!\!/S^1)$, the map (2.25) is in fact an isomorphism. So we have $H_3(rX;\mathbb{Z}) = 0$.

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Now since the map (2.25) is an isomorphism, to show that ψ induces an isomorphism on $H_4(-;\mathbb{Z})$ it is enough to show that the co-kernel of the map

$$H_4(LX;\mathbb{Z}) \to H_4(X;\mathbb{Z}) \oplus H_4(LX/\!\!/S^1;\mathbb{Z}) \tag{2.26}$$

is the $\mathbb{Z}$ summand in $H_4(LX/\!\!/S^1;\mathbb{Z})$ coming from the 0th row in the Serre spectral sequence. This is because, in that case, the composition

$$\mathrm{B}S^1 \to LX/\!\!/S^1 \to rX,$$

where the first map is the section of p, induces an isomorphism on $H_4(-;\mathbb{Z})$; and Lemma 2.20 implies that the composition

$$\mathrm{B}S^1 \to LX/\!\!/S^1 \to rX \xrightarrow{\psi} \mathrm{BHomeo}^+(\mathbb{R}^2)$$

induces a homology isomorphism.

To do this, from the second part of Corollary 2.22, we know that, in

$$H_3(LX;\mathbb{Z}) \xrightarrow{d_2} H_4(LX;\mathbb{Z}) \xrightarrow{\mathrm{ev}_*} H_4(X;\mathbb{Z}),$$

$d_2(H_3(LX;\mathbb{Z}))$ surjects to $H_4(X;\mathbb{Z})$ via ev_*. Since $H_4(LX/\!\!/S^1;\mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus H_4(LX;\mathbb{Z})/d_2(H_3(LX;\mathbb{Z}))$, the co-kernel of the map

$$H_4(LX;\mathbb{Z}) \to H_4(X;\mathbb{Z}) \oplus H_4(LX/\!\!/S^1;\mathbb{Z})$$

is the $\mathbb{Z}$ summand in $H_4(LX/\!\!/S^1;\mathbb{Z})$. Hence, ψ induces an isomorphism on $H_4(-;\mathbb{Z})$. $\qquad\square$

Question 2.27. Is there a 'discrete' Godbillon–Vey class similar to the case codimension 1 PL foliations in [GS87], to give a nontrivial map $H_5(\overline{\mathrm{B}\Gamma}_2^{\mathrm{PL}};\mathbb{Z}) \to \mathbb{R}$?

2.2 Homology of PL surface homeomorphisms made discrete

To relate the group homology of PL surface homeomorphisms to the homotopy type of $\overline{\mathrm{B}\Gamma}_2^{\mathrm{PL}}$, we first recall a version of the Mather–Thurston theorem that the author proved [Nar23, § 5]. Let M be an n-dimensional PL manifold possibly with a nonempty boundary. The topological group $\mathrm{PL}(M, \mathrm{rel}\,\partial)$ is the realization of the simplicial group $S_\bullet\mathrm{PL}(M, \mathrm{rel}\,\partial)$ whose k-simplices are given by the set of PL homeomorphisms of $\Delta^k \times M$ that commute with the projection to the first factor and whose supports are away from the boundary of M. We have the map $\mathrm{PL}(M, \mathrm{rel}\,\partial)^\delta \to \mathrm{PL}(M, \mathrm{rel}\,\partial)$ given by the inclusion of 0-simplices. This map induces the map between classifying spaces

$$\mathrm{BPL}(M, \mathrm{rel}\,\partial)^\delta \to \mathrm{BPL}(M, \mathrm{rel}\,\partial),$$

whose homotopy fiber is denoted by $\overline{\mathrm{BPL}(M, \mathrm{rel}\,\partial)}$. This homotopy fiber can also be described as the realization of the semi-simplicial set $S_\bullet(\overline{\mathrm{BPL}(M, \mathrm{rel}\,\partial)})$ whose k-simplices are given by the set of codimension n foliations on $\Delta^k \times M$ that are transverse to the fibers of the projection $\Delta^k \times M \to \Delta^k$ and the holonomies are compactly supported PL homeomorphisms of the fiber M. Note that $S_\bullet\mathrm{PL}(M, \mathrm{rel}\,\partial)$ acts levelwise on the simplices $S_\bullet(\overline{\mathrm{BPL}(M, \mathrm{rel}\,\partial)})$. Hence, we have an action of $\mathrm{PL}(M, \mathrm{rel}\,\partial)$ on $\overline{\mathrm{BPL}(M, \mathrm{rel}\,\partial)}$.

On the other hand, the space $\overline{\mathrm{BPL}(M, \mathrm{rel}\,\partial)}$ is related to the classifying space of the groupoid Γ_n^{PL} as follows. Recall that forgetting the germ of the foliation of foliated microbundles induces the map

$$\nu\colon \mathrm{B}\Gamma_n^{\mathrm{PL}} \to \mathrm{BPL}(\mathbb{R}^n)$$

between classifying spaces. Let $\tau_M\colon M \to \mathrm{BPL}(\mathbb{R}^n)$ be a map that classifies the tangent microbundle of M. Let $\mathrm{Sect}_\partial(\tau_M^*(\nu))$ be the space of sections of the pullback bundle $\tau_M^*(\nu)$

over M that are supported away from the boundary. The support of a section is measured with respect to a fixed base section. By the obstruction theory and the fact that the fiber of the bundle $\tau_M^*(\nu)$ over M is at least n-connected, the space of sections is connected. So different choices of a base section do not change the homotopy type of the compactly supported sections.

We recall from [Nar17, §1.2.2] how $\mathrm{PL}(M, \mathrm{rel}\,\partial)$ acts on $\mathrm{Sect}_\partial(\tau_M^*(\nu))$. The PL tangent microbundle of the PL manifold M is the microbundle

$$M \xrightarrow{\Delta} M \times M \xrightarrow{pr_1} M.$$

A germ of PL foliation c on $\Delta^p \times M \times M$ at $\Delta^p \times \mathrm{diag}M$ which is transverse to the fiber of the projection $id \times pr_1 : \Delta^p \times M \times M \to \Delta^p \times M$ is said to be horizontal at $x \in M$ if there exists a neighborhood U around x such that the restriction of the foliation c to $\Delta^p \times U \times U$ is induced by the projection $\Delta^p \times U \times U \to U$ on the last factor. By the support of c we mean the set of $x \in M$ where c is not horizontal.

Now we define the semi-simplicial set $S_\bullet(\mathrm{Sect}_c(\tau_M^*(\nu)))$ whose p-simplices are given as the set of germs of PL foliations on $\Delta^p \times M \times M$ at $\Delta^p \times \mathrm{diag}M$ which are transverse to the fiber of the projection $id \times pr_1 : \Delta^p \times M \times M \to \Delta^p \times M$ and have compact support. The realization of this semi-simplicial set gives a model for the compactly supported sections $\mathrm{Sect}_c(\tau_M^*(\nu))$. Similarly to the previous case, there is an obvious action of $S_\bullet(\mathrm{PL}(M, \mathrm{rel}\,\partial))$ on $S_\bullet(\mathrm{Sect}_c(\tau_M^*(\nu)))$.

In this model, there is a natural map [Nar17, §1.2.2]

$$\overline{\mathrm{BPL}(M, \mathrm{rel}\,\partial)} \to \mathrm{Sect}_\partial(\tau_M^*(\nu)) \tag{2.28}$$

that is $\mathrm{PL}(M, \mathrm{rel}\,\partial)$-equivariant, and we showed that it induces a homology isomorphism [Nar23].[2] Therefore, the induced map between the homotopy quotients of the actions of $\mathrm{PL}(M, \mathrm{rel}\,\partial)$ on both sides also induces a homology isomorphism. Hence, $\mathrm{BPL}^\delta(M, \mathrm{rel}\,\partial)$ is homology isomorphic to $\mathrm{Sect}_\partial(\tau_M^*(\nu))/\!\!/\mathrm{PL}(M, \mathrm{rel}\,\partial)$.

Proof of Theorem 1.5. Let Σ be an oriented closed surface possibly with nonempty boundary. To show that the map

$$\mathrm{BPL}_0^\delta(\Sigma, \mathrm{rel}\,\partial) \to \mathrm{BPL}_0(\Sigma, \mathrm{rel}\,\partial)$$

induces an isomorphism on $H_*(-;\mathbb{Z})$ for $* \leq 2$ and a surjection on $H_3(-;\mathbb{Z})$, it is enough to show that $H_*(\overline{\mathrm{BPL}(\Sigma, \mathrm{rel}\,\partial)}; \mathbb{Z})$ vanishes for $* \leq 2$. By the Mather–Thurston theorem described above, these groups are isomorphic to $H_*(\mathrm{Sect}_\partial(\tau_\Sigma^*(\nu)); \mathbb{Z})$. Hence, it is enough to show that $\mathrm{Sect}_\partial(\tau_\Sigma^*(\nu))$ is 2-connected. Note that, in general, if the fiber of a fibration $\pi: E \to M^n$ is k-connected, elementary obstruction theory argument implies that the space of section of π is $(k-n)$-connected. Now recall that the homotopy fiber of the fibration $\tau_\Sigma^*(\nu) \to \Sigma$ is $\overline{\mathrm{B}\Gamma}_2^{\mathrm{PL}}$ which is 4-connected by Theorem 1.2. Therefore, the space $\mathrm{Sect}_\partial(\tau_\Sigma^*(\nu))$ is 2-connected. $\square$

Remark 2.29. Calegari and Rolfsen proved in particular the local indicability of PL homeomorphisms [CR15, Theorem 3.3.1] of manifolds relative to the nonempty boundary. As a consequence of their local indicability result, one deduces that no finitely generated subgroup of $\mathrm{PL}_{c,0}^\delta(\Sigma)$ is a simple group. However, our theorem shows that the ambient group $\mathrm{PL}_{c,0}^\delta(\Sigma)$ is simple, which is similar to Thurston's stability for C^1-diffeomorphisms groups [Thu74c].

Remark 2.30. We know that $\mathrm{PL}_0(\Sigma, \mathrm{rel}\,\partial) \simeq \mathrm{Diff}_0(\Sigma, \mathrm{rel}\,\partial)$ [BH81, p. 8]. Given that the homotopy type of $\mathrm{Diff}_0(\Sigma, \mathrm{rel}\,\partial)$ is completely known for all surfaces [EE69], we could also compute

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[2] Gael Meigniez also told the author that his method in [Mei21] can be used to prove the PL version of the Mather–Thurston theorem.

the second group homology of $\mathrm{PL}_0^\delta(\Sigma, \mathrm{rel}\,\partial)$ with $\mathbb{F}_p$ coefficients. For example, if Σ is a hyperbolic surface, we obtain that $H_2(\mathrm{BPL}_0^\delta(\Sigma, \mathrm{rel}\,\partial); \mathbb{Z})$ is trivial.

In this dimension, it is known [BH81, p. 8] that $\mathrm{PL}^+(\mathbb{R}^2) \simeq \mathrm{SO}(2)$. Therefore, $H^*(\mathrm{BPL}^+(\mathbb{R}^2); \mathbb{Q})$ is generated by the Euler class $e \in H^2(\mathrm{BPL}^+(\mathbb{R}^2); \mathbb{Q})$. A consequence of our computation with Greenberg's model is the following nonvanishing result.

THEOREM 2.31. *The classes $\nu^*(e^k) \in H^{2k}(\mathrm{B\Gamma}_2^{\mathrm{PL}}; \mathbb{Q})$ are nontrivial for all k.*

This is in contrast to the smooth case. Since in the smooth case we also have the Euler class $\nu^*(e) \in H^2(\mathrm{B\Gamma}_2; \mathbb{Q})$ and as a consequence of the Bott vanishing theorem $\nu^*(e^4)$ vanishes in $H^8(\mathrm{B\Gamma}_2; \mathbb{Q})$. However, as we will see in the PL case, all the powers $\nu^*(e^k)$ are nontrivial. Instead of the identity component, if we consider the entire group $\mathrm{PL}^\delta(\Sigma, \mathrm{rel}\,\partial)$, using Theorem 2.31 and the method in [Nar17, Theorem 0.4], we can prove the following nonvanishing result in *the stable range*.

THEOREM 2.32. *Let Σ be a compact orientable surface. Then the map*

$$H^*(\mathrm{BPL}(\Sigma, \mathrm{rel}\,\partial); \mathbb{Q}) \to H^*(\mathrm{BPL}^\delta(\Sigma, \mathrm{rel}\,\partial); \mathbb{Q})$$

induces an injection when $* \le (2g(\Sigma) - 2)/3$ where $g(\Sigma)$ is the genus of the surface Σ.

As a consequence of the Madsen–Weiss theorem [MW07], $H^*(\mathrm{BPL}(\Sigma, \mathrm{rel}\,\partial); \mathbb{Q})$ is isomorphic to the polynomial ring $\mathbb{Q}[\kappa_1, \kappa_2, \dots]$ in the stable range, $* \le (2g(\Sigma) - 2)/3$. Here the κ_i are certain characteristic classes of surface bundles known as ith MMM classes whose degree is $2i$. This is also in contrast to the case of smooth diffeomorphisms. In particular, we have the following nonvanishing result.

COROLLARY 2.33. *The κ_i are all nontrivial in $H^*(\mathrm{BPL}^\delta(\Sigma, \mathrm{rel}\,\partial); \mathbb{Q})$ as long as $i \le (2g(\Sigma) - 2)/6$.*

Since the proof of Theorem 2.32 uses some background from [Nar17, §§ 1 and 2], we shall first explain how to adapt these techniques to the case of PL homeomorphisms of surfaces. The proofs in [Nar17] are formulated for diffeomorphism groups of surfaces, but since, in dimension 2, the diffeomorphism group of a surface has the same homotopy type as the PL homeomorphism group, we can use the results of [Nar17] as follows.

In dimension 2, we know [BH81, p. 8] that $\phi \colon \mathrm{BGL}_2^+(\mathbb{R}) \to \mathrm{BPL}_2^+(\mathbb{R})$ and $\eta \colon \mathrm{BPL}_2^+(\mathbb{R}) \to \mathrm{BTop}_2^+$ are weak homotopy equivalences where Top_2^+ is the group of orientation-preserving homeomorphisms of $\mathbb{R}^2$. Let ρ be a homotopy inverse to η. We shall consider the tangential structures

$$\nu \colon \mathrm{B\Gamma}_2^{\mathrm{PL}} \to \mathrm{BPL}^+(\mathbb{R}^2),$$

$$\nu^s \colon \phi^*(\nu) \to \mathrm{BGL}_2^+(\mathbb{R}),$$

$$\nu^t \colon \rho^*(\nu) \to \mathrm{BTop}_2^+,$$

where $\phi^*(\nu)$ and $\rho^*(\nu)$ are the pullbacks of the fibration induced by ν with the homotopy fiber $\overline{\mathrm{B\Gamma}}_2^{\mathrm{PL}}$. Hence, these fibrations are all fiber homotopy equivalent. Let $\tau_\Sigma^s, \tau_\Sigma^p$ and τ_Σ^t be the map classifying the tangent (micro)bundles in the smooth, PL and topological category. respectively.

Given the above homotopy equivalences, the space of sections $\mathrm{Sect}_\partial((\tau_\Sigma^s)^*(\nu^s))$, $\mathrm{Sect}_\partial((\tau_\Sigma^p)^*(\nu))$ and $\mathrm{Sect}_\partial((\tau_\Sigma^t)^*(\nu^t))$ are also all homotopy equivalent. Now since we have

$\mathrm{Diff}(\Sigma, \mathrm{rel}\,\partial) \simeq \mathrm{Homeo}(\Sigma, \mathrm{rel}\,\partial) \simeq \mathrm{PL}(\Sigma, \mathrm{rel}\,\partial)$ [BH81, p. 8], we get a zigzag of weak homotopy equivalences

$$|S.(\mathrm{Sect}_c((\tau_\Sigma^s)^*(\nu^s)))|/\!\!/ |S.(\mathrm{Diff}(\Sigma, \mathrm{rel}\,\partial))| \qquad\qquad |S.(\mathrm{Sect}_c((\tau_\Sigma^p)^*(\nu)))|/\!\!/ |S.(\mathrm{PL}(\Sigma, \mathrm{rel}\,\partial))|$$

$$\simeq \qquad\qquad\qquad \simeq$$

$$|S.(\mathrm{Sect}_c((\tau_\Sigma^t)^*(\nu^t)))|/\!\!/ |S.(\mathrm{Homeo}(\Sigma, \mathrm{rel}\,\partial))|.$$

But as we explained, the analog of the Mather–Thurston theorem for PL homeomorphisms, the equivariance of the homology equivalence (2.28) implies that

$$|S_\bullet(\mathrm{Sect}_c((\tau_\Sigma^p)^*(\nu)))|/\!\!/ |S_\bullet(\mathrm{PL}(\Sigma, \mathrm{rel}\,\partial))|$$

is homology isomorphic to $\mathrm{BPL}^\delta(\Sigma, \mathrm{rel}\,\partial)$. Given the above zigzag, we have the following lemma.

LEMMA 2.34. *The classifying space* $\mathrm{BPL}^\delta(\Sigma, \mathrm{rel}\,\partial)$ *is homology isomorphic to* $|S_\bullet(\mathrm{Sect}_c((\tau_\Sigma^s)^*$
$(\nu^s)))|/\!\!/ |S_\bullet(\mathrm{Diff}(\Sigma, \mathrm{rel}\,\partial))|.$

On the other hand, there is a natural map [Nar17, (1.12)] from

$$|S_\bullet(\mathrm{Sect}_c((\tau_\Sigma^s)^*(\nu^s)))|/\!\!/ |S_\bullet(\mathrm{Diff}(\Sigma, \mathrm{rel}\,\partial))|$$

to the moduli space $\mathcal{M}^{\nu^s}(\Sigma)$ of tangential ν^s-structures on Σ defined in [Nar17, § 1.2.3]. The space $\mathcal{M}^{\nu^s}(\Sigma)$ has been very well studied, and we used the techniques of [RW16, GMTW09, GRW10] in [Nar17] to show that it exhibits homological stability. As we shall recall in the following proof, its stable homology is described in terms of the Madsen–Tillman spectrum of the corresponding tangential structure.

Proof of Theorems 2.31 and 2.32. The key point in Lemma 2.20 is that Greenberg's model for $\mathrm{B}\Gamma_2^{\mathrm{PL}}$ allows us, up to homotopy, to find a section for the map

$$\nu \colon \mathrm{B}\Gamma_2^{\mathrm{PL}} \to \mathrm{B}S^1.$$

Recall that a null-homotopic map $X \to \mathrm{B}S^1$ and the natural map $LX/\!\!/S^1 \to \mathrm{B}S^1$ induce a map $rX \to \mathrm{B}S^1$. But a section to the map $LX/\!\!/S^1 \to \mathrm{B}S^1$ induces a section for $rX \to \mathrm{B}S^1$. Therefore, $\nu^*(e^k) \in H^{2k}(\mathrm{B}\Gamma_2^{\mathrm{PL}}; \mathbb{Q})$ are nontrivial for all k.

Now let γ be the tautological 2-plane bundle over $\mathrm{B}S^1$ and let $\mathrm{MT}\nu^s$ be the Thom spectrum of the virtual bundle $(\nu^s)^*(-\gamma)$. And let $\Omega_0^\infty \mathrm{MT}\nu^s$ be the base point component of the infinite loop space associated with this Thom spectrum. Then exactly the same method as in [Nar17, Theorem 0.4] goes through to show that there is a map

$$\mathrm{BPL}(\Sigma, \mathrm{rel}\,\partial)^\delta \to \Omega_0^\infty \mathrm{MT}\nu^s,$$

which is homology isomorphism up to degrees $* \le (2g(\Sigma) - 2)/3$. Hence, we obtain

$$H^*(\mathrm{BPL}(\Sigma, \mathrm{rel}\,\partial)^\delta; \mathbb{Q}) \cong \mathrm{Sym}^*(H^{*>2}(\mathrm{B}\Gamma_2^{\mathrm{PL}}; \mathbb{Q})[-2])$$

in degrees $* \le (2g(\Sigma) - 2)/3$ where the right-hand side is the polynomial ring with the generators in the graded vector space $H^{*>2}(\mathrm{B}\Gamma_2^{\mathrm{PL}}; \mathbb{Q})$ which is shifted by degree 2. Since by the Madsen–Weiss theorem $H^*(\mathrm{BPL}(\Sigma, \mathrm{rel}\,\partial); \mathbb{Q})$ is isomorphic to $\mathrm{Sym}^*(H^{*>2}(\mathrm{B}S^1; \mathbb{Q})[-2])$ in the

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same stable range and we know that

$$H^*(\mathrm{B}S^1; \mathbb{Q}) \to H^*(\mathrm{B}\Gamma_2^{\mathrm{PL}}; \mathbb{Q})$$

is injective, then so is

$$H^*(\mathrm{BPL}(\Sigma, \mathrm{rel}\,\partial); \mathbb{Q}) \to H^*(\mathrm{BPL}(\Sigma, \mathrm{rel}\,\partial)^\delta; \mathbb{Q})$$

in stable range. $\qquad\square$

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