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The generic rank of the Baum–Bott map for foliations of the projective plane

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Abstract

Our main result says that the generic rank of the Baum–Bott map for foliations of degree d, $d \geqslant 2$, of the projective plane is $d^2 + d$. This answers a question of Gómez-Mont and Luengo and shows that are no other universal relations between the Baum–Bott indexes of a foliation of $\mathbb{P}^2$ besides the Baum–Bott formula. We also define the *Camacho–Sad field* for foliations on surfaces and prove its invariance under the pull-back by meromorphic maps. As an application we prove that a generic foliation of degree $d \geqslant 2$ is not the pull-back of a foliation of smaller degree. In Appendix A we show that the monodromy of the singular set of the universal foliation with *very ample* cotangent bundle is the full symmetric group.

Contents

1. Introduction and statement of results

1.1 The Baum–Bott map

One of the most basic invariant for singularities of holomorphic foliations of surfaces is the *Baum–Bott index*: if $\mathcal{F}$ is a germ of holomorphic foliation of $(\mathbb{C}^2, 0)$ induced by a holomorphic 1-form $\omega = A(x, y)\,dy - B(x, y)\,dx$ with an isolated singularity at 0, then the Baum–Bott index of $\mathcal{F}$ at 0 is defined as

$$\mathrm{BB}(\mathcal{F}, 0) = \frac{1}{(2\pi i)^2}\int_\Gamma \eta \wedge d\eta$$

where η is any $(1, 0)$-form (C^∞ on a punctured neighborhood of $0 \in \mathbb{C}^2$) satisfying $d\omega = \eta \wedge \omega$ and Γ is the boundary of a small ball around 0 (see, for instance, [Bru04]). When the dual vector field $X = A(x, y)\partial_x + B(x, y)\partial_y$ has an invertible linear part, i.e. $\det(DX(0)) \neq 0$, a simple computation

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shows that

$$\mathrm{BB}(\mathcal{F},0)=\frac{\mathrm{tr}^2(DX(0))}{\det(DX(0))}.$$

Singularities with invertible linear part are usually called *simple singularities.*

Let S be a compact complex surface S. A singular foliation by curves $\mathcal{F}$ on S can be defined by a global holomorphic section of $TS\otimes\mathcal{L}$, for a suitable line bundle $\mathcal{L}$. This line bundle $\mathcal{L}$ is the cotangent bundle of $\mathcal{F}$ and is usually denoted by $T^*_{\mathcal{F}}$. We will denote by $\mathbb{F}\mathrm{ol}(\mathcal{L})$ the space of foliations on S with cotangent bundle $\mathcal{L}$, i.e.

$$\mathbb{F}\mathrm{ol}(\mathcal{L})=\mathbb{P}\mathrm{H}^0(S,TS\otimes\mathcal{L}).$$

For any $\mathcal{F}\in\mathbb{F}\mathrm{ol}(\mathcal{L})$ with isolated singularities, $\mathrm{sing}(\mathcal{F})$, the *singular set* of $\mathcal{F}$, contains $N(\mathcal{L})=c_2(TS\otimes\mathcal{L})$ singularities counted with multiplicities.

If there exists a foliation $\mathcal{F}_0\in\mathbb{F}\mathrm{ol}(\mathcal{L})$ with only simple singularities then the set $U\subset\mathbb{F}\mathrm{ol}(\mathcal{L})$ of foliations with only simple singularities is an open Zariski set. In this case any foliation $\mathcal{F}\in\mathbb{F}\mathrm{ol}(\mathcal{L})$ has exactly $N(\mathcal{L})=N$ singularities. If $\mathrm{sing}(\mathcal{F}_0)=\{p_1,\dots,p_N\}$, then there exist a neighborhood $V\subset U$ and holomorphic maps $\gamma_1,\dots,\gamma_N\colon V\to S$ such that $\gamma_j(\mathcal{F}_0)=p_j$ and, for any $\mathcal{F}\in V$, we have $\mathrm{sing}(\mathcal{F})=\{\gamma_1(\mathcal{F}),\dots,\gamma_N(\mathcal{F})\}$. In this case, we can define a holomorphic map $\mathrm{BB}\colon V\to\mathbb{C}^N$ by

$$\mathrm{BB}(\mathcal{F})=(\mathrm{BB}(\mathcal{F},\gamma_1(\mathcal{F})),\dots,\mathrm{BB}(\mathcal{F},\gamma_N(\mathcal{F}))).$$

We will call the map BB the *local Baum–Bott* map. We observe that it is possible to extend the domain of BB to U, if we symmetrize the coordinates in $\mathbb{C}^N$. More precisely, if we denote by $\mathbb{C}^N/S_N$ the quotient of $\mathbb{C}^N$ by the equivalence relation which identifies two points $(z_1,\dots,z_N)$ and $(z_{\sigma(1)},\dots,z_{\sigma(N)})$, where $\sigma\in S_N$ (the symmetric group in N elements), then we define $\mathbb{BB}\colon U\to\mathbb{C}^N/S_N$ by

$$\mathbb{BB}(\mathcal{F})=[\mathrm{BB}(\mathcal{F},p_1),\dots,\mathrm{BB}(\mathcal{F},p_N)],$$

where $\mathrm{sing}(\mathcal{F})=\{p_1,\dots,p_N\}$ and $[\lambda_1,\dots,\lambda_N]$ denotes the class of $(\lambda_1,\dots,\lambda_N)$ in $\mathbb{C}^N/S_N$. Of course, this map can be extended to a rational map

$$\mathbb{BB}\colon\mathbb{F}\mathrm{ol}(\mathcal{L})\dashrightarrow(\mathbb{P}^1)^N/S_N\cong\mathbb{P}^N$$

which we will call the *global Baum–Bott map*.

The well-known Baum–Bott index theorem [BB70] (first proved by Chern [Che73] in the case of foliations with only simple singularities) says that for a foliation $\mathcal{F}$ with isolated singularities of compact surface S,

$$N_{\mathcal{F}}\cdot N_{\mathcal{F}}=\sum_{p\in\mathrm{sing}(\mathcal{F})}\mathrm{BB}(\mathcal{F},p),$$

where $N_{\mathcal{F}}$ is the normal bundle of $\mathcal{F}$, i.e. $N_{\mathcal{F}}=T^*_{\mathcal{F}}\otimes KS^*$ with KS being the canonical bundle of S. In particular the maximal rank of $\mathbb{BB}$ on $\mathbb{F}\mathrm{ol}(\mathcal{L})$ is always less than $N(\mathcal{L})$ and the Baum–Bott map is never dominant: the closure of its image has codimension at least one.

In this paper we are interested in the generic rank of the Baum–Bott map just defined for foliations of the projective plane. Of course, the generic rank of the local and global Baum–Bott maps coincide. Recall that the degree of a foliation $\mathcal{F}$ of $\mathbb{P}^2$, denoted by $\deg(\mathcal{F})$, is defined as the number d of tangencies of a generic line with $\mathcal{F}$ and that $\mathcal{F}$ has $N(d):=N(T^*_{\mathcal{F}})=d^2+d+1$ singularities counted with multiplicities.

For foliations of degree 0 of $\mathbb{P}^2$ we have just one singularity and its index is determined by Baum–Bott's theorem. For foliations of degree 1 we have three singularities (counted with multiplicities) and every foliation admits an invariant line. Camacho–Sad index theorem imposes an extra condition

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on the Baum–Bott indexes and thus the rank of the Baum–Bott map is one, see [GL97]. A natural problem, proposed by Gomez-Mont and Luengo in [GL97], is the following.

Question 1. When $d \geqslant 2$, are there other *hidden* relations between the Baum–Bott indexes of a degree d foliation of the projective plane? In other terms, what is the generic rank of the Baum–Bott map for foliations of projective plane?

Our first result says that the only *universal* relation among the Baum–Bott indexes is Baum–Bott's formula.

Theorem 1. *If $d \geqslant 2$, then the maximal rank of the Baum–Bott map for degree d foliations of $\mathbb{P}^2$ is $N(d) - 1 = d^2 + d$.*

An immediate consequence of Theorem 1 is the following.

Corollary 1. *If $d \geqslant 2$, then the dimension of the generic fiber of the map $\mathbb{BB}\colon \mathbb{F}\mathrm{ol}(d) \dashrightarrow \mathbb{P}^N$ is $3d+2$.*

In fact, one has just to remark that $\dim \mathbb{F}\mathrm{ol}(d) = (d+1)(d+3) - 1$. We do not know whether the generic fiber of the Baum–Bott map is irreducible or not.

1.2 The rank at Jouanolou's foliations

In general, it does not seem to be an easy problem to compute the rank of the Baum–Bott map at a specific foliation. For $\mathcal{J}_d$, the degree d Jouanolou foliation (cf. § 3 for the definition), we are able to determine the rank.

Theorem 2. *For any $d \geqslant 2$, the rank of the local Baum–Bott map at $\mathcal{J}_d$ is*

$$\frac{d^2 + 7d - 6}{2}.$$

In particular, if $d = 2, 3$, then $\mathrm{rk}(\mathrm{BB}, \mathcal{J}_d) = d^2 + d$, and if $d \geqslant 4$, then $\mathrm{rk}(\mathrm{BB}, \mathcal{J}_d) < d^2 + d$.

Note that at these points the rank of the global Baum–Bott map is strictly less then the rank of the local Baum–Bott map: since all of the singularities of $\mathcal{J}_d$ have the same Baum–Bott indexes, then $\mathrm{BB}(\mathcal{J}_d) \in (\mathbb{P}^1)^{N(d)}$ is on the critical set of the symmetrization

$$(\mathbb{P}^1)^{N(d)} \to \mathbb{P}^{N(d)}.$$

1.3 The Camacho–Sad field

Another local index often considered in the theory of holomorphic foliations is the so-called *Camacho–Sad index* of a foliation $\mathcal{F}$ with respect to a separatrix C through a singular point p. Suppose that the germ of $\mathcal{F}$ at $p \in C$ is represented by a germ of holomorphic 1-form ω and that $(f = 0)$ is a reduced equation of the germ of C at p. Then there exist germs $g, h \in \mathcal{O}_p$ and a germ of holomorphic 1-form η at p such that $g\omega = h \cdot df + f \cdot \eta$ and $g, h|_C \not\equiv 0$ (cf. [Suw95] or [Bru04]). The Camacho–Sad index of $\mathcal{F}$ at p with respect to C, is defined as

$$\mathrm{CS}(\mathcal{F}, C, p) = \mathrm{Res}_p\left(-\frac{\eta}{h}\right) = \frac{1}{2\pi i}\int_\gamma -\frac{\eta}{h},$$

where γ is a union of small circles positively oriented around p, one for each local irreducible branch of the germ of C at p.

If p is a reduced and simple singularity of $\mathcal{F}$, i.e. we have two distinct non-zero eigenvalues at p, say λ_1 and $\lambda_2 \neq 0$, such that $\lambda_1/\lambda_2 \notin \mathbb{Q}_+$, then it is known that $\mathcal{F}$ has exactly two local separatrices, say Σ_j, $j = 1, 2$, tangent to the eigenspace associated to λ_j. In this case, we have

$$\begin{aligned} \mathrm{CS}(\mathcal{F}, \Sigma_1, p) &= \lambda_2/\lambda_1, \\ \mathrm{CS}(\mathcal{F}, \Sigma_2, p) &= \lambda_1/\lambda_2, \\ \mathrm{BB}(\mathcal{F}, p) &= \mathrm{CS}(\mathcal{F}, \Sigma_1, p) + \mathrm{CS}(\mathcal{F}, \Sigma_2, p) + 2. \end{aligned} \tag{1}$$

If p is reduced and non-simple singularity, i.e. p is a saddle-node singularity, then, in general, one has just one analytic local separatrix, which is tangent to the eigenspace of the non-zero eigenvalue. The Camacho–Sad index with respect to this separatrix is zero (cf. [Bru04] or [CS82]). In the direction of the zero eigenvalue there is always a unique formal separatrix (which sometimes is convergent). This follows from the formal normal form of the saddle-node (cf. [MR82]): the foliation is formally equivalent to that induced by

$$\omega = x^{k+1}\, dy - y(1 + \lambda \cdot x^k)\, dx,$$

where $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. When there exists an analytic separatrix tangent to the eigendirection of the eigenvalue zero, then its Camacho–Sad index is λ. Even if this separatrix is formal, it can be proved that the number λ is invariant by formal diffeomorphisms (cf. [MR82]). Therefore, we can define its Camacho–Sad index as λ.

On the other hand, Seidenberg's resolution theorem asserts that for any foliation $\mathcal{F}$ on a surface S there exists finite composition of punctual blow-ups, say $\pi \colon M \to S$, such that the foliation $\tilde{\mathcal{F}} := \Pi^*(\mathcal{F})$ (the strict transform) on M has only reduced singularities. The foliation $\tilde{\mathcal{F}}$ is usually called a *resolution* of $\mathcal{F}$.

Definition 1. Let $\mathcal{F}$ be a foliation on a compact surface S. We define its *Camacho–Sad field*, denoted by $\mathbb{K}(\mathcal{F})$, as follows.

- *Reduced case.* All singularities of $\mathcal{F}$ are either reduced or saddle-nodes. Let $\mathrm{sing}(\mathcal{F}) = \{p_1, \dots, p_k\}$ and let Σ_j^i, $i = 1, 2$, be the two separatrices of $\mathcal{F}$ through p_j (formal or not), $j = 1, \dots, k$. Then we define
$$\mathbb{K}(\mathcal{F}) = \mathbb{Q}(\mathrm{CS}(\mathcal{F}, \Sigma_1^1, p_1), \mathrm{CS}(\mathcal{F}, \Sigma_1^2, p_1), \dots, \mathrm{CS}(\mathcal{F}, \Sigma_k^2, p_k)).$$
- *General case.* We take any resolution $\tilde{\mathcal{F}}$ of $\mathcal{F}$ and define $\mathbb{K}(\mathcal{F}) = \mathbb{K}(\tilde{\mathcal{F}})$.

We invite the reader to verify that the definition above does not depend on the chosen resolution using the following facts.

(i) There exists a minimal resolution, that is, a resolution with the minimal number of blowing-ups.
(ii) When we blow-up in a reduced and simple singularity with Camacho–Sad indexes with respect to the separatrixes λ and λ^{-1}, then two new simple and reduced singularities appear and their Camacho–Sad indexes are $\lambda - 1$, $1/(\lambda - 1)$, $\lambda^{-1} - 1$ and $\lambda/(1 - \lambda)$.
(iii) When we blow-up at a saddle node with Camacho–Sad indexes 0 and λ, then two new singularities appear, one saddle-node with Camacho–Sad indexes 0 and $\lambda - 1$ and a simple singularity with both Camacho–Sad indexes equal to -1.

We will say that a subset of $\mathbb{F}\mathrm{ol}(d)$ is generic if its complement has zero Lebesgue measure. The next corollary is in fact a reformulation of Theorem 1 in terms of the concept just defined.

Corollary 2. *If $d \geqslant 2$, then there exists a generic subset $G(d) \subset \mathbb{F}\mathrm{ol}(d)$ such that for any $\mathcal{F} \in G(d)$ the transcendence degree of $\mathbb{K}(\mathcal{F})$ over $\mathbb{Q}$ is $d^2 + d$.*

Our main result concerning the Camacho–Sad field is the following.

Theorem 3. *Let M and S be two complex compact and connected surfaces, $\mathcal{F}$ a foliation on S and $\phi : M \dashrightarrow S$ a meromorphic map. Suppose that ϕ has generic rank two. Then $\mathbb{K}(\phi^*(\mathcal{F})) = \mathbb{K}(\mathcal{F})$.*

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One of our motivations to introduce the Camacho–Sad field was to prove the following corollary.

Corollary 3. *The generic foliation of degree $d \geqslant 2$ is not the pull-back by a rational map of a foliation of smaller degree.*

1.4 Monodromy

In Appendix A we prove that the monodromy of the singular set of a generic family of holomorphic foliations is the full symmetric group. An immediate corollary is that the functions $\gamma_1, \dots, \gamma_N : V \subset \mathbb{F}\mathrm{ol}(d) \to \mathbb{P}^2$ used to parametrize the singularities in the proof of Theorem 1 although algebraic are not solvable by radicals when $d \geqslant 2$, i.e. they cannot be expressed in terms of combinations of radicals of rational functions on $\mathbb{F}\mathrm{ol}(d)$.

2. The generic rank of Baum–Bott's map

2.1 Some words about the notation

Let $\mathbb{F}\mathrm{ol}(d)$ be the space of foliations of degree d on $\mathbb{P}^2$, $d \geqslant 0$. A foliation of degree d on $\mathbb{P}^2$, can be expressed in an affine coordinate system $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2$, by a polynomial vector field on $\mathbb{C}^2$ of the form $X = P(x, y)\partial_x + Q(x, y)\partial_y$, where

$$\begin{cases} P(x,y) = p(x,y) + x \cdot g(x,y) \\ Q(x,y) = q(x,y) + y \cdot g(x,y) \end{cases} \tag{2}$$

with $\max(\deg(p), \deg(q)) \leqslant d$ and g is a homogeneous polynomial of degree d.

We will denote by $\mathbb{R}(d) \subset \mathbb{F}\mathrm{ol}(d)$ the Zariski dense subset of foliations $\mathcal{F}$ of degree d with all singularities simple. If $\mathcal{F} \in \mathbb{F}\mathrm{ol}(d)$, then $N\mathcal{F} = \mathcal{O}(d+2)$. Thus, the Baum–Bott theorem mentioned in the introduction says that

$$\sum_{p \in \mathrm{sing}\mathcal{F}} \mathrm{BB}(\mathcal{F}, p) = (d+2)^2,$$

for every $\mathcal{F} \in \mathbb{F}\mathrm{ol}(d)$ with isolated singularities. We recall that $\mathbb{R}(d)$ is open and dense in $\mathbb{F}\mathrm{ol}(d)$, cf., for instance, [Lin88]. Recall that for any $\mathcal{F}_0 \in \mathbb{R}(d)$, $\#(\mathrm{sing}(\mathcal{F}_0)) = d^2 + d + 1$.

2.2 Idea of the proof of Theorem 1 and the key lemma

The proof of Theorem 1 will be by induction on $d \geqslant 2$. The result for $d = 2$ is due to Guillot (cf. [Gui06]). Note that Theorem 2 contains, in particular, a new proof of Guillot's result.

Before entering into the details of the proof, let us give a rough idea of it. We start with $\mathcal{F} \in \mathbb{F}\mathrm{ol}(d-1)$, a degree $d-1$ foliation on $\mathbb{P}^2$, such that $\mathcal{F}$ has $(d-1)^2 + (d-1) + 1$ distinct singularities and the Baum–Bott map at $\mathcal{F}$ has rank $(d-1)^2 + (d-1)$. After multiplying $\mathcal{F}$ by a sufficiently general line $\ell \subset \mathbb{P}^2$ we obtain an element of $\mathbb{F}\mathrm{ol}(d)$ with codimension one singular set. Let $\mathcal{G} \in \mathbb{F}\mathrm{ol}(d)$ be such that the tangency of $\mathcal{F}$ and $\mathcal{G}$ is a reduced curve intersecting ℓ transversely. If we consider the pencil of foliations generated by $\ell\mathcal{F}$ and $\mathcal{G}$, then we can assume that the elements of this pencil that are sufficiently close to $\ell\mathcal{F}$ have $(d-1)^2 + (d-1) + 1$ singularities close to $\mathrm{sing}(\mathcal{F})$ and $2d$ singularities close to ℓ. Noting the elements of such a pencil by $\{\mathcal{F}_t\}_{t \in \mathbb{P}^1}$, $\mathcal{F}_0 = \ell\mathcal{F}$, simple computations shows that

$$\lim_{t \to 0} \mathrm{BB}(\mathcal{F}_t, p(t)) = \infty,$$

for every holomorphic map $p : \Delta(0, \epsilon) \to \mathbb{P}^2$ such that $p(0) \in \ell$ and $p(t) \in \mathrm{sing}(\mathcal{F}_t)$. We point that such maps are completely determined by the initial condition $p(0)$ and that $p(0)$ must belong to $\ell \cap \mathrm{tang}(\mathcal{F}, \mathcal{G})$. Moreover, there exists a logarithmic 1-form $\eta = \eta(\mathcal{F}, \mathcal{G})$ on $\ell \cong \mathbb{P}^1$ with polar set

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equal to $\ell \cap \operatorname{tang}(\mathcal{F}, \mathcal{G})$ such that

$$\lim_{t \to 0} t \cdot \mathrm{BB}(\mathcal{F}_t, p(t)) = \operatorname{Res}(\eta, p(0)).$$

With some extra work one can prove that with a suitable choice of $\mathcal{G}$ we can prescribe the residues of $\eta(\mathcal{F}, \mathcal{G})$, i.e. for every set $\mathcal{S}$ of $2d$ complex numbers subjected to the only condition that their sum is equal to zero there exists $\mathcal{G}$ such that the residues of $\eta(\mathcal{F}, \mathcal{G})$ are $\mathcal{S}$. A moment of thought shows that

$$\operatorname{rank}(\mathrm{BB}(\mathcal{F}_\epsilon)) \geqslant \underbrace{(d-1)^2 + (d-1)}_{\operatorname{rank}(\mathrm{BB}(\mathcal{F}))} + \underbrace{2d-1}_{\text{residues of } \eta} = d^2 + d - 1$$

for $\epsilon > 0$ sufficiently small. This is almost what we want to prove except that on the right-hand side we need $d^2 + d$ instead of $d^2 + d - \mathbf{1}$. Anyway, the argument sketched above gives a good *first-order approximation* of our proof.

In the actual proof, the induction step will be reduced to the following lemma.

Lemma 2.1. *Let $F = (G, H) : \mathbb{D}^* \times \mathbb{D}^{k-1} \times \mathbb{D}^\ell \to \mathbb{C}^k \times \mathbb{C}^\ell$ be a holomorphic map. Denote the variables in $\mathbb{D} \times \mathbb{D}^{k-1} \times \mathbb{D}^\ell$ by $(s, Z, T) = (s, z_1, \ldots, z_{k-1}, t_1, \ldots, t_\ell)$. Suppose that:*

(a) *H extends to a holomorphic function on $\mathbb{D} \times \mathbb{D}^{k-1} \times \mathbb{D}^\ell$ and*

$$\frac{\partial H}{\partial z_j}(0, Z, T) = 0, \quad \forall j = 1, \ldots, k-1;$$

(b) *G is of the form*

$$G(s, Z, T) = \frac{1}{s}[A(Z, T) + s \cdot R(s, X, T)],$$

where $A = (A_1, \ldots, A_k)\colon \mathbb{D}^{k-1} \times \mathbb{D}^\ell \to \mathbb{C}^k$ and $R\colon \mathbb{D}^k \times \mathbb{D}^\ell \to \mathbb{C}^k$ are holomorphic;

(c) *there exists $Z_0 \in \mathbb{D}^{k-1}$ satisfying $\det(M(Z_0, 0)) \neq 0$, where $M(Z, T)$ is the $k \times k$ matrix*

$$\begin{bmatrix} A(Z,T) \\ \dfrac{\partial A}{\partial z_1}(Z,T) \\ \vdots \\ \dfrac{\partial A}{\partial z_{k-1}}(Z,T) \end{bmatrix} := \begin{bmatrix} A_1(Z,T) & A_2(Z,T) & \cdots & A_k(Z,T) \\ \dfrac{\partial A_1}{\partial z_1}(Z,T) & \dfrac{\partial A_2}{\partial z_1}(Z,T) & \cdots & \dfrac{\partial A_k}{\partial z_1}(Z,T) \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial A_1}{\partial z_{k-1}}(Z,T) & \dfrac{\partial A_2}{\partial z_{k-1}}(Z,T) & \cdots & \dfrac{\partial A_k}{\partial z_{k-1}}(Z,T) \end{bmatrix};$$

(d) *for $Z_0 \in \mathbb{D}^{k-1}$ we have that $\operatorname{rk}(H_{Z_0}, 0) = \ell$, where $H_{Z_0}(T) = H(0, Z_0, T)$.*

Then there exists $r > 0$ such that $\operatorname{rk}(F, (s_0, Z_0, 0)) = k + \ell$ for every s_0 with $0 < |s_0| < r$.

Proof. Let $\Delta(s,Z,T)$ be given by

$$\Delta(s,Z,T) = \det \begin{bmatrix} \dfrac{\partial G}{\partial s} & \dfrac{\partial H}{\partial s} \\ \dfrac{\partial G}{\partial z_1} & \dfrac{\partial H}{\partial z_1} \\ \vdots & \vdots \\ \dfrac{\partial G}{\partial z_{k-1}} & \dfrac{\partial H}{\partial z_{k-1}} \\ \dfrac{\partial G}{\partial t_1} & \dfrac{\partial H}{\partial t_1} \\ \vdots & \vdots \\ \dfrac{\partial G}{\partial t_\ell} & \dfrac{\partial H}{\partial t_\ell} \end{bmatrix}.$$

Using part (b), we get the following relations:

$$\frac{\partial G}{\partial s}(s,Z,T) = -\frac{1}{s^2}A(Z,T) + C(s,Z,T),$$

$$\frac{\partial G}{\partial z_j}(s,Z,T) = \frac{1}{s}\frac{\partial A}{\partial z_j}(Z,T) + D_j(s,X,T),$$

$$\frac{\partial G}{\partial t_i}(s,X,T) = \frac{1}{s}\frac{\partial A}{\partial t_i}(Z,T) + E_i(s,Z,T),$$

where $C = \partial R/\partial s$, $D_j = \partial R/\partial z_j$ and $E_i = \partial R/\partial t_i$.

These relations imply that

$$\Delta(s,Z,T) = \det \begin{bmatrix} -\dfrac{1}{s^2}A(Z,T) + C(s,Z,T) & \dfrac{\partial H}{\partial s} \\ \dfrac{1}{s}\dfrac{\partial A}{\partial z_1}(Z,T) + D_1(s,Z,T) & \dfrac{\partial H}{\partial z_1}(s,Z,T) \\ \vdots & \vdots \\ \dfrac{1}{s}\dfrac{\partial A}{\partial z_{k-1}}(Z,T) + D_{k-1}(s,X,T) & \dfrac{\partial H}{\partial z_{k-1}}(s,Z,T) \\ \dfrac{1}{s}\dfrac{\partial A}{\partial t_1}(Z,T) + E_1(s,Z,T) & \dfrac{\partial H}{\partial t_1}(s,Z,T) \\ \vdots & \vdots \\ \dfrac{1}{s}\dfrac{\partial A}{\partial t_\ell}(Z,T) + E_\ell(s,Z,T) & \dfrac{\partial H}{\partial t_\ell}(s,Z,T) \end{bmatrix}$$

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$$
= \frac{1}{s^k} \det \begin{bmatrix} -\frac{1}{s} A(Z,T) + s \cdot C(s,Z,T) & \frac{\partial H}{\partial s} \\ \frac{\partial A}{\partial z_1}(Z,T) + s \cdot D_1(s,Z,T) & \frac{\partial H}{\partial z_1}(s,Z,T) \\ \vdots & \vdots \\ \frac{\partial A}{\partial z_{k-1}}(Z,T) + s \cdot D_{k-1}(s,X,T) & \frac{\partial H}{\partial z_{k-1}}(s,Z,T) \\ \frac{\partial A}{\partial t_1}(Z,T) + s \cdot E_1(s,Z,T) & \frac{\partial H}{\partial t_1}(s,Z,T) \\ \vdots & \vdots \\ \frac{\partial A}{\partial t_\ell}(Z,T) + s \cdot E_\ell(s,Z,T) & \frac{\partial H}{\partial t_\ell}(s,Z,T) \end{bmatrix}
$$

$$
= \frac{1}{s^{k+1}} \det \begin{bmatrix} -A(Z,T) + s^2 \cdot C(s,Z,T) & s \cdot \frac{\partial H}{\partial s} \\ \frac{\partial A}{\partial z_1}(Z,T) + s \cdot D_1(s,Z,T) & \frac{\partial H}{\partial z_1}(s,Z,T) \\ \vdots & \vdots \\ \frac{\partial A}{\partial z_{k-1}}(Z,T) + s \cdot D_{k-1}(s,X,T) & \frac{\partial H}{\partial z_{k-1}}(s,Z,T) \\ \frac{\partial A}{\partial t_1}(Z,T) + s \cdot E_1(s,Z,T) & \frac{\partial H}{\partial t_1}(s,Z,T) \\ \vdots & \vdots \\ \frac{\partial A}{\partial t_\ell}(Z,T) + s \cdot E_\ell(s,Z,T) & \frac{\partial H}{\partial t_\ell}(s,Z,T) \end{bmatrix}.
$$

Hence, using part (a), we deduce that $\lim_{s\to 0} s^{k+1} \cdot \Delta(s,Z,T)$ is equal to

$$
\det \begin{bmatrix} -A(Z,T) & 0 \\ \frac{\partial A}{\partial z_1}(Z,T) & 0 \\ \vdots & \vdots \\ \frac{\partial A}{\partial z_{k-1}}(Z,T) & 0 \\ \frac{\partial A}{\partial t_1}(Z,T) & \frac{\partial H}{\partial t_1}(0,Z,T) \\ \vdots & \vdots \\ \frac{\partial A}{\partial t_\ell}(Z,T) & \frac{\partial H}{\partial t_\ell}(0,Z,T) \end{bmatrix} = -\det(M(Z,T)) \cdot \det\left(\frac{\partial H_i}{\partial t_j}(0,Z,T)\right).
$$

In other words, if we set $\phi(s, Z, T) = -s^{k+1} \cdot \Delta(s, Z, T)$, then ϕ extends continuously to $s = 0$ as

$$\phi(0, Z, T) = \det(M(Z, T)) \cdot \det\left(\frac{\partial H_i}{\partial t_j}(0, Z, T)\right)_{1 \leqslant i,j \leqslant \ell}.$$

It follows from parts (c) and (d) that $\phi(0, Z_0, 0) \neq 0$. Thus, there exists $r > 0$ such that, if $0 < |s| \leqslant r$, then $\Delta(s, Z_0, 0) \neq 0$. □

Now we will work to construct a family of foliations with Baum–Bott map fitting in the above setup.

2.3 Construction of the family

Let us consider the following situation: let $\mathcal{F}_0 \in \mathbb{R}(d-1)$ be a foliation of degree $d - 1 \geqslant 2$, let L be a line on $\mathbb{P}^2$ and let $E = (\mathbb{C}^2, (x, y))$ be an affine coordinate system in $\mathbb{P}^2$, such that we have the following.

(I) $\mathrm{rk}(\mathrm{BB}, \mathcal{F}_0) = (d-1)^2 + d - 1 = d^2 - d := \ell$.

(II) $\mathrm{sing}(\mathcal{F}_0) \cap L = \emptyset$ and $\mathrm{sing}(\mathcal{F}_0) = \{q_1^0, \dots, q_{\ell+1}^0\} \subset \mathbb{C}^2 \subset \mathbb{P}^2$.

(III) $\mathcal{F}_0$ is defined on E by the polynomial vector field

$$X_0 := P_0(x, y)\partial_x + Q_0(x, y)\partial_y,$$

where $P_0(x, y) = P^0(x, y) + x \cdot g(x, y)$, $Q_0(x, y) = Q^0(x, y) + y \cdot g(x, y)$, $\deg(P^0) = \deg(Q^0) = d-1$ and $g(x, y)$ is a homogeneous polynomial of degree $d-1$. We will assume that $g(x, 0) \not\equiv 0$, i.e. the line at infinite of this affine coordinate system is not invariant for $\mathcal{F}_0$.

(IV) $L = (y = 0)$. In particular the polynomials $P(x) := P_0(x, 0)$ and $Q(x) := Q_0(x, 0)$ are relatively prime, that is $\gcd(P(x), Q(x)) = 1$.

(V) $\deg(P(x)) = d$ and $\deg(Q(x)) = d - 1$. This condition is generic and it implies that all tangencies of $\mathcal{F}_0$ with the line L are contained in $\mathbb{C}^2 \cap L$, because these tangencies are given by $(y = P(x) = 0)$.

Let V be a neighborhood of $\mathcal{F}_0$ in $\mathbb{R}(d-1)$ such that there exist holomorphic maps $q_1^0, \dots, q_{\ell+1}^0 \colon V \to \mathbb{C}^2$ with $q_j^0(\mathcal{F}_0) = q_j^0$, $j = 1, \dots, \ell + 1$, and $\mathrm{sing}(\mathcal{F}) = \{q_1^0(\mathcal{F}), \dots, q_{\ell+1}^0(\mathcal{F})\}$. We can take V sufficiently small in order to assure that $q_j^0(\mathcal{F}) \cap (y = 0) = \emptyset$ for all $j = 1, \dots, \ell+1$ and all $\mathcal{F} \in V$.

Since, by hypothesis, $\mathrm{rk}(\mathrm{BB}, \mathcal{F}_0) = d^2 - d = \ell$, there exist polynomial vector fields of the form (2), $X_1, \dots, X_\ell$, $X_i = P_i\partial_x + Q_i\partial_y$, with the following additional properties.

(VI) For any $T = (t_1, \dots, t_\ell) \in \mathbb{D}^\ell$, $X_T := X_0 + \sum_{i=1}^{\ell} t_i \cdot X_i \in V$.
In this situation, we can define $H_1 \colon \mathbb{D}^\ell \to \mathbb{C}^\ell$, by

$$H_1(T) = (\mathrm{BB}(X_T, q_1^0(X_T)), \dots, \mathrm{BB}(X_T, q_\ell^0(X_T))).$$

It follows from property (I) that we can assume the following.

(VII) $\mathrm{rk}(H_1, 0) = d^2 - d = \ell$.
Next, we will see how to obtain foliations $\mathcal{F} \in \mathbb{R}(d)$ such that $\mathrm{rk}(\mathrm{BB}, \mathcal{F}) = d^2 + d$. We will consider the vector field $y \cdot X_0$ as a foliation, say $\tilde{\mathcal{F}}_0$, of degree d with a line of singularities. Let $p(x), q(x) \in \mathbb{C}[x]$ be polynomials with the following properties.

(VIII) $p(x)$ in monic of degree $d + 1$ and $q(x)$ has degree at most d.

We will set $Z(x, y) = p(x)\partial_x + (q(x) + y \cdot x^d)\partial_y$. Note that this vector field defines an element in $\mathbb{F}\mathrm{ol}(d)$. Moreover, the space of such vector fields has dimension $2d$. Consider the family of

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foliations $(\mathcal{F}(s,Z,T))_{s,Z,T}$ of degree d on $\mathbb{P}^2$, which are defined on E by the polynomial vector field

$$X(s,Z,T) = y \cdot \left(X_0 + \sum_{i=1}^{\ell} t_i \cdot X_i \right) + s \cdot Z.$$

Note that the components of $X(s,Z,T)$ are

$$\begin{cases} W_1 := y(P_0(x,y) + \sum_i t_i \cdot P_i(x,y)) + s \cdot p(x) \\ W_2 := y(Q_0(x,y) + \sum_i t_i \cdot Q_i(x,y) + s \cdot (q(x) + y \cdot x^d)). \end{cases}$$

For $s \neq 0$ with sufficiently small modulus and Z, T fixed, the singularities of $\mathcal{F}(s,Z,T)$ are contained in the affine curve $\{F_{(Z,T)}(x,y) = 0\} \subset \mathbb{C}^2$, where $F_{(Z,T)}(x,y)$ is equal to

$$p(x) \cdot \left[Q_0(x,y) + \sum_i t_i \cdot Q_i(x,y) \right] - (q(x) + y \cdot x^d) \cdot \left[P_0(x,y) + \sum_i t_i \cdot P_i(x,y) \right].$$

Since P and Q are relatively prime we have the following.

LEMMA 2.2. *Given a polynomial $f(x) \in \mathbb{C}[x]$ of degree $2d$ there exist unique polynomials $p(x), q(x) \in \mathbb{C}[x]$ such that*

$$\deg(p) = d+1, \quad \deg(q) \leqslant d-2 \quad \textit{and} \quad f(x) = p(x)Q(x) - q(x)P(x).$$

Proof. In fact, since $\gcd(P(x), Q(x)) = 1$, there exist $a(x), b(x) \in \mathbb{C}[x]$ such that

$$a(x) \cdot Q(x) - b(x) \cdot P(x) = 1 \quad \Longrightarrow \quad (f \cdot a)(x) \cdot Q(x) - (f \cdot b)(x) \cdot P(x) = f(x).$$

Dividing $f \cdot b(x)$ by $Q(x)$ we get $f \cdot b = g \cdot Q + q$, where $\deg(q) \leqslant d-2$. Thus,

$$f = (f \cdot a - g \cdot P)Q - qP =: pQ - qP \quad \Longrightarrow \quad p \cdot Q = f + q \cdot P.$$

Since $\deg(q \cdot P) = \deg(q) + \deg(P) \leqslant 2d - 1$, we have $\deg(f + q \cdot P) = 2d$. This implies that $2d = \deg(p \cdot Q) = \deg(p) + d - 1$, and so $\deg(p) = d+1$. If we have another solution $p_1 \cdot Q - q_1 \cdot P = f$, with $\deg(p_1) = d+1$ and $\deg(q_1) \leqslant d-2$, then

$$(p - p_1)Q = (q - q_1)P \quad \Longrightarrow \quad Q | q - q_1 \quad \text{and} \quad \deg(Q) > \deg(q - q_1),$$

which implies that $q = q_1$ and $p = p_1$. $\square$

Similar arguments also prove the following lemma.

LEMMA 2.3. *Let $P_k = \{g \in \mathbb{C}[x] \mid \deg(g) \leqslant k\}$ and consider the linear map $\Phi\colon P_{d+1} \times P_{d-2} \to P_{2d}$ given by $\Phi(p,q) = p \cdot Q - q \cdot P$. Then Φ is an isomorphism.*

After setting $f_{(Z,T)}(x) = F_{(Z,T)}(x, 0)$ we can take Z_0 in such a way that we have the following property.

(IX) The polynomial $f_{(Z_0,0)}(x)$ has simple roots and has degree $2d$.

Let $(p(x), q(x)) \in P_{d+1} \times P_{d-2}$ be such that $p(x)$ is monic and $Z = p(x)\partial_x + (q(x) + y \cdot x^d)\partial_y$. Then, we can write $p(x) = x^{d+1} + \sum_{j=0}^{d} z_{j+1} \cdot x^j$ and $q(x) = \sum_{j=0}^{d-2} z_{d+2+j} \cdot x^j$. Consider the space of vector fields Z as above, parametrized by $(z_1, \ldots, z_{2d}) \in \mathbb{C}^{2d}$. In what follows, we will use this parametrization and the notation $Z = (z_1, \ldots, z_{2d})$.

2.4 Applying the key lemma I: first properties

Next we will describe how to apply Lemma 2.1 to the family $(s,Z,T) \mapsto X(s,Z,T)$. The first step is the following.

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Lemma 2.4. *Let $Z_0 = p_0(x)\partial_x + (q_0(x) + y \cdot x^d)\partial_y$ be such that property (IX) is satisfied and let $\{x_1^0, \dots, x_{2d}^0\}$ be the roots of $f_{(Z_0,0)}(x) = 0$. Then there exist neighborhoods $D = D(0, r)$ of $0 \in \mathbb{C}$, U of Z_0, D^ℓ of $0 \in \mathbb{C}^\ell$ and holomorphic functions*

$$q_i : D \times U \times D^\ell \to \mathbb{C}^2, \quad i = 1, \dots, d^2 - d + 1 = \ell + 1$$
$$p_j : D \times U \times D^\ell \to \mathbb{C}^2, \quad j = 1, \dots, 2d,$$

with the following properties.

(a) *For any $(Z, T) \in U \times D^\ell$ the equation $f_{(Z,T)}(x) = 0$ has $2d$ simple roots, say $x_1(Z, T), \dots, x_{2d}(Z, T)$, such that $x_i \colon U \times D^\ell \to \mathbb{C}$ is holomorphic and $x_i(Z_0, 0) = x_i^0$ for all $i = 1, \dots, 2d$.*

(b) *$p_j(0, Z, T) = (x_j(Z, T), 0)$ for every $j = 1, \dots, 2d$ and for every $(Z, T) \in U \times D^\ell$.*

(c) *$q_i(0, 0, T) = q_i^0(T)$ for all $T \in \mathbb{D}^\ell$ and all $i = 1, \dots, \ell + 1$. In particular, $q_i(0, 0, 0) = q_i^0$ for all $i = 1, \dots, \ell + 1$ and*

$$\operatorname{sing}(X_T) = \{q_1(0, 0, T), \dots, q_{\ell+1}(0, 0, T)\},$$

for all $T \in U$.

(d) *For $(s, Z, T) \in D \times U \times \mathbb{D}^\ell$, $s \neq 0$, we have that $\operatorname{sing}(\mathcal{F}(s, Z, T))$ is equal to $\{p_1(s, Z, T), \dots, p_{2d}(s, Z, T), q_1(s, Z, T), \dots, q_{\ell+1}(s, Z, T)\}$.*

(e) *If $H_i(s, Z, T)$ denotes the Baum–Bott index of $\mathcal{F}(s, Z, T)$ at the point $q_i(s, Z, T)$, $i = 1, \dots, \ell+1$, then*

$$\frac{\partial H_i}{\partial z_r}(0, Z, T) \equiv 0, \quad \forall 1 \leqslant i \leqslant \ell + 1 \quad \text{and} \quad 1 \leqslant r \leqslant 2d.$$

(f) *For every $(s, T) \in D \times \mathbb{D}^\ell$, with $s \neq 0$, then $p_j(s, Z, T)$ is a simple singularity of $\mathcal{F}(s, Z, T)$. Furthermore, if $G_j(s, Z, T)$ denotes the Baum–Bott index of $\mathcal{F}(s, Z, T)$ at the singularity $p_j(s, Z_0, T)$, then*

$$\lim s \cdot G_j(s, Z, T) = \frac{Q_T^2(x_j(Z, T), 0)}{f'_{(Z,T)}(x_j(Z, T))} := A_j(Z, T). \tag{3}$$

Proof. The lemma is a consequence of the implicit function theorem (IFT) applied in several cases. In part (a) we apply the IFT to the function

$$(x, Z, T) \in \mathbb{C} \times P_{d+1} \times P_{d-2} \times \mathbb{C}^d \mapsto f_{(Z,T)}(x) \in \mathbb{C}$$

at the points $(x_{i0}, Z_0, 0)$, $i = 1, \dots, 2d$, where x_{i0}, $i = 1, \dots, 2d$, are the roots of $f_{(Z_0,0)}(x) = 0$. We leave the details for the reader.

For the existence of the functions $q_1, \dots, q_{\ell+1}$, defined in a neighborhood of $(0, Z_0, 0)$ in $\mathbb{C} \times P_{d+1} \times P_{d-2} \times \mathbb{C}^{\ell+1}$, we apply the IFT at the points $(x_i^0, y_i^0, 0, Z_0, 0)$, where $q_i^0 := (x_i^0, y_i^0) \in \mathbb{C}^2$, $1 \leqslant i \leqslant \ell+1$, are the singularities of $\mathcal{F}_0$, to the function $W(x, y, s, Z, T) = (W_1(x, y, s, Z, T), W_2(x, y, s, Z, T))$ defined as

$$\left(y\left(P_0(x, y) + \sum_i t_i P_i(x, y)\right) + sp(x), y\left(Q_0(x, y) + \sum_i t_i Q_i(x, y)\right) + s(q(x) + yx^d)\right).$$

In order to prove that $\det(\partial W/\partial x, \partial W/\partial y)(x_i^0, y_i^0, 0, Z_0, 0) \neq 0$ just observe that $W(x, y, 0, Z_0, 0) = (y \cdot P_0(x, y), y \cdot Q_0(x, y))$, q_i^0 is a non-degenerate singularity of $\mathcal{F}_0$ and that $y_i^0 \neq 0$ (see property (II)). We leave the details for the reader. Note that we can choose the neighborhood $V := D \times U \times D^\ell$ of $(0, Z_0, 0)$ in such a way that $q_i(s, Z, T) \notin (y = 0)$ for all $(s, Z, T) \in V$.

Let us prove part (e). Since $W_1(x, y, s, Z, T)$ and $W_2(x, y, s, Z, T)$ are the components of $X(s, Z, T)$, we have to compute $H_i(0, Z, T) = \mathrm{BB}(X(0, Z, T), q_i(0, Z, T))$. Note that $W_1(x, y, 0, Z, T) = y \cdot P_T(x, y)$ and $W_2(x, y, 0, Z, T) = y \cdot Q_T(x, y)$. This implies that $q_i(0, Z, T) = q_i(0, 0, T)$

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and, since $q_i(0,Z,T) \notin (y=0)$, then

$$H_i(0,Z,T) = \mathrm{BB}(y(P_T\partial_x + Q_T\partial_y), q_i(0,0,T)) = \mathrm{BB}(P_T\partial_x + Q_T\partial_y, q_i(0,0,T)).$$

This proves part (e).

Let us prove the existence of the functions $p_1,\ldots,p_{2d}$. As we have observed before, if $s \neq 0$, then $\mathrm{sing}(\mathcal{F}(s,Z,T)) \cap \mathbb{C}^2 \subset (F_{(Z,T)} = 0)$. Let $W = (W_1, W_2)$ be as above. If we set $P_T = P_0 + \sum_i t_i \cdot P_i$ and $Q_T = Q_0 + \sum_i t_i \cdot Q_i$, then we can write

$$W = (W_1, W_2) = (y \cdot P_T + s \cdot p(x), y \cdot Q_T + s \cdot (q(x) + y \cdot x^d)).$$

As the reader can check

$$(W=0) = (W_1 = F_{(Z,T)} = 0) = (W_2 = F_{(Z,T)} = 0).$$

Therefore, we have to apply the IFT at the points $(x_{i0}, 0, 0, Z_0, 0)$ to one of the functions

$$(x,y,s,Z,T) \mapsto (W_j(x,y,s,Z,T), F_{(Z,T)}(x,y)) = \Phi_j(x,y,s,Z,T), \quad j=1 \text{ or } j=2.$$

Note that

$$\Phi_1(x,y,0,Z,T) = (y \cdot P_T(x,y), F_{(Z,T)}(x,y)).$$

Therefore, $\det(\partial\Phi_1/\partial x, \partial\Phi_1/\partial y)(x,0,0,Z_0,0)$ is equal to

$$\det\begin{pmatrix} 0 & P_0(x,0) \\ f'_{(Z_0,0)}(x) & * \end{pmatrix} = -P(x) \cdot f'_{(Z_0,0)}(x).$$

Similarly,

$$\det(\partial\Phi_2/\partial x, \partial\Phi_2/\partial y)(x,0,0,Z_0,0) = -Q(x) \cdot f'_{(Z_0,0)}(x).$$

It follows from property (IV) that either $P(x_i^0) \neq 0$ or $Q(x_i^0) \neq 0$. Since $f_{(Z_0,0)}$ has simple roots, we can apply the IFT to obtain the function p_i.

Set $p_i(s,Z,T) = (x_i(s,Z,T), y_i(s,Z,T))$.

Assertion 2.1. *For every $i \in \{1,\ldots,2d\}$ we have $y_i(s,Z,T) = s \cdot u_i(s,Z,T)$, where u_i is holomorphic and $F_{Z,T}(x_i(0,Z,T),0) = f_{(Z,T)}(x_i(0,Z,T)) = 0$. In particular, $x_i(0,Z,T) = x_i(Z,T)$ (in the notation of part (a)). Moreover, if $P_0(x_i^0,0) = P(x_i^0) \neq 0$ and we take the neighborhood V to be small, then*

$$u_i(0,Z,T) = -\frac{p(x_i(Z,T))}{P_T(x_i(Z,T),0)}. \tag{4}$$

Similarly, if $Q_0(x_{i0},0) \neq 0$ and we take V to be small, then

$$u_i(0,Z,T) = -\frac{q(x_i(Z,T))}{Q_T(x_i(Z,T),0)}. \tag{5}$$

In any case, we have that

$$\begin{cases} u_i(0,Z,T) \cdot Q_T(x_i(Z,T)) + q(x_i(Z,T)) = 0 \\ u_i(0,Z,T) \cdot P_T(x_i(Z,T),0) + p(x_i(Z,T)) = 0 \end{cases} \tag{6}$$

for all $(0,Z,T) \in V$.

Proof of the assertion. Let us suppose that $P(x_i^0) \neq 0$. If we take V to be small, then $P_T(x_i(s,Z,T), y_i(s,Z,T)) \neq 0$ for all $(s,Z,T) \in V$. It follows that

$$y_i \cdot P_T(x_i,y_i) + s \cdot p(x_i) \equiv 0 \quad \Longrightarrow \quad y_i(0,Z,T) = 0$$

and

$$\frac{\partial y_i}{\partial s}(0,Z,T) \cdot P_T(x_i(Z,T),0) + p(x_i(Z,T)) \equiv 0.$$

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Since $u_i(0,Z,T) = (\partial y_i/\partial s)(0,Z,T)$, this implies (4). The proofs of (5) and (6) are left for the reader. ☐

Let us continue the proof of Lemma 2.4 by proving part (f). We will prove first that the singularities $p_i(s,Z,T)$ are non-degenerate for $s \neq 0$. Denote by J the Jacobian matrix

$$J = \begin{pmatrix} \dfrac{\partial W_1}{\partial x} & \dfrac{\partial W_1}{\partial y} \\ \dfrac{\partial W_2}{\partial x} & \dfrac{\partial W_2}{\partial y} \end{pmatrix}.$$

First we prove, for all $i = 1,\dots,2d$, that $\det(J(p_i(s,Z,T),s,Z,T)) \neq 0$ whenever $s \neq 0$ and $(s, Z - Z_0, T)$ has a small norm. Since $W_1 = y \cdot P_T + s \cdot p$ and $W_2 = y \cdot Q_T + s \cdot (q + y \cdot x^d)$, by a direct computation, we get that $\det(J(p_i,s,Z,T))$ is equal to

$$\begin{aligned} &W_{1x} \cdot W_{2y} - W_{1y} \cdot W_{2x} \\ &\quad = [(yP_{Tx} + sp')(Q_T + yQ_{Ty} + sx^d) - (P_T + y \cdot P_{Ty})(yQ_{Tx} + sq' + dsyx^{d-1})](p_i(s,Z,T)) \\ &\quad = s[(u_iP_{Tx} + p')(Q_T + su_iQ_{Ty} + sx^d) - (P_T + su_iP_{Ty})(u_iQ_{Tx} + q' + dsu_ix^{d-1})](x_i,y_i). \end{aligned}$$

Therefore if we define $\Delta(Z,T) := \lim s^{-1} \det(J(p_i(s,Z,T),s,Z,T))$, then

$$\Delta(Z,T) = [(u_i \cdot P_{Tx} + p') \cdot Q_T - P_T \cdot (u_i \cdot Q_{Tx} + q')](p_i(0,Z,T)).$$

On the other hand, (6) implies that $\Delta(Z,T)$ is equal to

$$\begin{aligned} &[(p' \cdot Q_T - u_i \cdot P_T \cdot Q_{Tx}) - (P_T \cdot q' - u_i \cdot P_{Tx} \cdot Q_T)](p_i(0,Z,T)) \\ &\quad = [(p' \cdot Q_T + p \cdot Q_{Tx}) - (P_T \cdot q' + P_{Tx} \cdot q)](p_i(0,Z,T)) \\ &\quad = \frac{\partial}{\partial x}[p \cdot Q_T - q \cdot P_T](p_i(0,Z,T)) \\ &\quad = f'_{(Z,T)}(x_i(Z,T)). \end{aligned}$$

If we take the neighborhood V of $(0,Z_0,0)$ to be small, then the polynomial $f_{(Z,T)}$ has simple roots, for every $(0,Z,T) \in V$. Since $x_i(0,Z,T) = x_i(Z,T)$ is a root of $f_{(Z,T)}$, we get that $\Delta(Z,T) = f'_{(Z,T)}(x_i(Z,T)) \neq 0$. Hence, $\det(J(p_i(s,Z,T),s,Z,T)) \neq 0$ for small $|s| > 0$. It remains to prove (3) in part (f). Since

$$G_i(s,Z,T) = \frac{\operatorname{tr}^2(J(p_i(s,Z,T),s,Z,T))}{\det(J(p_i(s,Z,T),s,Z,T))}$$

and

$$\operatorname{tr}(J(p_i(s,Z,T),s,Z,T)) = [s \cdot u_i \cdot P_{Tx} + s \cdot p' + Q_T + s \cdot u_i \cdot Q_{Ty} + s \cdot x^d](p_i(s,Z,T))$$

we get

$$\lim \operatorname{tr}^2(J(p_i(s,Z,T),s,Z,T)) = Q_T^2(x_i(Z,T))$$

and

$$\begin{aligned} \lim \frac{1}{s} G_i(s,Z,T) &= \lim \frac{\operatorname{tr}^2(J(p_i(s,Z,T),s,Z,T))}{s \cdot \det(J(p_i(s,Z,T),s,Z,T))} \\ &= \frac{Q_T^2(x_i(Z,T),0)}{f'_{(Z,T)}(x_i(Z,T))}. \end{aligned}$$

This finishes the proof of the lemma. ☐

To apply Lemma 2.1 we set $\mathrm{BB}(s,Z,T)$ equal to $(G(s,Z,T), H(s,Z,T))$, i.e.

$$\mathrm{BB}(s,Z,T) = (G_1(s,Z,T),\dots,G_{2d}(s,Z,T),H_1(s,Z,T),\dots,H_{d^2-d}(s,Z,T)).$$

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We are going to prove that we can choose Z_0 in such a way that, for $|s| > 0$ small, $\mathrm{rk}(\mathrm{BB}, (s, Z_0, 0)) = d^2 + d$.

It follows from property (VII) and from Lemma 2.4(e) that H satisfies the hypotheses (a) and (d) of Lemma 2.1. We have also seen that

$$G(s, Z, T) = \frac{1}{s}[A(Z, T) + s \cdot R(s, Z, T)],$$

where R is holomorphic,

$$A(Z, T) = \lim s \cdot G(s, Z, T) = (A_1(Z, T), \dots, A_{2d}(Z, T))$$

and

$$A_j(Z, T) = \frac{Q_T^2(x_j(Z, T), 0)}{f'_{(Z,T)}(x_j(Z, T))}.$$

In order to finish the proof, it is sufficient to prove that there exists Z_0 and $j \in \{1, \dots, 2d\}$ such that $\det(M_j(Z_0)) \neq 0$, where

$$M_j(Z) = \left[A^{\mathrm{T}}(Z, 0), \frac{\partial A^{\mathrm{T}}}{\partial z_1}(Z, 0), \dots, \frac{\partial A^{\mathrm{T}}}{\partial z_{j-1}}(Z, 0), \frac{\partial A^{\mathrm{T}}}{\partial z_{j+1}}(Z, 0), \dots, \frac{\partial A^{\mathrm{T}}}{\partial z_{2d}}(Z, 0)\right].$$

In the above expression, for $C \in \mathbb{C}^{2d}$, we are denoting by C^{T} the transpose of C, that is, we are considering the transpose of the matrix given in Lemma 2.1(c).

2.5 Applying the key lemma II: fine tuning

According to Lemma 2.3, the map $\Phi\colon P_{d+1} \times P_{d-2} \to P_{2d}$ defined by $\Phi(Z) = \Phi(p, q) = p \cdot Q - q \cdot P := f$ is an isomorphism. On the other hand, observe that

$$A_j(Z, 0) = \frac{Q_0^2(x_j(Z), 0)}{f'_Z(x_j(Z))} = \frac{Q^2(x_j(Z))}{f'_Z(x_j(Z))},$$

where $x_1(Z) := x_1(Z, 0), \dots, x_{2d}(Z) := x_{2d}(Z, 0)$ are the roots of $f_Z := f_{(Z,0)}$.

The idea is to use Lemma 2.3 to parametrize the space P_{2d} by the roots of f_Z instead of the coefficients $(z_1, \dots, z_{2d})$ of $Z = (p, q)$. We have seen before that $\deg(p \cdot Q - q \cdot P) = \deg(p \cdot Q) = 2d$. Since we are free to choose one of the coefficients of Q, we will suppose that it is monic of degree $d - 1$. This implies that $f_Z = p \cdot Q - q \cdot P$ is monic (see property (VIII)). Therefore, we can write

$$f_Z(x) = (x - x_1(Z)) \cdots (x - x_{2d}(Z))$$

and the map $\rho(Z) = (x_1(Z), \dots, x_{2d}(Z))$ is a biholomorphism in a neighborhood of Z_0. Let ζ be the local inverse of ρ, defined in a neighborhood W of $(x_1(Z_0), \dots, x_{2d}(Z_0))$. Set $C = A \circ \zeta\colon W \to \mathbb{C}^{2d}$. If $X = (x_1, \dots, x_{2d})$, then

$$f_{\zeta(X)}(x) := f_X(x) = (x - x_1) \cdots (x - x_{2d}).$$

Therefore, $C(X) = (C_1(X), \dots, C_{2d}(X))$, where

$$C_j(X) = A_j(\zeta(X)) = \frac{Q^2(x_j)}{f'_X(x_j)}.$$

Let $N(X)$ be the $2d \times 2d$ matrix defined by

$$N(X) = \left[C^{\mathrm{T}}(X), \frac{\partial C^{\mathrm{T}}}{\partial x_2}(X), \dots, \frac{\partial C^{\mathrm{T}}}{\partial x_{2d}}(X)\right].$$

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We assert that it is enough to prove that $\det(N(X)) \not\equiv 0$. In fact, since $C(X) = A \circ \zeta(X)$ we get

$$\frac{\partial C}{\partial x_j} = \sum_{i=1}^{2d} \frac{\partial A}{\partial z_i} \circ \zeta \frac{\partial \zeta_i}{\partial x_j} = \sum_{i=1}^{2d} \frac{\partial A}{\partial z_i} \frac{\partial \zeta_i}{\partial x_j},$$

where in the third expression we have omitted the composition with ζ. This implies that

$$\begin{aligned}
\det(N) &= \det\left[A, \sum_{i_2=1}^{2d} \frac{\partial A}{\partial z_{i_2}} \frac{\partial \zeta_{i_2}}{\partial x_2}, \dots, \sum_{i_{2d}=1}^{2d} \frac{\partial A}{\partial z_{i_{2d}}} \frac{\partial \zeta_{i_{2d}}}{\partial x_{2d}}\right] \\
&= \sum_{i_2,\dots,i_{2d}} \frac{\partial \zeta_{i_2}}{\partial x_2} \cdots \frac{\partial \zeta_{i_{2d}}}{\partial x_{2d}} \det\left[A, \frac{\partial A}{\partial z_{i_2}}, \dots, \frac{\partial A}{\partial z_{i_{2d}}}\right] \\
&= \sum_{j=1}^{2d} \Phi_j \cdot \det(M_j \circ \zeta),
\end{aligned}$$

where

$$\Phi_j = \pm \det\left(\frac{\partial \zeta_i}{\partial x_k}\right)_{1 \leqslant i \leqslant 2d, i \neq j, 2 \leqslant k \leqslant 2d}.$$

In particular, if $\det(N(X)) \not\equiv 0$, then $\det(M_j(Z)) \not\equiv 0$, for some $j \in \{1, \dots, 2d\}$.

To conclude the proof of the theorem it remains to show that $\det(N(X)) \not\equiv 0$. Recall that $Q(x)$ is a monic polynomial of degree $d-1$ and $C(X) = (C_1(X), \dots, C_{2d}(X))$, where

$$C_j(X) = C_j(x_1, \dots, x_{2d}) = \frac{Q^2(x_j)}{f'_X(x_j)} = \frac{Q^2(x_j)}{\prod_{i \neq j}(x_j - x_i)} \tag{7}$$

because $f_X(x) = \prod_{i=1}^{2d}(x - x_i)$. Fix $x_0 \in \mathbb{C}$ which is not a root of $Q(x) = 0$ and a neighborhood $D := D(x_0, r)$ such that $Q(x) \neq 0$ for all $x \in D$. We will work in the open set $U \subset \mathbb{C}^{2d}$ defined by

$$U = \{(x_1, \dots, x_{2d}) \mid x_i \neq x_j \text{ if } i \neq j\}.$$

If $X \in U$, then $C_j(X) \neq 0$ and

$$\det(N(X)) = C_1(X) \cdots C_{2d}(X) \cdot \det(K(X)),$$

where K is the matrix

$$K = \begin{bmatrix} 1 & \dots & 1 \\ \dfrac{\partial C_1}{\partial x_2} \Big/ C_1 & \dots & \dfrac{\partial C_{2d}}{\partial x_2} \Big/ C_{2d} \\ \vdots & \dots & \vdots \\ \dfrac{\partial C_1}{\partial x_{2d}} \Big/ C_1 & \dots & \dfrac{\partial C_{2d}}{\partial x_{2d}} \Big/ C_{2d} \end{bmatrix}.$$

It follows from (7) that

$$\frac{\partial C_j / \partial x_i}{C_j}(X) = \begin{cases} \dfrac{1}{x_i - x_j}, & \text{if } i \neq j. \\ \dfrac{2Q'(x_j)}{Q(x_j)} + \displaystyle\sum_{i \neq j} \frac{1}{x_i - x_j}, & \text{if } i = j. \end{cases}$$

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In particular, if we denote $\phi_j = 2Q'(x_j)/Q(x_j)$, $j = 2, \dots, 2d$, then, for any $X \in U$, we have the following expression for $K(X)$

$$\begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ \dfrac{1}{x_2 - x_1} & \phi_2 + \displaystyle\sum_{i \neq 2} \dfrac{1}{x_i - x_2} & \dots & \dfrac{1}{x_2 - x_{2d-1}} & \dfrac{1}{x_2 - x_{2d}} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \dfrac{1}{x_{2d-1} - x_1} & \dfrac{1}{x_{2d-1} - x_2} & \dots & \phi_{2d-1} + \displaystyle\sum_{i \neq 2d-1} \dfrac{1}{x_i - x_{2d-1}} & \dfrac{1}{x_{2d-1} - x_{2d}} \\ \dfrac{1}{x_{2d} - x_1} & \dfrac{1}{x_{2d} - x_2} & \dots & \dfrac{1}{x_{2d} - x_{2d-1}} & \phi_{2d} + \displaystyle\sum_{i \neq 2d} \dfrac{1}{x_i - x_{2d}} \end{bmatrix}.$$

Now, define

$$\Delta_1(x_1, \dots, x_{2d-1}) := \lim_{x_{2d} \to x_1} (x_1 - x_{2d}) \cdot \det(K(X))$$

and, inductively,

$$\Delta_j(x_1, \dots, x_{2d-j}) := \lim_{x_{2d-j+1} \to x_1} (x_1 - x_{2d-j+1}) \cdot \Delta_{j-1}(x_1, \dots, x_{2d-j+1}).$$

We will prove that $\Delta_{2d-1}(x_1) = (2d)! \neq 0$ and this fact will imply that $\det(N(X)) \not\equiv 0$. As the reader can check, $\Delta_1(x_1, \dots, x_{2d-1})$ is equal to

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ \dfrac{1}{x_2 - x_1} & \phi_2 + \displaystyle\sum_{i=3}^{2d-1} \dfrac{1}{x_i - x_2} + \dfrac{2}{x_1 - x_2} & \dots & \dfrac{1}{x_2 - x_{2d-1}} & \dfrac{1}{x_2 - x_1} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \dfrac{1}{x_{2d-1} - x_1} & \dfrac{1}{x_{2d-1} - x_2} & \dots & \phi_{2d-1} + \displaystyle\sum_{i=2}^{2d-2} \dfrac{1}{x_i - x_{2d-1}} + \dfrac{2}{x_1 - x_{2d-1}} & \dfrac{1}{x_{2d-1} - x_1} \\ -1 & 0 & \dots & 0 & 1 \end{vmatrix},$$

where $|\cdot|$ denotes the determinant. If we sum the first column with the last in the above determinant, we get

$$\begin{vmatrix} 2 & 1 & \dots & 1 \\ \dfrac{2}{x_2 - x_1} & \phi_2 + \displaystyle\sum_{i=3}^{2d-1} \dfrac{1}{x_i - x_2} + \dfrac{2}{x_1 - x_2} & \dots & \dfrac{1}{x_2 - x_{2d-1}} \\ \vdots & \vdots & \dots & \vdots \\ \dfrac{2}{x_{2d-1} - x_1} & \dfrac{1}{x_{2d-1} - x_2} & \dots & \phi_{2d-1} + \displaystyle\sum_{i=2}^{2d-2} \dfrac{1}{x_i - x_{2d-1}} + \dfrac{2}{x_1 - x_{2d-1}} \end{vmatrix}.$$

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By a similar argument, we have that $\Delta_2(x_1,\dots,x_{2d-2})$ is equal to

$$\begin{vmatrix} 2 & 1 & \dots & 1 & 1 \\ \dfrac{2}{x_2-x_1} & \phi_2+\displaystyle\sum_{i=3}^{2d-2}\frac{1}{x_i-x_2}+\frac{3}{x_1-x_2} & \dots & \dfrac{1}{x_2-x_{2d-2}} & \dfrac{1}{x_2-x_1} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \dfrac{2}{x_{2d-2}-x_1} & \dfrac{1}{x_{2d-2}-x_2} & \dots & \phi_{2d-2}+\displaystyle\sum_{i=2}^{2d-3}\frac{1}{x_i-x_{2d-2}}+\frac{3}{x_1-x_{2d-2}} & \dfrac{1}{x_{2d-2}-x_1} \\ -2 & 0 & \dots & 0 & 2 \end{vmatrix},$$

or, more succinctly,

$$2\cdot\begin{vmatrix} 3 & 1 & \dots & 1 \\ \dfrac{3}{x_2-x_1} & \phi_2+\displaystyle\sum_{i=3}^{2d-2}\frac{1}{x_i-x_2}+\frac{3}{x_1-x_2} & \dots & \dfrac{1}{x_2-x_{2d-2}} \\ \vdots & \vdots & \dots & \vdots \\ \dfrac{3}{x_{2d-2}-x_1} & \dfrac{1}{x_{2d-2}-x_2} & \dots & \phi_{2d-2}+\displaystyle\sum_{i=2}^{2d-3}\frac{1}{x_i-x_{2d-2}}+\frac{3}{x_1-x_{2d-2}} \end{vmatrix}.$$

Similarly, $\Delta_3(x_1,\dots,x_{2d-3})$ is equal to

$$6\cdot\begin{vmatrix} 4 & 1 & \dots & 1 \\ \dfrac{4}{x_2-x_1} & \phi_2+\displaystyle\sum_{i=3}^{2d-3}\frac{1}{x_i-x_2}+\frac{4}{x_1-x_2} & \dots & \dfrac{1}{x_2-x_{2d-3}} \\ \vdots & \vdots & \dots & \vdots \\ \dfrac{4}{x_{2d-3}-x_1} & \dfrac{1}{x_{2d-3}-x_2} & \dots & \phi_{2d-3}+\displaystyle\sum_{i=2}^{2d-4}\frac{1}{x_i-x_{2d-3}}+\frac{4}{x_1-x_{2d-3}} \end{vmatrix}.$$

Proceeding in this way we see that $\Delta_j(x_1,\dots,x_{2d-j})$ is given by

$$j!\cdot\begin{vmatrix} j+1 & 1 & \dots & 1 \\ \dfrac{j+1}{x_2-x_1} & \phi_2+\displaystyle\sum_{i=3}^{2d-j}\frac{1}{x_i-x_2}+\frac{j+1}{x_1-x_2} & \dots & \dfrac{1}{x_2-x_{2d-j}} \\ \vdots & \vdots & \dots & \vdots \\ \dfrac{j+1}{x_{2d-j}-x_1} & \dfrac{1}{x_{2d-j}-x_2} & \dots & \phi_{2d-j}+\displaystyle\sum_{i=2}^{2d-j-1}\frac{1}{x_i-x_{2d-j}}+\frac{j+1}{x_1-x_{2d-j}} \end{vmatrix}.$$

In particular,

$$\Delta_{2d-2}(x_1,x_2)=(2d-2)!\cdot\begin{vmatrix} 2d-1 & 1 \\ \dfrac{2d-1}{x_2-x_1} & \phi_2+\dfrac{2d-1}{x_1-x_2} \end{vmatrix}.$$

Hence,

$$\Delta_{2d-1}(x_1)=\lim_{x_2\to x_1}(x_1-x_2)\cdot\Delta_{2d-2}(x_1,x_2)=(2d-2)!\begin{vmatrix} 2d-1 & 1 \\ 1-2d & 2d-1 \end{vmatrix}=(2d)!.$$

This finishes the proof of Theorem 1.

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3. The rank at Jouanolou's foliations

Jouanolou's foliations are the first examples of foliations of $\mathbb{P}^2$ without invariant algebraic curves, (cf. [Jou79]). They can be defined as follows: for every integer d, $d \geqslant 2$, the degree d Jouanolou foliation, $\mathcal{J}_d$, is induced in affine coordinates $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2$ by the vector field

$$X_d(x, y) = (1 - x \cdot y^d)\partial_x + (x^d - y^{d+1})\partial_y = \partial_x + x^d\partial_y - y^d \cdot R,$$

where $R = x\partial_x + y\partial_y$ is the radial vector field on $\mathbb{C}^2$.

Most of arguments proving that $\mathcal{J}_d$ has no invariant algebraic curves take advantage of the highly symmetrical character of $\mathcal{J}_d$: $\mathrm{Aut}(\mathcal{J}_d)$, the automorphism group of $\mathcal{J}_d$, is a semi-direct product of a cyclic group of order 3 and a cyclic group of order d^2+d+1. If β is a primitive (d^2+d+1)th root of the unity, then generators of $\mathrm{Aut}(\mathcal{J}_d)$, in the affine coordinates $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2$, are

$$A : (x, y) \mapsto (\beta^{-d}x, \beta y),$$
$$B : (x, y) \mapsto (y^{-1}, xy^{-1}).$$

The singular set of $\mathcal{J}_d$ is equal to

$$\mathrm{sing}(\mathcal{J}_d) = \{p_j \mid p_j = A^{j-1}(1, 1), 1 \leqslant j \leqslant d^2 + d + 1\},$$

i.e. it is the orbit of the point $p_1 = (1, 1)$ under the action on $\mathbb{P}^2$ of the subgroup of $\mathrm{Aut}(\mathcal{J}_d)$ generated by A. It follows that all of the singularities of $\mathcal{J}_d$ are isomorphic simple singularities with Baum–Bott index

$$\frac{(d+2)^2}{d^2+d+1}.$$

We will also take advantage of $\mathrm{Aut}(\mathcal{J}_d)$ to determine the rank of the Baum–Bott map at $\mathcal{J}_d$. Instead of considering the Baum–Bott map as defined from $\mathbb{F}\mathrm{ol}(d)$ to $\mathbb{P}^{d^2+d+1}$ we will consider it defined from $V_d = \mathrm{H}^0(\mathbb{P}^2, T\mathbb{P}^2(d-1))$ to the same target. Our problem translates to compute the rank at X_d.

It will be convenient to consider V_d as the $\mathbb{C}$-vector space generated by the set

$$\mathcal{P}_d = \{x^i \cdot y^j\partial_x, x^k \cdot y^\ell\partial_y, x^m \cdot y^n \cdot R \mid 0 \leqslant i+j, k+\ell \leqslant d \text{ and } m+n = d\}.$$

Note that all of the elements in $\mathcal{P}_d$ are eigenvectors of $A^* : V_d \to V_d$, where $A^*(X) = DA^{-1} \cdot X \circ A$. Explicitly, we have

$$A^*(x^i \cdot y^j\partial_x) = \beta^{j-d(i-1)} \cdot x^i \cdot y^j\partial_x,$$
$$A^*(x^k \cdot y^\ell\partial_y) = \beta^{\ell-1-dk} \cdot x^k \cdot y^\ell\partial_y,$$
$$A^*(x^m \cdot y^n \cdot R) = \beta^{n-dm} \cdot x^m \cdot y^n \cdot R.$$

The invariance of $\mathcal{J}_d$ under A is expressed in the formula

$$A^*(X_d) = \beta^d \cdot X_d.$$

Since β is a primitive (d^2+d+1)th root of unity, A^* has at most d^2+d+1 maximal eigenspaces. If we denote by E_j, $1 \leqslant j \leqslant d^2+d+1$, the maximal eigenspace associated to the eigenvalue β^j, then

$$V_d = \bigoplus_{j=1}^{d^2+d+1} E_j.$$

Now, let U be a neighborhood of X_d in V_d and $\gamma_j \colon U \to \mathbb{P}^2$, $j = 1 \dots d^2+d+1$, be holomorphic maps such that $\gamma_j(X_d) = p_j$ and

$$\mathrm{sing}(\mathcal{F}(X)) = \{\gamma_1(X), \dots, \gamma_{d^2+d+1}(X)\},$$

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for every $X \in U$. Compute the rank of the Baum–Bott map which is equivalent to computing the rank of $B = (B_1, \ldots, B_{d^2+d+1}) : U \to \mathbb{C}^{d^2+d+1}$ given by

$$B_j(X) = \mathrm{BB}(X, \gamma_j(X)) = \frac{\mathrm{tr}^2}{\det}(DX(\gamma_j(X))).$$

3.1 The rank of B at X_d

By definition the rank of B at X_d is the rank of the liner map $T := DB(X_d) : V_d \to \mathbb{C}^{d^2+d+1}$, the derivative of B at X_d. If we denote by $T_j := DB_j(X_d)$, $1 \leqslant j \leqslant d^2 + d + 1$, then the next lemma describes some useful relations between A^* and T_j.

Lemma 3.1. *For any $Y \in V_d$*

$$T_j(A^*(Y)) = \beta^d \cdot T_{j+1}(Y), \tag{8}$$

where $1 \leqslant j \leqslant d^2 + d + 1$, and $T_{d^2+d+1} = T_0$. In particular:

(a) $A^*(\ker(T)) = \ker(T)$;

(b) *if we set $K_j := E_j \cap \ker(T)$, $j = 1, \ldots, d^2 + d + 1$, then*

$$\ker(T) = \bigoplus_{j=1}^{d^2+d+1} K_j;$$

(c) $E_j \cap \ker(T_1) = K_j$, *for all $j = 1, \ldots, d^2 + d + 1$;*

(d) *let $k = \#\{j \mid T_1|_{E_j} \not\equiv 0\}$, then $\mathrm{rk}(T) = \mathrm{rk}(\mathrm{BB}, \mathcal{J}_d) = k$.*

Proof. Observe first that for any $Y \in V$, we have that the foliations induced by $A^*(X_d + Y)$ and $X_d + \beta^{-d} \cdot A^*Y$ are equal, i.e.

$$\mathcal{F}(A^*(X_d + Y)) = \mathcal{F}(X_d + \beta^{-d} \cdot A^*(Y)).$$

Moreover, since $A^*(X) = DA^{-1} \cdot X \circ A$,

$$p \in \mathrm{sing}(\mathcal{F}(A^*(X_d + Y))) \iff A(p) \in \mathrm{sing}(X_d + Y).$$

If we set $P_j(Y) = A^{-1}(\gamma_j(X_d + Y))$, then $P_j(0) = A^{-1}(p_j) = p_{j-1}$ and $P_j(Y) = \gamma_{j-1}(X_d + \beta^{-d} \cdot A^*(Y))$. Thus,

$$\gamma_j(X_d + Y) = A(\gamma_{j-1}(X_d + \beta^{-d} \cdot A^*(Y))),$$

for all $Y \in V_d$ sufficiently small where, by convention, we set $\gamma_0 = \gamma_{d^2+d+1}$. Now we can easily verify that

$$\begin{aligned} B_j(X_d + Y) &= \mathrm{BB}(X_d + Y, \gamma_j(X_d + Y)) \\ &= \mathrm{BB}(X_d + \beta^{-d} \cdot A^*(Y)), \gamma_{j-1}(X_d + \beta^{-d} \cdot A^*(Y)) \\ &= B_{j-1}(X_d + \beta^{-d} \cdot A^*(Y)). \end{aligned}$$

Hence,

$$T_j(Y) = DB_j(X_d) \cdot Y = DB_{j-1}(X_d) \cdot (\beta^{-d} \cdot A^*(Y)) = \beta^{-d} \cdot T_{j-1}(A^*(Y)).$$

This proves (8). Observe that (8) implies parts (a) and (b).

Relation (8) also implies that $T_1((A^*)^k(Y)) = \beta^{kd} \cdot T_{1+k}(Y)$. Thus, $Y \in E_j \cap \ker(T_1)$ if and only if

$$A^*(Y) = \beta^j \cdot Y \quad \text{and} \quad 0 = T_1(\beta^{kj} \cdot Y) = T_1((A^*)^k(Y)) = \beta^{kd} \cdot T_{1+k}(Y),$$

or, equivalently, $T_n(Y) = 0$ for all $n \in \{1, \ldots, d^2+d+1\}$ and $A^*(Y) = \beta^j \cdot Y$. Thus, we can conclude that

$$E_j \cap \ker(T_1) = E_j \cap K,$$

proving in this way part (c).

Let us prove part (d). Note that $\mathrm{rk}(B(X_d)) = \dim(\mathrm{Im}(T))$. Let $k = \#\{j \mid T_1|_{E_j} \not\equiv 0\}$ and $\{j \mid T_1|_{E_j} \not\equiv 0\} = \{j_1, \ldots, j_k\}$, where $j_1 < \cdots < j_k$. Choose $Y_1, \ldots, Y_k \in V_d$ such that $Y_i \in E_{j_i}$ and $T_1(Y_i) \neq 0$ for all $i = 1, \ldots, k$. It follows from (8) that

$$T_j(Y_i) = \beta^{-d} \cdot T_{j-1}(A^*(Y_i)) = \beta^{j_i - d} \cdot T_{j-1}(Y_i)$$

and, by induction, that

$$T_j(Y_i) = \beta^{(j_i-d)(j-1)} \cdot T_1(Y_i) \quad \Longrightarrow \quad T(Y_i) = T_1(Y_i) \cdot (1, \beta^{(j_i-d)}, \ldots, \beta^{(N-1)(j_i-d)}).$$

We want to prove that the vectors $T(Y_1), \ldots, T(Y_k) \in \mathbb{C}^N$ are linearly independent. Since $T_1(Y_i) \neq 0$ for all $i = 1, \ldots, k$, this is equivalent to proving that the vectors $(1, \beta^{(j_i-d)}, \beta^{2(j_i-d)}, \ldots, \beta^{(N-1)(j_i-d)}) \in \mathbb{C}^N$ are linearly independent. Observe that

$$\det \begin{vmatrix} 1 & \beta^{(j_1-d)} & \beta^{2(j_1-d)} & \ldots & \beta^{(k-1)(j_1-d)} \\ 1 & \beta^{(j_2-d)} & \beta^{2(j_2-d)} & \ldots & \beta^{(k-1)(j_2-d)} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & \beta^{(j_k-d)} & \beta^{2(j_k-d)} & \ldots & \beta^{(k-1)(j_k-d)} \end{vmatrix} = \prod_{r<s} (\beta^{j_s-d} - \beta^{j_r-d}) \neq 0,$$

because $\beta^{j_s-d} \neq \beta^{j_r-d}$ for $r < s$. This finishes the proof of the lemma. □

3.2 Maximal eigenspaces of A^*

Recall that $\mathcal{P}_d$ is a basis for V_d. We will denote by $\mathcal{P}_d(Y)$ the subset of $\mathcal{P}_d$ of the form

$$\mathcal{P}_d(Y) = \{x^i \cdot y^j \cdot Y \mid x^i \cdot y^j \cdot Y \in V_d, 0 \leqslant i+j \leqslant d\}.$$

In this notation we have that $\mathcal{P}_d$ is the disjoint union of $\mathcal{P}_d(\partial_x)$, $\mathcal{P}_d(\partial_y)$ and $\mathcal{P}_d(x\partial_x + y\partial_y)$.

Lemma 3.2. *Let $i, j \geqslant 0$ be such that $0 \leqslant i+j \leqslant d$ and $A^*(x^i \cdot y^j) = x^i \cdot y^j$. Then $i = j = 0$. In particular, given $Y \in V_d$, then the eigenvalues of $Y_1, Y_2 \in \mathcal{P}_d(Y)$ are distinct for $Y_1 \neq Y_2$.*

Proof. Note that $A^*(x^i \cdot y^j) = \beta^{j-i\cdot d} \cdot x^i \cdot y^j$. In particular, $A^*(x^i \cdot y^j) = x^i \cdot y^j$ if and only if

$$j - i \cdot d = 0 \mod N \iff (d+1)\cdot j + i = 0 \mod N \iff i = j = 0.$$

In the first equivalence we have used that $-d(d+1) = 1 \mod N$ and in the second that

$$0 \leqslant (d+1)\cdot j + i = d \cdot j + i + j \leqslant d(j+1) \leqslant d(d+1) = N-1 < N.$$

We leave the proof of the second part for the reader. □

In the next result we describe the dimensions of the maximal eigenspaces of A^*.

Lemma 3.3. *For any $j = 1, \ldots, d^2+d+1$ we have*

$$0 \leqslant \dim(E_j) \leqslant 3.$$

Moreover:

(a) *$\dim(E_d) = 3$ and $E_d \subset \ker(T)$;*

(b) *$\dim(E_j) = 3$ if and only if $j = d$;*

(c) *$\#\{j \mid E_j \neq \{0\}\} = (d^2+7d-4)/2$.*

Proof. Note that $\mathcal{P}_d(\partial_x)\cup\mathcal{P}_d(\partial_y)\cup\mathcal{P}_d(R)$, $(R=x\partial_x+y\partial_y)$, is a basis of V_d formed by eigenvectors of A^*. From Lemma 3.2, it follows that the vectors in $\mathcal{P}_d(\partial_x)$ have distinct eigenvalues. Analogously, the vectors in $\mathcal{P}_d(\partial_y)$ (respectively in $\mathcal{P}_d(R)$) have different eingenvalues. This implies that $0\leqslant \dim(E_j)\leqslant 3$.

If $\dim(E_j)=3$, then E_j must contain one vector in each part of the basis: $\mathcal{P}_d(\partial_x)$, $\mathcal{P}_d(\partial_y)$ and $\mathcal{P}_d(R)$.

Note that $E_d=\langle\partial_x, x^d\cdot\partial_y, y^d\cdot R\rangle$. Let us prove that $E_d\subset\ker(T)$. Let $C_{(s,t)}(x,y)=(s\cdot x, t\cdot y)$ and consider the family $X(r,s,t)\in V_d$ given by

$$X(r,s,t)=r\cdot C^*_{(s,t)}(X_d)=r\cdot s^{-1}\partial_x+r\cdot s^d\cdot t^{-1}\cdot x^d\cdot\partial_y+r\cdot t^d\cdot y^d\cdot R.$$

Of course, for $r,s,t\neq 0$ we have

$$B(X(r,s,t))=B(X_d).$$

This implies that the vectors $\partial/\partial x$, $x^d(\partial/\partial y)$ and $y^d\cdot R$ belong to $\ker(T)$. This proves part (a).

Let us prove part (b). Suppose that $\dim(E_r)=3$ for some $r\in\{1,\dots,d^2+d+1\}$. Then, we must have $E_r=\langle x^i\cdot y^j\cdot\partial_x, x^k\cdot y^\ell\cdot\partial_y, x^m\cdot y^n\cdot R\rangle$, where $0\leqslant i+j, k+\ell\leqslant d$ and $m+n=d$. This implies that

$$-d(i-1)+j=-d\cdot k+\ell-1=-d\cdot m+n=r \mod N. \tag{9}$$

Since $-d(d+1)=1 \mod N$, this implies that

$$\begin{aligned} i-1+(d+1)j=m+(d+1)n=d\cdot n+m+n=d(n+1) \mod N\\ \implies\quad d\cdot j+i+j-1=d(n+1) \mod N.\end{aligned}$$

Let us suppose by contradiction that $r\neq d$. In the case $i=j=0$ we have $r=d$, and so we must have $1\leqslant i+j\leqslant d$. This implies that

$$\begin{aligned} 0\leqslant d\cdot j+i+j-1\leqslant d\cdot j+d-1=d(j+1)-1\leqslant d(d+1)-1<N\\ \implies\quad d\cdot j+i+j-1=d(n+1),\end{aligned}$$

because $0<d(n+1)\leqslant d(d+1)<N$. Therefore, d divides $i+j-1$. Since $0\leqslant i+j-1\leqslant d-1$, we get $i+j=1$ and $j=n+1>0$. Hence, $i=0$, $j=n+1$ and $r=n-d\cdot m=n+1+d \mod N$. It follows that $d(m+1)+1=0 \mod N$, which implies that $i=0$, $j=1$, $m=d$, $n=0$ and $r=d+1$.

On the other hand this, together with (9), implies that

$$r=d+1=-d\cdot k+\ell-1 \mod N \quad\implies\quad d(k+1)+2=\ell \mod N.$$

We assert that this is impossible, if $0\leqslant k+\ell\leqslant d$. In fact, if $0\leqslant k\leqslant d-1$, then we would get

$$0<d(k+1)+2\leqslant d^2+2<N \quad\implies\quad \ell=d(k+1)+2 \quad\implies\quad \ell>d,$$

which is impossible. If $k=d$, then $\ell=0$ and we would get $d(d+1)+2=0 \mod N$, which is a contradiction. Therefore, $r=d$, which proves part (b).

It remains to prove part (c). Set $M=\#\{j\mid E_j\neq\{0\}\}$. It is clear that M is the number of different eigenvalues of A^*. Lemma 3.2 implies that all vectors in $P(\partial_x)$ have different eigenvalues. Since $\#(P(\partial_x))=(d+1)(d+2)/2$, we get this number of eigenvalues, such that the corresponding eigenvectors are in $P(\partial_x)$. Consider the function $\phi\colon\mathcal{P}_d(x\cdot\partial_x)\to\mathcal{P}_d(y\cdot\partial_y)$ defined by

$$\phi(x^i\cdot y^j\cdot\partial_x)=x^{i-1}\cdot y^{j+1}\cdot\partial_y.$$

A straightforward computation shows that, if $Y\in P(x\cdot\partial_x)$ is such that $A^*(Y)=\lambda\cdot Y$, then $A^*(\phi(Y))=\lambda\cdot\phi(Y)$. This proves that the eingenvectors of A^* in $\mathcal{P}_d(\partial_y)$ which correspond to new eigenvalues (not found in the previous set) must be in $\mathcal{P}_d(\partial_y)\setminus\mathcal{P}_d(y\cdot\partial_y)$. Therefore, they are of the form $x^k\cdot\partial_y$, where $0\leqslant k\leqslant d$. We assert that there are $d-1$ new eigenvalues in this set.

In fact, if $x^i \cdot y^j \cdot \partial_x$ and $x^k \cdot \partial_y$ have the same eigenvalue, then $-d(i-1)+j = -k \cdot d - 1 \mod N$. Thus,

$$i - 1 + (d+1)j = k - (d+1) \mod N,$$

which implies that

$$k = d(j+1) + i + j \mod N.$$

Of course, we have the known solution, $k = d$, $i = j = 0$, which corresponds to vectors in E_d. Another solution is $k = d-1$, $i = 0$ and $j = d$, as the reader can check. On the other hand, if $0 \leqslant j \leqslant d-1$, then

$$0 < d(j+1) + i + j \leqslant d^2 + d < N \quad \Longrightarrow \quad d(j+1) + i + j = k,$$

implying that

$$i = j = 0 \quad \text{and} \quad k = d.$$

Therefore, there are only two repeated eigenvalues and $d-1$ new in this set. The repeated eigenvalues correspond to E_d and E_{2d}.

It remains to find how many new eigenvalues we can find in the set $\mathcal{P}_d(R)$. Suppose first that we have a vector $x^m \cdot y^n \cdot R$ in $\mathcal{P}_d(R)$ with the same eigenvalue of a vector $x^i \cdot y^j \cdot \partial_x \in \mathcal{P}_d(\partial_x)$. This case was already considered in the proof of part (b). We have found two possibilities: $(i,j) = (0,0)$, $(m,n) = (0,d)$ (which corresponds to vectors in E_d) and $(i,j) = (0,1)$, $(m,n) = (d,0)$ (which corresponds to E_{d+1}). Suppose now that we have a vector $x^m \cdot y^n \cdot R$ in $\mathcal{P}_d(R)$ and a vector $x^k \cdot \partial_y$ in $\mathcal{P}_d(\partial_y)$ with the same eigenvalue. Then

$$-k \cdot d - 1 = -d \cdot m + n \mod N \quad \Longrightarrow \quad k - (d+1) = m + n(d+1) = d(n+1) \mod N$$

which implies that

$$k = d \cdot n + 2d + 1 \mod N.$$

We have the following two solutions of the above relation: $k = d$, $(m,n) = (0,d)$ (which corresponds to E_d) and $k = 0$, $(m,n) = (1, d-1)$. On the other hand, if $0 \leqslant n \leqslant d-2$, then

$$2d + 1 \leqslant d \cdot n + 2d + 1 \leqslant d^2 + 1 < N \quad \Longrightarrow \quad k = d \cdot n + 2d + 1 > d,$$

which contradicts $0 \leqslant k \leqslant d$. Therefore, there are two repeated solutions, which correspond to E_d and E_{d^2+d}. This implies that there is a total of three eigenvalues in $\mathcal{P}_d(R)$ which were already found in the previous sets. Since $\#(\mathcal{P}_d(R)) = d+1$, we find $d-2$ new eigenvalues corresponding to eigenvectors in the set $\mathcal{P}_d(R)$. Hence, the total number of eigenvalues of A^* is

$$M = \frac{(d+1)(d+2)}{2} + d - 1 + d - 2 = \frac{d^2 + 7d - 4}{2},$$

which proves the lemma. $\square$

In order to finish the proof of Theorem 2, it is sufficient to verify the following fact: for any $j \in \{0, \ldots, N-1\}$ such that $j \neq d$ and $E_j \neq \{0\}$, $T_1|_{E_j} \not\equiv 0$. To do this will need first to carry on a study of the local variation of the Baum–Bott index.

3.3 Local variation of the Baum–Bott index

We will consider the following situation: let X be a polynomial vector field in V_d and $p_0 \in \mathbb{C}^2$ be a non-degenerate singularity of X. Denote by X_1 the 1-jet of X at p_0, that is, $X_1 = DX(p_0)$. Let $U \subset V_d$ be a neighborhood of X such that there exists a holomorphic map $p\colon U \to \mathbb{C}^2$ with $p(X) = p_0$ and for any $Y \in U$, then $p(Y)$ is a non-degenerate singularity of Y. Let $B\colon U \to \mathbb{C}$ be defined by $B(Y) = \mathrm{BB}(Y, p(Y))$. We will prove the following result.

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LEMMA 3.4. *Suppose that the eigenvalues of X_1 are in the Poincaré domain and have no resonances. Let $Z \in V_d \cap \ker(DB(X))$, that is, $dB(X) \cdot Z = 0$. Then there exists $\lambda \in \mathbb{C}$ and a germ of holomorphic vector field Y at p_0, such that*

$$Z_{p_0} = \lambda \cdot X_{p_0} + [X_{p_0}, Y],$$

where in the above relation, X_{p_0} and Z_{p_0} denote the germs of the respective vector fields at p_0. In particular, if $Z(p_0) = 0$, then $Y(p_0) = 0$ and

$$Z_1 = \lambda \cdot X_1 + [X_1, Y_1],$$

where $Z_1 = DZ(p_0)$ and $Y_1 = DY(p_0)$.

Proof. Let $B\colon U \to \mathbb{C}$ be as before. Set $B(X) = b_0$ and let $S := B^{-1}(b_0)$. We will prove first that $DB(X) \not\equiv 0$. This will imply that we can suppose (by taking a smaller U) that S is a smooth codimension one sub-variety of U.

To simplify the notation, we will suppose that $p_0 = 0 \in \mathbb{C}^2$. In this case, we have $X = X_1 +$ higher-order terms, where in a suitable affine coordinate system,

$$X_1 = \lambda_1 \cdot x\partial_x + \lambda_2 \cdot y\partial_y, \quad \lambda_1, \lambda_2 \notin \mathbb{R}_- \text{ and } \lambda_2/\lambda_1, \lambda_1/\lambda_2 \notin \mathbb{N} \text{ (Poincaré conditions).}$$

Consider the curve $X(t)$ in V_d defined by

$$X(t) = X + t \cdot x\partial_x.$$

Then $X(0) = X$, $X(t)(0) \equiv 0 \in \mathbb{C}^2$ and $X(t)_1 = X_1 + t \cdot x\partial_x$, which implies that

$$B(X(t)) = \frac{(\lambda_1 + \lambda_2 + t)^2}{(\lambda_1 + t)\lambda_2}$$

and, consequently,

$$DB(X) \cdot (x\partial_x) = \frac{d}{dt} B(X(t))|_{t=0} = \frac{1 - (\lambda_2/\lambda_1)^2}{\lambda_2} \neq 0,$$

because $\lambda_2/\lambda_1 \neq \pm 1$. Therefore, we will suppose that S is smooth of codimension one.

Now, let $Z \in \ker(DB(X))$. Since S is smooth, there exists a real analytic curve $Y(t) \subset S$, $t \in (-\epsilon, \epsilon)$, such that $Y(0) = X$ and $(d/dt)Y(t)|_{t=0} = Z$. Therefore, we can write

$$Y(t) = X + t \cdot Z + \sum_{n=2}^{\infty} t^n \cdot Y_n, Y_n \in V_d, \quad \forall n \geqslant 2.$$

Set $p(t) := p(Y(t))$, so that $p(0) = p_0$ and $p(t)$ is a non-degenerate singularity of $Y(t)$. Let $\lambda_1(t)$ and $\lambda_2(t)$ be eigenvalues of $DY(t)(p(t))$, where we can suppose that $t \mapsto \lambda_j(t)$ is real analytic and $\lambda_j(0) = \lambda_j$ for $j = 1, 2$. Since $B(Y(t)) = b_o$ for all $t \in (-\epsilon, \epsilon)$, we get

$$b_o \equiv \frac{(\lambda_1(t) + \lambda_2(t))^2}{\lambda_1(t) \cdot \lambda_2(t)} \equiv \frac{(\lambda_1 + \lambda_2)^2}{\lambda_1 \cdot \lambda_2} \quad \Longrightarrow \quad \lambda_2(t)/\lambda_1(t) \equiv \lambda_2/\lambda_1, \quad \forall t \in (-\epsilon, \epsilon),$$

as the reader can check, by using the condition $\lambda_j(0) = \lambda_j$, $j = 1, 2$. This implies that,

$$\lambda_2(t)/\lambda_2 \equiv \lambda_1(t)/\lambda_1 := \phi(t),$$

where ϕ is real analytic and $\phi(0) = 1$. Now, we use the Poincaré conditions. It follows from Poincaré's linearization theorem that there exist $0 < \delta \leqslant \epsilon$, a neighborhood V of $0 \in \mathbb{C}^2$ and a real analytic map $\Psi\colon (-\delta, \delta) \times V \to \mathbb{C}^2$ with the following properties:

(i) $\Psi(t, 0) = p(t)$ for all $t \in (-\delta, \delta)$;

(ii) for all $t \in (-\delta, \delta)$, $\Psi_t(x, y) := \Psi(t, x, y)$ is a biholomorphism from V to $V(t) := \Psi_t(V)$ and $\Psi_0 = \mathrm{id}_V$ (the identity map);

(iii) for all $t \in (-\delta, \delta)$ we have $\Psi_t^*(Y(t)) = \phi(t) \cdot Y(0) = \phi(t) \cdot X$.

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Writing explicitly the last relation, we have

$$D\Psi_t^{-1} \cdot Y(t) \circ \Psi_t = \phi(t) \cdot X \quad \Longrightarrow \quad Y(t) \circ \Psi_t = \phi(t) \cdot D\Psi_t \cdot X. \tag{10}$$

Let $\Psi_t(x,y) = (\Psi_t^1(x,y), \Psi_t^2(x,y))$ and consider the vector field $W = P_1(\partial/\partial x) + P_2(\partial/\partial y)$, where

$$P_j(x,y) = \frac{\partial \Psi^j}{\partial t}(0,x,y), \quad j = 1,2.$$

Note that the components of W and $(\partial/\partial t)\Psi|_{t=0}$ coincide. Taking the partial derivative of both members of (10) with respect to t at $t=0$, we get

$$\begin{aligned} Z + DX \cdot W &= Z + DY(0) \cdot W \\ &= Y'(0) \circ \Psi_0 + DY(0) \circ \Psi_0 \cdot \frac{\partial \Psi_t}{\partial t}\bigg|_{t=0} \\ &= \phi'(0) \cdot D\Psi_0 \cdot X + \phi(0) \cdot D\left(\frac{\partial}{\partial t}\Psi|_{t=0}\right) \cdot X \\ &= \phi'(0) \cdot X + DW \cdot X. \end{aligned}$$

If we set $\lambda = \phi'(0)$, then we get

$$Z = \lambda \cdot X + DW \cdot X - DX \cdot W = \lambda \cdot X + [W, X].$$

This proves the first part of the lemma. We leave the proof of the second part for the reader. □

3.4 Conclusion of the proof of Theorem 2

Back to our original problem it remains to show that for any $j \in \{0, \dots, N-1\}$ such that $j \neq d$ and $E_j \neq \{0\}$, then $T_1|_{E_j} \not\equiv 0$. This will be achieved in the next result.

Lemma 3.5. *Let $W \in \mathcal{P}_d$ be such that $W \in \ker(T_1)$. Then $W \in E_d$.*

Proof. Let W be in $\mathcal{P}_d \cap \ker(T_1)$. We have three possible cases.

First case: $W = x^i \cdot y^j \partial_x$, where $0 \leqslant i + j \leqslant d$. Recall that $\partial_x \in \ker(T_1)$. We assert that, if $1 \leqslant i + j \leqslant d$, then $W \notin \ker(T_1)$.

In fact, set $Z = W - \partial_x = (x^i \cdot y^j - 1)\partial_x$. Since $T_1(\partial_x) = 0$, we have

$$T_1(W) = 0 \iff T_1(Z) = 0.$$

Recall that $T_1 = DB_1(X_d)$, $B_1(X) = \mathrm{BB}(\mathcal{F}(X), \gamma_1(X))$ and $\gamma_1(X_d) = (1,1) = p_1$. Since $Z(1,1) = 0$, it follows from Lemma 3.4 that it is enough to verify whether or not $Z_1 = DZ(1,1)$ belongs to the image of the linear map $\Psi\colon \mathbb{C} \times L_1 \to L_1$ defined by

$$\Psi(\lambda, Y_1) = \lambda \cdot X_1 + [X_1, Y_1],$$

where L_1 is the set of 1-jets of germs of holomorphic vector fields Y at $(1,1)$ such that $Y(1,1) = 0$. Note that L_1 is isomorphic to the set M_2, of 2×2 matrices, via the linear map $\Phi\colon L_1 \to M_2$ defined by

$$Y = P\partial_x + Q\partial_y \overset{\Phi}{\mapsto} DY(1,1) = \begin{bmatrix} \dfrac{\partial P}{\partial x}(1,1) & \dfrac{\partial P}{\partial y}(1,1) \\ \dfrac{\partial Q}{\partial x}(1,1) & \dfrac{\partial Q}{\partial x}(1,1) \end{bmatrix}.$$

The map Φ is an isomorphism of Lie algebras. We will call $\Phi(Y_1)$ the matrix form of Y_1 and, to simplify, we will keep the notation Y_1 instead of $\Phi(Y_1)$. Note that,

$$X_1 = \begin{bmatrix} -1 & -d \\ d & -(d+1) \end{bmatrix} \quad \text{and} \quad Z_1 = \begin{bmatrix} i & j \\ 0 & 0 \end{bmatrix}.$$

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Let $Y_1 = \left[\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right]$. Then, $\Psi(\lambda, Y_1) = \lambda \cdot X_1 + [X_1, Y_1]$ and

$$[X_1, Y_1] = Y_1 X_1 - X_1 Y_1 = \begin{bmatrix} d(\beta+\gamma) & d(\delta-\alpha-\beta) \\ d(\gamma+\delta-\alpha) & -d(\beta+\gamma)\end{bmatrix} := \begin{bmatrix} x & y \\ z & -x\end{bmatrix},$$

as the reader can check. In particular, we get $\mathrm{tr}([X_1, Y_1]) = 0$ and the following relation between the entries of $[X_1, Y_1]$

$$x = z - y. \tag{11}$$

Let us suppose that $Z_1 = \Psi(\lambda, Y_1)$. Since $\mathrm{tr}([X_1, Y_1]) = 0$, we get

$$i = \mathrm{tr}(Z_1) = \lambda \cdot \mathrm{tr}(X_1) = -\lambda \cdot (d+2) \quad \Longrightarrow \quad \lambda = -\frac{i}{d+2}.$$

This implies that the matrix $Z_1 + (i/(d+2))X_1$ must satisfy (11). On the other hand, we have,

$$Z_1 + \frac{i}{d+2} X_1 = \begin{bmatrix} \dfrac{(d+1)i}{d+2} & \dfrac{(d+2)j - d\cdot i}{d+2} \\ \dfrac{d\cdot i}{d+2} & -\dfrac{(d+1)i}{d+2}\end{bmatrix}.$$

Hence, $Z \in \ker(T_1)$ if and only if

$$\frac{(d+1)i}{d+2} = \frac{d\cdot i}{d+2} - \frac{(d+2)j - d\cdot i}{d+2}$$

if and only if

$$(d-1)i = (d+2)j.$$

The last relation implies that $d+2|i$, which implies that $i = 0$ and $j = 0$, which contradicts the assumption $i + j \geqslant 1$.

Second case: $W = x^k \cdot y^\ell \partial_y$, where $0 \leqslant k + \ell \leqslant d$. Recall that $x^d \partial_y \in \ker(T_1)$. We assert that, if $0 \leqslant k \leqslant d-1$ and $0 \leqslant k+\ell \leqslant d$, then $W \notin \ker(T_1)$.

The idea is the same as in the first case. Let $Z = W - x^d\partial_y = (x^k \cdot y^\ell - x^d)\partial_y$. Since $x^d\partial_y \in \ker(T_1)$, then $W \in \ker(T_1) \iff Z \in \ker(T_1)$. In this case, we have $Z(1,1) = 0$ and

$$Z_1 = \begin{bmatrix} 0 & 0 \\ k-d & \ell\end{bmatrix} \quad \Longrightarrow \quad \lambda = -\frac{\ell}{d+2} \quad \Longrightarrow \quad Z_1 - \lambda X_1 = \begin{bmatrix} -\dfrac{\ell}{d+2} & -\dfrac{d\cdot \ell}{d+2} \\ \dfrac{d\cdot\ell + (k-d)(d+2)}{d+2} & \dfrac{\ell}{d+2}\end{bmatrix}.$$

Hence, $Z \in \ker(T_1)$ if and only if

$$-\frac{\ell}{d+2} = \frac{d\cdot\ell + (k-d)(d+2)}{d+2} + \frac{d\cdot\ell}{d+2} \iff (d-k)(d+2) = (2d+1)\ell.$$

As the reader can check, if $0 \leqslant k+\ell \leqslant d$, then the last relation is possible only for $k = d$ and $\ell = 0$, which proves the assertion.

Third case: $W = x^m y^n R$, where $m + n = d$. Recall that $y^d \cdot R \in \ker(T_1)$. We assert that, if $0 \leqslant n \leqslant d-1$, then $W \notin \ker(T_1)$.

In this case, if $Z = W - y^d \cdot R = (x^m \cdot y^n - y^d)\cdot R$, then $Z(1,1) = 0$ and

$$Z_1 = \begin{bmatrix} m & n-d \\ m & n-d\end{bmatrix} \quad \Longrightarrow \quad \mathrm{tr}(Z_1) = 0$$

and

$$\lambda = 0 \quad \Longrightarrow \quad m = m - (n-d) \quad \Longrightarrow \quad n = d \text{ and } m = 0.$$

This finishes the proof of the lemma and of Theorem 2. ∎

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4. The Camacho–Sad field

4.1 Preliminaries

Let M and S be two complex compact surfaces, $\phi\colon M \dashrightarrow S$ a meromorphic map and $\mathcal{F}$ a foliation on S. We want to prove that $\mathbb{K}(\phi^*(\mathcal{F})) = \mathbb{K}(\mathcal{F})$. We will use the notation $\mathcal{G} := \phi^*(\mathcal{F})$. As was sketched in the introduction, the theorem is true when ϕ consists of a sequence of blowing-ups. This fact allows us to reduce the problem to the case where $\mathcal{F}$ and $\mathcal{G}$ are reduced and ϕ is holomorphic. Thus, from now on, we will suppose that the foliations $\mathcal{F}$ and $\mathcal{G} = \phi^*(\mathcal{F})$ are reduced and that $\phi\colon M \to S$ is holomorphic. Before going on, let us fix some notation.

Let $\mathcal{H}$ be a reduced foliation on a compact surface V. Given $p \in V$ we will associate a field, $\mathbb{K}(\mathcal{H}, p)$, as follows: let X be a holomorphic vector field which represents $\mathcal{H}$ in a neighborhood of p. When $p \in \operatorname{sing}(\mathcal{H})$, we will denote by λ_1, λ_2 the eigenvalues of $DX(p)$. We have the following three possibilities.

- $p \in \operatorname{sing}(\mathcal{F})$, $\lambda_1, \lambda_2 \neq 0$ and $\lambda_2/\lambda_1 \notin \mathbb{Q}_+$. In this case, $\mathcal{H}$ has two local separatrixes Σ_1 and Σ_2 through p and $\mathrm{CS}(\mathcal{H}, \Sigma_1, p) = \lambda_2/\lambda_1$, $\mathrm{CS}(\mathcal{H}, \Sigma_2, p) = \lambda_1/\lambda_2$. In this case, we set: $\mathbb{K}(\mathcal{H}, p) = \mathbb{Q}(\lambda_2/\lambda_1) = \mathbb{Q}(\lambda_1/\lambda_2)$.
- $\lambda_1 = 0$ and $\lambda_2 \neq 0$. We will suppose that $\lambda_2 = 1$. In this case, $\mathcal{H}$ has one local analytic separatrix Σ_2 through p, tangent to the eigenspace of $\lambda_2 = 1$ and $\mathrm{CS}(\mathcal{H}, \Sigma_2, p) = 0$. The separatrix Σ_1, tangent to the eigenspace of $\lambda_1 = 0$ is formal, in general, but X is formally equivalent to the vector field $Y := x^{k+1}\partial_x + y(1 + \lambda x^k)\partial_y$. We have $\mathrm{CS}(\mathcal{H}, \Sigma_1, p) = \lambda$ (by definition) and we set $\mathbb{K}(\mathcal{H}, p) = \mathbb{Q}(\lambda)$.
- $p \notin \operatorname{sing}(\mathcal{F})$. In this case, we set $\mathbb{K}(\mathcal{F}, p) = \mathbb{Q}$.

In general, if $\emptyset \neq A \subset V$ and $A \cap \operatorname{sing}(\mathcal{H}) = \{p_1, \ldots, p_k\}$, we set

$$\mathbb{K}(\mathcal{H}, A) = \mathbb{Q}(\mathbb{K}(\mathcal{H}, p_1), \ldots, \mathbb{K}(\mathcal{H}, p_k)).$$

When $A \cap \operatorname{sing}(\mathcal{H}) = \emptyset$ we set $\mathbb{K}(\mathcal{H}, A) = \mathbb{Q}$. In this notation, we also have that:

- $\mathbb{K}(\mathcal{H}) = \mathbb{K}(\mathcal{H}, V)$;
- if $A, B \subset V$, then $\mathbb{K}(\mathcal{H}, A \cup B) = \mathbb{Q}(\mathbb{K}(\mathcal{H}, A), \mathbb{K}(\mathcal{H}, B))$.

The next result implies Theorem 3.

Lemma 4.1. *For any $p \in S$, we have*

$$\mathbb{K}(\phi^*(\mathcal{F}), \phi^{-1}(p)) = \mathbb{K}(\mathcal{F}, p).$$

We first note that $\phi^{-1}(p) \neq \emptyset$, because the generic rank of ϕ is two, which implies that ϕ is surjective. Moreover, $\phi^{-1}(p)$ is an analytic subset whose connected components have dimension zero (points) or one (curves). In fact, we will prove that for any connected component C of $\phi^{-1}(p)$ we have

$$\mathbb{K}(\phi^*(\mathcal{F}), C) = \mathbb{K}(\mathcal{F}, p).$$

Clearly this fact implies the lemma. Before going on, we will state some remarks and preliminary results.

Remark 4.1. Let Z be vector field representing $\mathcal{F}$ in a sufficiently small neighborhood U of a point $p \in S$. Locally, and up to an analytic change of coordinates, we have three possibilities.

(1) p is not a singularity of $\mathcal{F}$. In this case, $\mathbb{K}(\mathcal{F}, p) = \mathbb{Q}$. We can suppose that $Z = \partial_y$. In particular, $\mathcal{F}$ has a local holomorphic first integral (y) and has just one local separatrix through p: the curve $y = 0$.

(2) p is a reduced and simple singularity of $\mathcal{F}$ and the eigenvalues of $DZ(p)$ are $\lambda_1, \lambda_2 \neq 0$. In this case, $\lambda_2/\lambda_1 \notin \mathbb{Q}_+$ and $\mathbb{K}(\mathcal{F}, p) = \mathbb{Q}(\lambda_2/\lambda_1)$. The foliation $\mathcal{F}$ has two local separatrices through p, which are smooth and transversal at p. We can suppose that they are $(x = 0)$ and $(y = 0)$ and that

$$Z = \lambda_1 \cdot x\partial_x + \lambda_2 \cdot y(1 + R(x, y))\partial_y, \tag{12}$$

where $R(0, 0) = 0$.

(3) p is a saddle-node of $\mathcal{F}$ and we can suppose that the eigenvalues of $DZ(p)$ are 0 and 1. In this case, Z is formally equivalent at p to the vector field $\hat{Z} = x^{k+1}\partial_x + \hat{y}(1+\lambda\cdot x^k)\partial_{\hat{y}}$, where $k \geqslant 1$, and $\mathbb{K}(\mathcal{F}, p) = \mathbb{Q}(\lambda)$. Here, we will use Dulac's normal form (cf. [MR82]). For every $m \geqslant k+1$ there exists a holomorphic coordinate system $(U, (x, y))$ such that $x(p) = y(p) = 0$ and $\mathcal{F}$ is defined by

$$Z = x^{k+1}\partial_x + [y(1 + \lambda \cdot x^k) + R(x, y)]\partial_y. \tag{13}$$

where the m-jet of R is zero at $0 \in \mathbb{C}^2$. When $\mathcal{F}$ has two local analytic separatrices through p, we can suppose that y divides R. When it has just one analytic separatrix, then it also has a formal one, given by $\hat{y} = 0$, where $\hat{y}$ is a divergent series of the form (cf. [MR82]):

$$\hat{y} = y - \sum_{j=r+1}^{\infty} a_j x^j. \tag{14}$$

We break down the proof of Lemma 4.1 in three cases.

Proof of Lemma 4.1 (First case: p is not a singularity of $\mathcal{F}$). Here $\mathcal{F}$ admits a holomorphic first integral in a neighborhood of p. If $g \in \mathcal{O}_p$ is such holomorphic first integrals then ϕ^*g is an holomorphic first integral for $\mathcal{G} = \phi^*\mathcal{F}$ in a neighborhood of $\phi^{-1}(p)$. Thus, $\mathbb{K}(\mathcal{G}, \phi^{-1}(p)) = \mathbb{Q}$. □

From now on, we will suppose that $p \in \operatorname{sing}(\mathcal{F})$. In the next results, we will consider the following situation: let $q \in \phi^{-1}(p) \cap \operatorname{sing}(\mathcal{G})$. Suppose that $\mathcal{G}$ has a local analytic separatrix $\tilde{\Sigma}$ through q such that $\phi(\tilde{\Sigma}) \neq \{p\}$. In this case, $\phi(\tilde{\Sigma}) := \Sigma$ is a local analytic separatrix of $\mathcal{F}$ through p.

Lemma 4.2. *In the above situation, we have:*

(a) $\mathrm{CS}(\mathcal{G}, \tilde{\Sigma}, q) \in \mathbb{Q}(\mathrm{CS}(\mathcal{F}, \Sigma, p))$;

(b) *if* $\mathbb{K}(\mathcal{F}, p) = \mathbb{Q}(\mathrm{CS}(\mathcal{F}, \Sigma, p))$, *then* $\mathbb{K}(\mathcal{F}, p) = \mathbb{Q}(\mathrm{CS}(\mathcal{G}, \tilde{\Sigma}, q))$.

Proof. Let $(f = 0)$ be a reduced equation Σ and write

$$g \cdot \omega = h \cdot df + f \cdot \mu, \tag{15}$$

where $g, h|_\Sigma \neq 0$. From the definition, we have

$$\mathrm{CS}(\mathcal{F}, \Sigma, p) = \frac{1}{2\pi i} \int_\gamma -\frac{\mu}{h},$$

where γ is a small circle in Σ around p, positively oriented. Note that $\phi^*(\omega) = \tilde{k}\cdot\theta_q$, where $\tilde{k} \in \mathcal{O}_q$ and θ_q represents the germ of $\mathcal{G}$ at q. Let $\tilde{f} = 0$ be a reduced equation of $\tilde{\Sigma}$. Since $\phi(\tilde{\Sigma}) = \Sigma = (f = 0)$, we get

$$\phi^*(f) = f \circ \phi = \tilde{g} \cdot \tilde{f}^m,$$

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where $m \geqslant 1$ and $\tilde{g}|_{\tilde{\Sigma}} \neq 0$. It follows from (15) that

$$\begin{aligned}\phi^*(g)\cdot\tilde{k}\cdot\theta_q &= \phi^*(h)\cdot d(\tilde{g}\cdot\tilde{f}^m)+\tilde{g}\cdot\tilde{f}^m\cdot\phi^*(\mu)\\ \Longrightarrow \quad \frac{\phi^*(g)\cdot\tilde{k}}{m\cdot\phi^*(h)\cdot\tilde{g}\cdot\tilde{f}^m}\cdot\theta_q &= \frac{d\tilde{f}}{\tilde{f}}+\frac{1}{m}\left[\frac{d\tilde{g}}{\tilde{g}}+\phi^*\left(\frac{\mu}{h}\right)\right]\\ \Longrightarrow \quad \mathrm{CS}(\mathcal{G},\tilde{\Sigma},q) &= -\frac{1}{2\pi i}\int_{\tilde{\gamma}}\frac{1}{m}\left[\frac{d\tilde{g}}{\tilde{g}}+\phi^*\left(\frac{\mu}{h}\right)\right],\end{aligned}$$

where $\tilde{\gamma}$ is a small circle in $\tilde{\Sigma}$ around q. Note that $\phi(\tilde{\gamma}) = \gamma^n$, where $n \geqslant 1$. Observe also that $(1/2\pi i)\int_{\tilde{\gamma}}(d\tilde{g}/\tilde{g}) = \ell \in \mathbb{Z}$. Hence,

$$\mathrm{CS}(\mathcal{G},\tilde{\Sigma},q) = -\frac{\ell}{m}+\frac{1}{m}\frac{1}{2\pi i}\int_{\gamma^n}-\frac{\mu}{h} = \frac{1}{m}(-\ell+n\cdot\mathrm{CS}(\mathcal{F},\Sigma,p)) \in \mathbb{Q}(\mathrm{CS}(\mathcal{F},\Sigma,p)).$$

Since $n \neq 0$, we get also that

$$\mathbb{Q}(\mathrm{CS}(\mathcal{G},\tilde{\Sigma},q)) = \mathbb{Q}(\mathrm{CS}(\mathcal{F},\Sigma,p)),$$

which implies part (b). □

Remark 4.2. The above result is true in the general case, that is, even if the map ϕ is meromorphic and the separatrices $\tilde{\Sigma}$ and Σ are singular.

Remark 4.3. If the connected component C of $\phi^{-1}(p)$ is a curve, then all irreducible components of C are invariant for the foliation $\mathcal{G}$, otherwise p would be a diacritical singularity of $\mathcal{F}$. Moreover, all of the singular points of C are nodes.

Proof of Lemma 4.1 (Second case: p is a singularity with two analytic separatrices). We will prove that every connected component C of $\phi^{-1}(p)$ is such that

$$\mathbb{K}(\mathcal{G},C) = \mathbb{K}(\mathcal{F},p).$$

First of all, observe that for one of the two separatrices, say Σ, we have

$$\mathbb{K}(\mathcal{F},p) = \mathbb{Q}(\mathrm{CS}(\mathcal{F},\Sigma,p)).$$

Let W be a neighborhood of C. Note that $\phi^{-1}(\Sigma)\cap W$ is a germ of analytic set around $\phi^{-1}(p)$, different from $\phi^{-1}(p)$. Each component of $\phi^{-1}(\Sigma)\setminus\phi^{-1}(p)$ is a curve biholomorphic to $\mathbb{D}^*$, whose closure contains a unique point in $\phi^{-1}(p)$. Let $\tilde{\Sigma}$ be a closure of some of these components and set $\tilde{\Sigma}\cap\phi^{-1}(p) = \{q\}$. It follows from Lemma 4.2(b) that

$$\mathbb{K}(\mathcal{G},q) = \mathbb{K}(\mathcal{F},p).$$

This implies that

$$\mathbb{K}(\mathcal{G},C) \supset \mathbb{K}(\mathcal{G},q) = \mathbb{K}(\mathcal{F},p).$$

It remains to prove that, for any $q \in \mathrm{sing}(\mathcal{G})\cap C$, then $\mathbb{K}(\mathcal{G},q) \subset \mathbb{K}(\mathcal{F},p)$. If C has dimension zero, that is, $C = q$, the above argument shows that $\mathbb{K}(\mathcal{G},C) = \mathbb{K}(\mathcal{F},p)$.

From now on, we will suppose that C is a curve. The next result implies the second case of Lemma 4.1. □

Lemma 4.3. *Let $q \in C\cap\mathrm{sing}(\mathcal{G})$ and $\tilde{\Sigma}_1$ be a separatrix of $\mathcal{G}$ through q. Then $\tilde{\Sigma}_1$ is analytic and*

$$\mathrm{CS}(\mathcal{G},\tilde{\Sigma}_1,q) \in \mathbb{K}(\mathcal{F},p).$$

Proof. Suppose first that q is a smooth point of C and that $\tilde{\Sigma}_1 \not\subset C$. If $\tilde{\Sigma}_1$ is a formal separatrix of $\mathcal{G}$ which is non-convergent, then $\mathcal{F}$ would have a formal non-convergent separatrix at p contrary to

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our assumptions. This $\tilde{\Sigma}_1$ is analytic. Thus, Lemma 4.2 implies that

$$\mathrm{CS}(\mathcal{G}, \tilde{\Sigma}_1, q) = \mathrm{CS}(\mathcal{F}, \phi(\tilde{\Sigma}_1), p) \in \mathbb{K}(\mathcal{F}, p)$$

and we are done in this case.

Let us suppose now that $\tilde{\Sigma}_1 \subset C$. In this case, $\tilde{\Sigma}_1$ is analytic and smooth, but $\phi(\tilde{\Sigma}_1) = \{p\}$ and we cannot use Lemma 4.2 directly. The result will follow from the lemma below. □

Lemma 4.4. *In the above situation, there is a bimeromorphism $\psi\colon \hat{S} \to S$ (a sequence of blowing-ups) such that, if we set $\hat{\phi} := \psi^{-1} \circ \phi\colon M \dashrightarrow \hat{S}$, $\hat{\mathcal{F}} = \psi^*(\mathcal{F})$ and $D = \psi^{-1}(p)$, then:*

(a) *there exists $\hat{p} \in D \cap \mathrm{sing}(\hat{\mathcal{F}})$ and a separatrix $\Sigma_1 \subset D$ of $\hat{\mathcal{F}}$ through $\hat{p}$ such that $\hat{\phi}(\tilde{\Sigma}_1) = \Sigma_1$;*

(b) $\mathrm{CS}(\mathcal{G}, \tilde{\Sigma}_1, q) \in \mathbb{K}(\hat{\mathcal{F}}, \hat{p}) \subset \mathbb{K}(\mathcal{F}, p)$.

Proof. Let $\tilde{\Sigma}_2$ be the other separatrix of $\mathcal{G}$ through $q \in M$ and let $(V, (u, v))$ be a local coordinate system around q such that $u(q) = v(q) = 0$, $\mathrm{sing}(\mathcal{G}) \cap V = \{q\}$, $\tilde{\Sigma}_1 = (u = 0)$, $\tilde{\Sigma}_2 = (v = 0)$, $V = \{(u, v) \mid |u|, |v| < \epsilon\}$ and $\phi(V) \subset U$.

Write the germ of ϕ at q as $\phi_q = (X_q, Y_q)$. Since $\phi(\tilde{\Sigma}_1) = \{p\}$ it follows that $X_q(u, v) = u^m \cdot f(u, v)$ and $Y_q(u, v) = u^n \cdot g(u, v)$, where $m, n \geqslant 1$, $f, g \in \mathcal{O}_q$ and $f(0, v), g(0, v) \not\equiv 0$. For $|c| < \epsilon$, let γ_c be the germ at p of the curve $u \mapsto \phi(u, c)$. Note that γ_0 might be a point (if $\phi(\tilde{\Sigma}_2) = \{p\}$), however if we take a smaller $\epsilon > 0$, then we can suppose that γ_c is a curve, for all $0 < |c| < \epsilon$. Moreover, there is a sequence of blowing-ups $\psi\colon \hat{S} \to S$ such that, if $D = \psi^{-1}(p)$ and ϵ is small enough, then:

(i) $\psi\colon \hat{S} \setminus D \to S \setminus \{p\}$ is a bimeromorphism;

(ii) there is a divisor $D_1 \subset D$ such that, for all $0 < |c| < \epsilon$, the strict transform $\hat{\gamma}_c$ of γ_c meets D_1 in a unique point, say $p(c)$;

(iii) if $c_1 \neq c_2$ and $0 \neq c_1, c_2$, then $p(c_1) \neq p(c_2)$; in particular, the map $c \in \{z \mid 0 < |z| < \epsilon\} \simeq \mathbb{D}^* \mapsto p(c) \in D_1$ is a holomorphic embedding.

The sequence of blowing-ups ψ is a simultaneous resolution of the germs γ_c, $0 < |c| < \epsilon$. We leave the details for the reader. In this case, it follows from Picard's theorem that there exist $\lim_{c \to 0} p(c) = \hat{p} \in D_1$. Moreover, if $\hat{\mathcal{F}} = \psi^*(\mathcal{F})$, then the germ Σ_1 of D_1 at $\hat{p}$, is a separatrix of $\hat{\mathcal{F}}$ through $\hat{p}$ and $\psi^{-1} \circ \phi(\tilde{\Sigma}_1) = \Sigma_1$. This proves part (a).

Let us prove part (b). Note first that

$$\mathrm{CS}(\hat{\mathcal{F}}, \Sigma_1, \hat{p}) \in \mathbb{K}(\mathcal{F}, p) \quad \Longrightarrow \quad \mathbb{Q}(\mathrm{CS}(\hat{\mathcal{F}}, \Sigma_1, \hat{p})) \subset \mathbb{K}(\mathcal{F}, p),$$

because ψ is a sequence of blowing-ups (see the introduction). On the other hand, Lemma 4.2 implies that

$$\mathrm{CS}(\mathcal{G}, \tilde{\Sigma}_1, q) \in \mathbb{Q}(\mathrm{CS}(\hat{\mathcal{F}}, \Sigma_1, \hat{p})).$$

This finishes the proof. □

To finish the proof of Lemma 4.1 it remains to treat just one case.

Proof of Lemma 4.1 (Third case: p is singular with just one analytic separatrix). In this case, $\mathcal{F}$ has a normal form like in (13) of Remark 4.1: for every $r \geqslant k + 1$ there exists a local coordinate system $(U, (x, y))$ where $\mathcal{F}$ is represented by

$$\omega = x^{k+1}\, dy - [y(1 + \lambda \cdot x^k) + R(x, y)]\, dx, \tag{16}$$

where $k \geqslant 1$ and $j_0^r(R) = 0$. Let C be a connected component of $\phi^{-1}(p)$ and consider a sufficiently small neighborhood W of C. We denote by Σ_1 the non-convergent separatrix and by Σ_2 the convergent separatrix. In the coordinate system $(U, (x, y))$ we have $\Sigma_2 = (x = 0)$ and Σ_1 is given by

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the divergent series

$$y = \sum_{j=r+1}^{\infty} a_j x^j.$$

As before, the proof consists of proving that:

(I) for any $q \in C \cap \operatorname{sing}(\mathcal{G})$ we have $\mathbb{K}(\mathcal{G}, q) \subset \mathbb{K}(\mathcal{F}, p)$; and

(II) there exists $q_0 \in C \cap \operatorname{sing}(\mathcal{G})$ such that $\mathbb{K}(\mathcal{G}, q_0) = \mathbb{K}(\mathcal{F}, p)$.

Proof of part (I). Let us consider first the case where the two separatrices through q are analytic. Let $\tilde{\Sigma}$ be one of these separatrices. It is sufficient to prove that $\mathrm{CS}(\mathcal{G}, \tilde{\Sigma}, q) \in \mathbb{K}(\mathcal{F}, p)$.

In fact, if $\phi(\tilde{\Sigma}) \neq \{p\}$, then $\phi(\tilde{\Sigma})$ is a curve and $\phi(\tilde{\Sigma}) \subset \Sigma_2$. Since $\mathrm{CS}(\mathcal{F}, \Sigma_2, p) = 0$, we get from Lemma 4.2 that $\mathrm{CS}(\mathcal{G}, \tilde{\Sigma}, q) \in \mathbb{Q}$, as asserted. On the other hand, if $\phi(\tilde{\Sigma}) = \{p\}$, then the assertion follows from Lemma 4.4(b).

Let us suppose now that there is a non-convergent separatrix, say $\tilde{\Sigma}_1$, and a convergent separatrix, say $\tilde{\Sigma}_2$, through q. We assert that there is a coordinate system $(V, (u, v))$ around q such that $u(q) = v(q) = 0$, $\phi(V) \subset W$ and $\phi|_V(u, v) = (X(u, v), Y(u, v))$, where:

(i) $X(u, v) = u^m$, $m \geqslant 1$;

(ii) $Y(u, v) = u^n \cdot v$, where $n = 0$ if $C = \{q\}$ and $n \geqslant 1$ if C is a curve.

In fact, we can write $\phi_{|W} = (X, Y)$, where $X, Y : W \to \mathbb{C}$ are holomorphic functions and $X(q) = Y(q) = 0$. Let X_q and Y_q be the germs of X and Y at q. Since $\Sigma_2 = (x = 0)$ is invariant for $\mathcal{F}$, the irreducible components of $(X_q = 0)$ are local analytic separatrices of $\mathcal{G}$ through q. This implies that $(X_q = 0) = \tilde{\Sigma}_2$. Choose a local coordinate system (u, v) around q such that $\tilde{\Sigma}_2 = (u = 0)$. In this case, we get $X_q = u^m \cdot g$, where $m \geqslant 1$ and $g \in \mathcal{O}_q^*$. If we consider the local change of variables $u_1 = u \cdot g^{1/m}$, then $X_q = u_1^m$, and so we can suppose $X_q = u^m$. In this coordinate system we must have $Y_q = u^n \cdot Y_1$, where $Y_1 \in \mathcal{O}_q$. If C is a curve, then $\tilde{\Sigma}_2 \subset C$ (by Remark 4.3) and $n \geqslant 1$. If $C = \{q\}$, then $n = 0$ and $Y(0, v) \not\equiv 0$. We assert that $Y_v(0, 0) \neq 0$. Note that this implies that, after a holomorphic change of variables, we can suppose that $Y_1(u, v) = v$.

In fact, to say that the formal separatrix $\hat{y} := y - \sum_j a_j x^j$ is invariant for $\mathcal{F}$ is equivalent to

$$d\hat{y} \wedge \omega = \hat{f} \cdot \hat{y} \cdot dx \wedge dy, \tag{17}$$

where $\hat{f} \in \hat{\mathcal{O}}_p$ and $\hat{\mathcal{O}}_p$ denotes the ring of formal power series at p. Consider the formal power series

$$u^n \cdot \hat{Y}_1 := \hat{Y}(u, v) := \phi^*(\hat{y}) = u^n \biggl(Y_1(u, v) - \sum_{j \geqslant r+1} a_j u^{mj-n} \biggr), \tag{18}$$

where $\hat{Y}_1 \in \hat{\mathcal{O}}_q$ if we take r large enough. Let $\hat{Y}_1 = g_1^{n_1}, \ldots, g_s^{n_s}$ be the decomposition of $\hat{Y}_1$ into irreducible factors of $\hat{\mathcal{O}}_q$. Write $\phi^*(\omega) = h \cdot \theta_q$, where θ_q represents the germ of $\mathcal{G}$ at q. It follows from (17) that

$$\begin{aligned} h \biggl[n \cdot g_1 \cdots g_s \, du + u \biggl(\sum_j n_j \cdot g_1 \cdots g_{j-1} \cdot g_{j+1} \cdots g_s \cdot dg_j \biggr) \biggr] \wedge \theta_q \\ = \Delta \cdot \tilde{f} \circ \phi \cdot u \cdot g_1 \cdots g_s \, du \wedge dv, \end{aligned}$$

where $\Delta = X_u \cdot Y_v - X_v \cdot Y_u = u^{m+n-1} \cdot Y_{1v}$. We assert that h divides Δ in the $\mathcal{O}_q$.

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In fact, as the reader can check, we have $\phi^*(\omega) = u^{m+n-1}(A\,dv - B\,du)$, where

$$A = u^{km+1}\cdot Y_{1v}$$
$$B = m\cdot Y_1\left(1+\left(\lambda-\frac{n}{m}\right)\cdot u^{km}\right) + u^{km+1}\cdot T(u,v)$$

and $T \in \mathcal{O}_q$. This implies that $h = u^{m+n-1}\cdot h_1$, where any factor of h_1 is also a factor Y_{1v}, because u does not divides B. Therefore, h divides Δ.

It follows that

$$\left[n\cdot g_1\cdots g_s\,du + u\left(\sum_j n_j g_1\cdots g_{j-1}\cdot g_{j+1}\cdots g_s\,dg_j\right)\right]\wedge\theta_q = \hat{f}\cdot u\cdot g_1\cdots g_s\,du\wedge dv,$$

where $\hat{f}\in\hat{\mathcal{O}}_q$. Hence, all factors $g_1,\dots,g_s$ and $(u=0)$ are invariant for $\mathcal{G}$. Since $\mathcal{G}$ has only two separatrices through q, we get that $s=1$ and g_1 is the formal separatrix of $\mathcal{G}$ through q. Since $\mathcal{G}$ is reduced, we get $g_{1v}(0)\neq 0$ and $\hat{Y}_1 = g^s$, where $g = g_1$ and $s = n_1$. It follows from (18) that

$$Y_{1v} = \hat{Y}_{1v} = sg^{s-1}g_v.$$

Therefore, $Y_{1v}(0)=0$ if and only if $s>1$. Suppose, by contradiction, that $s>1$. Since $g_v(0)\neq 0$, by the formal Weierstrass' theorem we can write $g = f\cdot(v-h(u))$, where $f\in\hat{\mathcal{O}}_q$, $f(0)\neq 0$ and $h(u)$ is a power series. Therefore, if we set $k = s\cdot f^{s-1}\cdot g_v$, then we have $k\in\hat{\mathcal{O}}_q$, $k(0)\neq 0$ and $Y_{1v} = k\cdot(v-h(u))^{s-1}$. This implies that the germ of analytic set $(Y_{1v}=0)$ (which is not empty), is also given by $(v-h(u)=0)$, and so, $h(u)$ is convergent. However, this is a contradiction, because $\phi(v-h(u)=0) = (\hat{y}=0)$, which is divergent. Hence $s=1$ and $Y_{1v}(0)\neq 0$.

Let us finish the proof of part (I). Since $X(u,v)=u^m$ and $Y(u,v)=u^n\cdot v$, we get from (16) that $\phi^*(\omega) = u^{m+n-1}\cdot\theta_q$, where, given $\ell > mk+1$,

$$\theta_q = u^{km+1}\,dv - m\left[v\left(1+\left(\lambda-\frac{n}{m}\right)\cdot u^{km}\right)+\tilde{R}(u,v)\right]du$$

and $\tilde{R}(u,v) = u^{-n}\cdot R(u^m, u^n\cdot v)\in u^\ell\cdot\mathcal{O}_q$, if r is large enough. This implies that the formal normal form of $\mathcal{G}$ at q is given by

$$u^{km+1}\,dv - m\left[v\left(1+\left(\lambda-\frac{n}{m}\right)u^{km}\right)\right]du \quad\Longrightarrow\quad \mathbb{K}(\mathcal{G},q) = \mathbb{Q}(m\lambda-n) = \mathbb{Q}(\lambda) = \mathbb{K}(\mathcal{F},p).$$

Proof of part (II). We will suppose that C is a curve. The case where C is a point will be left for the reader. It follows from the proof of part (I) that it is sufficient to find a point $q\in C\cap\operatorname{sing}(\mathcal{G})$ with a non-convergent separatrix. Let W be a sufficiently small neighborhood of C. Consider the curve $C_1 := \phi^{-1}(y=0)\cap W$. Since $\phi(C_1) = (y=0)\neq\{p\}$, it follows that $C_1\setminus C\neq\emptyset$. Moreover, if δ is a component of $C_1\setminus C$, then δ is biholomorphic to $\mathbb{D}^*$ and $\overline{\delta}\cap C$ is a point, say q. We will denote by δ_q the germ of δ at q. We assert that $\mathcal{G}$ has a non-convergent separatrix through q.

We will see at the end that q is smooth point of C. Let us suppose this fact for a moment. Since $\phi(C)=\{p\}$, there exists a coordinate system $(V,(u,v))$ such that $V\subset W$, $u(q)=v(q)=0$ and $C\cap V = (u=0)$. In this case, the germ of ϕ at q can be written as

$$\phi_q(u,v) = (X_q(u,v), Y_q(u,v)) = (u^m X_1(u,v), u^n Y_1(u,v)),$$

where $X_1, Y_1\in\mathcal{O}_q$ and $X_1(0,v), Y_1(0,v)\not\equiv 0$. Note that $Y_1(0,0)=0$ and $\delta_q\subset(Y_1=0)$. On the other hand, since $(x=0)$ is an analytic separatrix of $\mathcal{F}$ through p, $X_1(0,0)\neq 0$, because otherwise q would be a node of C. This implies that, after a holomorphic change of variables, we can suppose that $X(u,v)=u^m$. It follows that the formal series

$$\hat{Y}_1 = \frac{1}{u^n}\Big(Y - \sum_{j\geqslant r+1} a_j X^j\Big) = Y_1 - \sum_j a_j u^{jm-n}$$

defines a formal separatrix of $\mathcal{G}$ through q (see the proof of part (I)).

It remains to prove that q is a smooth point of C. Suppose by contradiction that q is a node of C. The idea is to prove that in this case $\mathcal{G}$ has more than two separatrices through q, which is not possible for a reduced foliation. Let $(V,(u,v))$ be a coordinate system such that $C\cap V = (u\cdot v=0)$. In this case, we can write $X_q(u,v)=u^m\cdot v^\ell\cdot X_1(u,v)$ and $Y_q(u,v)=u^n\cdot v^s\cdot Y_1(u,v)$, where $X_1(0,v), Y_1(0,v), X_1(u,0), Y_1(u,0)\not\equiv 0$ and $m,n,\ell,s\in\mathbb{N}$. As before, we must have $X_1(0,0)\neq 0$, because $(x=0)$ is an analytic separatrix through p. Hence, after a holomorphic change of variables, we can suppose that $X(u,v)=u^m\cdot v^\ell$. If $r\gg 1$, then we get the formal power series

$$\hat{Y}_1 = \frac{1}{u^n\cdot v^s}\Big(Y - \sum_{j\geqslant r+1} a_j u^{jm}\cdot v^{j\ell}\Big) = Y_1 - \sum_{j\geqslant r+1} a_j u^{jm-n}\cdot v^{j\ell-s}\in\hat{\mathcal{O}}_q.$$

Note that $\hat{Y}_1(0,0)=0$. This implies that all irreducible components of $\hat{Y}_1$ in the ring $\hat{\mathcal{O}}_q$ are invariant for $\mathcal{G}$ (see the proof of part (I)). Since u and v do not divide $\hat{Y}_1$ in $\hat{\mathcal{O}}_q$, $\mathcal{G}$ has more than two separatrices through q: $(u=0)$, $(v=0)$ and the irreducible components of $\hat{Y}_1$. This finishes the proof of the third case of Lemma 4.1 and of Theorem 3. ∎

4.2 Proof of Corollary 2

If $\mathbb{BB}:\mathbb{F}\mathrm{ol}(d)\dashrightarrow\mathbb{P}^{d^2+d+1}$ is the global Baum–Bott map, then by Theorem 1 it follows that the closure of its image is a hypersurface H. Clearly this hypersurface is defined over $\mathbb{Q}$. This is sufficient to assure that there exists a generic set $U\subset H$, such that the field generated by the quotients of the coordinates of $p=[p_0:\dots:p_{d^2+d+2}]$ has transcendence degree $d^2+d=\dim H$ for every $p\in U$.

Since the Camacho–Sad index and the Baum–Bott index of a simple singularity are algebraically dependent, if we take $G(d)=\mathbb{BB}^{-1}(U)\cap\mathbb{R}(d)$, then, for every $\mathcal{F}\in G(d)$, the transcendence degree of $\mathbb{K}(\mathcal{F})=d^2+d$. Moreover, since U is dense in the image of $\mathbb{BB}$ we have that $G(d)$ is also generic. ∎

4.3 A basic property of the Camacho–Sad field and the Proof of Corollary 3

We will derive Corollary 3 from corollary 2 and the basic property of the Camacho–Sad field described in the next proposition. Here we will use the terminology and notation of [Bru04, ch. 1].

Proposition 4.1. *Let $\mathcal{F}$ be foliation of compact surface S with isolated singularities and cotangent bundle isomorphic to $\mathcal{L}$. The transcendence degree of $\mathbb{K}(\mathcal{F})$ over $\mathbb{Q}$ is at most $c_2(TS\otimes\mathcal{L})-1$.*

Proof. If all of the singularities are simple, i.e. they all have multiplicity one, then the result is an immediate consequence of Baum–Bott's formula.

Suppose now that there is a singularity p of $\mathcal{F}$ with multiplicity $\mu(p)\geqslant 2$. We have three possibilities:

(1) p is a saddle-node;

(2) p is a singularity without linear part;

(3) p is a nilpotent singularity.

In case (1) we have already seen that the transcendence degree of $\mathbb{K}(\mathcal{F},p)$ is at most 1.

In case (2) we can apply the Van den Essen formula (cf. [Bru04, p. 13]) to see that after blowing up the sum of the Milnor numbers over the singularities on the exceptional divisor is strictly less than $\mu(p)$.

In case (3) the argument is more involved. After blowing-up a nilpotent singularity only one singularity q appears at the exceptional divisor. We have two possibilities.

(3.1) q is a singularity without a linear part: after blowing up q, two or three singularities appear at the exceptional divisor. The important fact is that the sum of its Milnor numbers is equal to $\mu(p)$. Thus, here without further ado we have that the transcendence degree of $\mathbb{K}(\mathcal{F}, p)$ is at most $\mu(p)$.

(3.2) q is (again) a nilpotent singularity: blowing up q we obtain a singularity without a linear part and after blowing up again we obtain three singularities with non-nilpotent linear part. It follows from Camacho–Sad index theorem that in this case $\mathbb{K}(\mathcal{F}, p) = \mathbb{Q}$.

An induction argument shows that the transcendence degree of $\mathbb{K}(\mathcal{F})$ is at most the sum of Milnor numbers of singularities of $\mathcal{F}$ which is equal to $c_2(TS \otimes \mathcal{L})$.

To conclude we analyze the two cases independently. In the first case, saddle-nodes do not appear in $\tilde{\mathcal{F}}$, the resolution of $\mathcal{F}$. So, at the end all of the singularities of $\tilde{\mathcal{F}}$ are simple and from (1) and Baum–Bott's formula we have that the transcendence degree of $\mathbb{K}(\mathcal{F})$ is at most $c_2(TS \otimes \mathcal{L}) - 1$. In the second case, at least one saddle-node appears at the resolution. Since they have multiplicity at least 2 and contribute to the transcendence degree with at most 1, the result also follows in this case. $\square$

Proof of Corollary 3. Corollary 3 follows from Theorem 3 and Corollary 2 combined with the proposition above. $\square$

5. An example

As already noted in the introduction the dimension of the generic fiber is given by $\dim \mathbb{F}\mathrm{ol}(d) - (d^2 + d) = 3d + 2$. It would be interesting to *classify* the exceptional fibers of the Baum–Bott map, i.e. fibers with dimension at least $3d + 3$.

Example 5.1. Let $\mathcal{F}_0$ be a foliation on $\mathbb{P}^2$ with a meromorphic first integral of the type F/L^{d+1}, where F and L are homogeneous, F of degree $d+1$ and L of degree one. In an affine coordinate system $\mathbb{C}^2$ where L is the line at infinity, the foliation is defined by $dF = 0$ and, so, it is of degree d. If F is generic, then $\mathcal{F}_0$ has d^2 simple singularities on $\mathbb{C}^2$, all of them with Baum–Bott index zero, and $d+1$ singularities at the line L, all of them with Baum–Bott index $(d+2)^2/(d+1)$. In fact, we will see in the next result that the fiber of BB containing $\mathcal{F}_0$ has dimension greater than $3d + 2$.

Proposition 5.1. *Let $\mathcal{F}$ be a degree d foliation of $\mathbb{P}^2$ with at least d^2 simple singularities with Baum–Bott index zero. Then $\mathcal{F}$ is a pencil generated by C and $(d+1)L$, where C has degree $d+1$ and L is a line. In particular, the fiber of the Baum–Bott map containing $\mathcal{F}$ has dimension*

$$\binom{d+3}{2} + 2.$$

Proof. We will start by proving that $\mathcal{F}$ has an invariant line. Consider an affine coordinate system $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2$, such that all singularities of $\mathcal{F}$ are contained in $\mathbb{C}^2$. In particular, the line at infinity is not invariant for $\mathcal{F}$. Recall that $\mathcal{F}$ is induced by a vector field X of the form

$$X = (a + xg)\partial_x + (b + yg)\partial_y,$$

where a, b are polynomials with $\deg(a), \deg(b) \leqslant d$ and g is a non-identically zero degree d homogeneous polynomial.

Let I be the ideal generated by $a + xg$ and $\mathrm{div}(X)$, where

$$\mathrm{div}(X) = \frac{\partial}{\partial x}(a + xg) + \frac{\partial}{\partial y}(b + yg) = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + (d+2)g.$$

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Note that, for any singularity p of $\mathcal{F}$ with Baum–Bott index zero, we have $\mathrm{div}(X)(p) = 0$. By Bezout's theorem we have that $V(I) = \{p \in \mathbb{P}^2 \mid f(p) = 0\ \forall f \in I\}$ has degree $\deg(\mathrm{div}(X))$ $\deg(a + xg) = d(d+1)$, i.e. $V(I)$ has $d^2 + d$ points (counted with multiplicity): d of these points are at infinity and they correspond to the intersection of the curve $\{g = 0\}$ (which is a union of lines) with the line at infinity; the other d^2 correspond to the singularities of X in $\mathbb{C}^2$ with vanishing trace, i.e. with Baum–Bott index zero. In particular, the closure of the curves $a + x.g = 0$ and $\mathrm{div}(X) = 0$ intersect transversely in $\mathbb{P}^2$.

Since $b+y\cdot g$ vanishes on all points of $V(I)$ it must belong to I. Keeping in mind that $\deg(b+y\cdot g) = \deg(a+x\cdot g) = \deg(\mathrm{div}(X))+1$ we can apply Noether's lemma to see that there exists $\ell_1, \ell_2 \in \mathbb{C}[x, y]$ such that $\deg(\ell_1) = \deg(\ell_2) = 1$ and

$$X(\ell_1) = \ell_2 \cdot \mathrm{div}(X).$$

Note that the left-hand side of the equation above vanishes at all singularities of X. This implies that all the singularities of $\mathcal{F}$ with Baum–Bott index distinct from zero have to be in ℓ_2. Comparing the homogeneous terms of degree $d + 1$ of the equation, one obtains that

$$g\biggl(\frac{\partial \ell_1}{\partial x}x + \frac{\partial \ell_1}{\partial y}y\biggr) = (d+2)g\biggl(\frac{\partial \ell_2}{\partial x}x + \frac{\partial \ell_2}{\partial y}y\biggr).$$

Thus, $\ell_1 - (d+2)\ell_2 \in \mathbb{C}$ and, consequently,

$$X(\ell_2) = \frac{1}{d+2} \cdot \mathrm{div}(X) \cdot \ell_2,$$

proving that ℓ_2 is invariant.

Let us choose an affine coordinate system where the line at infinity is invariant and

$$X = a\partial_x + b\partial_y,$$

with $\deg(a) = \deg(b) = d$. We claim that $\mathrm{div}(X) \equiv 0$. Let I be the ideal generated by $\mathrm{div}(X)$ and a. If $\mathrm{div}(X) \not\equiv 0$, then $\mathrm{div}(X)$ has degree at most $d - 1$ and $V(I)$ in this case has degree at most $d(d-1)$. Since $V(I)$ has to vanish at d^2 points we get $\mathrm{div}(X) \equiv 0$.

The condition $\mathrm{div}(X) = 0$ is equivalent to the closedness of the polynomial 1-form $\omega = b\,dx - a\,dy$. So $\omega = dF$ for some polynomial F of degree $d+1$, i.e. $\mathcal{F}$ is a pencil generated by F and L^{d+1}, where F has degree $d + 1$ and L is the line at infinity.

We conclude that the fiber of the Baum–Bott map that contains $\mathcal{F}$ can be parametrized as

$$(F, L) \in \mathcal{P}_{d+1} \times \mathcal{P}_1 \mapsto \mathcal{F}(F/L^{d+1}),$$

where $\mathcal{P}_j$ denotes the set of homogeneous polynomials on $\mathbb{C}^3$ of degree j and $\mathcal{F}(G)$ the foliation with first integral G. Note that $\mathcal{F}(F/L^{d+1})$ is defined in homogeneous coordinates by the 1-form

$$\omega(F, L) = L \cdot dF - (d+1) \cdot F \cdot dL.$$

On the other hand, the reader can check that $\omega(F, L) = \omega(F_1, L_1)$ if and only if $(F_1, L_1) = \lambda \cdot (F, L)$, where $\lambda \in \mathbb{C}^*$. This implies that the dimension of the fiber of the Baum–Bott map that contains $\mathcal{F}$ has dimension $\dim(\mathbb{P}(\mathcal{P}_{d+1} \times \mathcal{P}_1)) = \binom{d+3}{2} + 2$. □

6. Some remarks and problems

6.1 The image of the Baum–Bott map

If F and L are generic, then the singularities of $\mathcal{F}(F/L^{d+1})$ are all simple. Moreover, there are two kinds of singularities: the d^2 singularities with Baum–Bott index zero and the $d + 1$ singularities in the line L, all of them with Baum–Bott index $(d+2)^2/(d+1)$. In particular, we see that $\mathbb{BB}(\mathbb{R}(d))$

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is not the whole hyperplane given by the Baum–Bott theorem. In fact, any point of the form $(0,\ldots,0,\lambda_1,\ldots,\lambda_{d+1})$, where $\sum_j \lambda_j = (d+2)^2$ and $\lambda_1 \neq (d+2)^2/(d+1)$ is not in $\mathrm{BB}(\mathbb{R}(d))$.

It would be interesting to describe $\mathrm{BB}(\mathbb{R}(d))$ or, more specifically, give a criterion to decide whether or not a point $[b_1,\ldots,b_N]$ belongs to $\mathrm{BB}(\mathbb{R}(d))$.

6.2 Affine versions of Theorem 1

Let $L \subset \mathbb{P}^2$ be a line and $\mathbb{F}\mathrm{ol}_L(d)$ be the space of foliations of degree d which leave L invariant. If $\mathcal{F} \in \mathbb{F}\mathrm{ol}_L(d)$ has only simple singularities, it is known (cf. [Bru04]) that L contains $(d+1)$ singularities and that

$$\sum_{p\in\mathrm{sing}(\mathcal{F})\cap L} \mathrm{CS}(\mathcal{F},L,p) = C\cdot C = 1.$$

This implies, in particular, that the maximal rank of $\mathrm{BB}|_{\mathbb{F}\mathrm{ol}_L(d)}$ is less than d^2+d. When $d \geqslant 2$, is the maximal rank of $\mathrm{BB}|_{\mathbb{F}\mathrm{ol}_L(d)}$ equal to d^2+d-1? If C is a smooth curve, what can be said about the generic rank of $\mathrm{BB}|_{\mathbb{F}\mathrm{ol}_C(d)}$ for $d \gg 0$? We believe that our strategy of proof should work on these situations.

6.3 The fibers of the Baum–Bott map

Recall that the dimension of the generic fiber of the global Baum–Bott for degree d foliations of $\mathbb{P}^2$ is $3d+2$. How many irreducible components does it have and what is its degree as an algebraic subset of $\mathbb{F}\mathrm{ol}(d)$?

6.4 Other surfaces

For an arbitrary compact complex surface S and an arbitrary non-negative integer k we have that the number of singularities (counted with multiplicities) of a foliation in $\mathbb{F}\mathrm{ol}(S,\mathcal{L})$ with isolated singularities is given by

$$c_2(TS\otimes\mathcal{L}^{\otimes k}) = k^2\cdot c_1(\mathcal{L})^2 + k\cdot c_1(\mathcal{L})\cdot c_1(S) + c_2(S).$$

On the other hand if $\mathcal{L}$ is an ample line bundle and $k \gg 0$, then, combining the Hirzebruch–Riemann–Roch theorem with Serre's vanishing theorem (see [BHPV04]), we have that dim $\mathbb{F}\mathrm{ol}(S,\mathcal{L}^{\otimes k}) = h^0(TS\otimes\mathcal{L}^{\otimes k}) - 1$ is equal to

$$\tfrac{1}{2}(c_1^2(TS\otimes\mathcal{L}^{\otimes k}) - 2c_2(TS\otimes\mathcal{L}^{\otimes k})) + \tfrac{1}{2}c_1(TS\otimes\mathcal{L}^{\otimes k})\cdot c_1(S) + 2\chi(S) - 1.$$

Straightforward manipulations shows that the dimension $\mathbb{F}\mathrm{ol}(S,\mathcal{L}^{\otimes k})$ is

$$k^2c_1(\mathcal{L})^2 + 2kc_1(\mathcal{L})\cdot c_1(S) + c_1^2(S) - c_2(S) + 2\chi(S) - c_2(S) - 1.$$

Thus, we have that $\dim\mathbb{F}\mathrm{ol}(S,\mathcal{L}^{\otimes k}) - c_2(TS\otimes\mathcal{L}^{\otimes k})$ is equal to

$$kc_1(\mathcal{L})\cdot c_1(S) + (c_1^2(S) - c_2(S) + 2\chi(S) - 1).$$

If $c_1(\mathcal{L})\cdot c_1(S) < 0$ (this happens, for example, when S is of general type), then

$$\dim\mathbb{F}\mathrm{ol}(S,\mathcal{L}^{\otimes k}) - c_2(TS\otimes\mathcal{L}^{\otimes k}) < 0,$$

for $k \gg 0$, i.e. we have more singularities than foliations. In particular, we have other relations between the Baum–Bott indexes besides the Baum–Bott formula. It would be really interesting to understand the nature of these relations. For instance, one could ask how they change when S and $\mathcal{L}$ are deformed. Another natural problem, motivated by the calculations above, is to know whether the Baum–Bott map for foliations of surfaces of general type with very ample cotangent bundle is generically finite.

6.5 Endomorphisms and foliations on $\mathbb{P}^n$

In [Gui04], Baum–Bott-like formulas are worked out for endomorphisms of projective spaces. There, by dimension counting, the existence of extra unknown relations among such multipliers is shown. An analogous phenomena also happens with one-dimensional foliations of $\mathbb{P}^n$, $n \geqslant 3$. Are these extra relations produced by some *index* formula? We refer to [Gui04] for a more complete discussion.

Appendix A. On the monodromy of the singular set

Let M be a projective manifold of dimension m, Θ_M be the tangent sheaf of M and $\mathcal{L}$ be a line bundle over M. The space of foliations by curves on M with cotangent bundle isomorphic to $\mathcal{L}$, denoted by $\mathrm{Fol}(M,\mathcal{L}) = \mathrm{Fol}(\mathcal{L})$, can be identified with the projectivization of the global sections of the bundle $\Theta_M \otimes \mathcal{L}$, i.e.

$$\mathrm{Fol}(\mathcal{L}) = \mathbb{P}\mathrm{H}^0(M, \Theta_M \otimes \mathcal{L}).$$

Over the product of $\mathrm{Fol}(\mathcal{L})$ with M we consider the natural foliation $\mathcal{F}(\mathcal{L})$ characterized by the property that the restriction of $\mathcal{F}(\mathcal{L})$ to the fiber over $\mathcal{F}$ under the natural projection $\pi : \mathrm{Fol}(\mathcal{L}) \times M \to \mathrm{Fol}(\mathcal{L})$ coincides with $\mathcal{F}$, i.e.

$$\mathcal{F}(\mathcal{L})_{|\pi^{-1}(\mathcal{F})} = \mathcal{F}.$$

We denote by $\mathcal{S}(\mathcal{L})$ the singular set of $\mathcal{F}(\mathcal{L})$.

Suppose that all of the irreducible components of $S(\mathcal{L})$ are of the same dimension and that $\pi = \pi_{|\mathcal{S}(\mathcal{L})} : \mathcal{S}(\mathcal{L}) \to \mathrm{Fol}(\mathcal{L})$ is generically finite. If we denote by $\Delta(\mathcal{L})$ the discriminant of the π, then for every foliation $\mathcal{F} \in \mathcal{F}(\mathcal{L}) \setminus \Delta(\mathcal{L})$ we can lift closed paths contained in $\mathcal{F}(\mathcal{L}) \setminus \Delta(\mathcal{L})$ and with base point $\mathcal{F}$ to $\mathcal{S}(\mathcal{L})$ inducing a representation

$$\Phi(\mathcal{F}) : \pi_1(\mathcal{F}(\mathcal{L}) \setminus \Delta(\mathcal{L}), \mathcal{F}) \to \mathrm{Perm}(\mathrm{sing}(\mathcal{F})).$$

Of course, if we choose another foliation $\mathcal{F}' \in \mathcal{F}(\mathcal{L}) \setminus \Delta(\mathcal{L})$ as a base point for the lifting of paths we obtain $\Phi(\mathcal{F}')$ which is conjugated to $\Phi(\mathcal{F})$. Therefore, we will say that the *monodromy of the singular set of* $\mathcal{F}(\mathcal{L})$ is a subgroup of the symmetric group on k elements, where k is the cardinality of $\mathrm{sing}(\mathcal{F})$, given by the image of $\Phi(\mathcal{F})$ up to conjugacy.

The aim of this appendix is to prove the following.

Theorem A1. *Let $\mathcal{L}$ be an ample line bundle over a projective manifold M of dimension m. For $k \gg 0$, the monodromy of the singular set of $\mathrm{Fol}(\mathcal{L}^{\otimes k})$ is the full symmetric group in $c_m(\Theta_M \otimes \mathcal{L})$ elements.*

We remark that the strategy of the proof is very similar to those presented in [Cuk99] and [Har79]. The careful reader will note that over $\mathbb{P}^n$ the result is valid for foliations of degree at least 2.

Proof of Theorem A1. Let $S \subset M \times \mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})$ be the singular set, i.e.

$$S = \{(p, \mathcal{F}) \mid p \in \mathrm{sing}(\mathcal{F})\}.$$

The set S can also be described as the projectivization of the kernel of the map of vector bundles

$$\begin{aligned} M \times \mathrm{H}^0(M, \Theta_M \otimes \mathcal{L}^{\otimes k}) &\to TM \otimes \mathcal{L}^{\otimes k} \\ (p, X) &\mapsto X(p). \end{aligned}$$

Since $k \gg 0$ and $\mathcal{L}$ is ample it follows from Serre's vanishing theorem that $\Theta_M \otimes \mathcal{L}^{\otimes k}$ is generated by global sections. In particular, the above map has constant rank and its kernel is a sub-bundle of $M \times \mathrm{H}^0(M, \Theta_M \otimes \mathcal{L}^{\otimes k})$ of codimension equal to $\dim M$. It follows that $S \subset M \times \mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})$ is a smooth irreducible subvariety and that the projection $\pi : S \to \mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})$ is surjective and generically finite. The irreducibility of S implies that the monodromy of π is 1-transitive.

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First step: the monodromy group is 2*-transitive.* Let p be an arbitrary point in M and let $\mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})_p \subset \mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})$ be the set of foliations having p as a singularity. If

$$S_p = \{(q, \mathcal{F}) \in M \setminus \{p\} \times \mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})_p \mid q \in \mathrm{sing}(\mathcal{F})\},$$

then, as before, S_p is the projectivization of the kernel of Φ,

$$\begin{aligned} \Phi : U \times V &\to TU \otimes \mathcal{L}^{\otimes k} \\ (z, X) &\mapsto X(z) \end{aligned}$$

where $U = M \setminus \{p\}$, $V = \mathrm{H}^0(M, \Theta_{M,p} \otimes \mathcal{L}^{\otimes k})$ and $\Theta_{M,p}$ is the subsheaf of Θ_M generated by vector fields vanishing at p. Clearly $\Theta_{M,p}$ is a coherent sheaf and, hence, we can again apply Serre's vanishing theorem to assure that S_p is a smooth irreducible subvariety of $M \setminus \{p\} \times \mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})_p$ and that $\pi_p : S_p \to F\mathrm{ol}(\mathcal{L}^{\otimes k})_p$ is surjective and generically finite. As before the monodromy of π_p is, thus, transitive.

Let G be the monodromy group of π and (p_1, q_1) and (p_2, q_2) be two pairs of the points in $M \times M$. Then, from the 1-transivity of G, there exists $\alpha \in G$ such that $\alpha(p_1) = p_2$. From the discussion above on the monodromy of π_p it follows that there exists $\beta \in G$ such that $\beta(p_2) = p_2$ and $\beta(q_1) = q_2$.

We have just proved that G, the monodromy group of π, is 2-transitive.

Second step: the monodromy group contains a transposition. First consider the local situation. Let X and Y be germs of holomorphic vector fields on a neighborhood of $0 \in \mathbb{C}^2$. Suppose that 0 is a singularity of multiplicity 2 of X and that $Y(0) \neq 0$. Consider the equation

$$(X + tY)(s(t)) = 0$$

with boundary value $s(0) = 0$ where $s \in \mathbb{C}[[t]]$ is a formal power series. Deriving with respect to t, we obtain that

$$DX(s(0)) \cdot s'(0) + Y(0) = 0.$$

When $Y(0)$ is not contained in the image of $DX(0)$ then the above equation has no solutions and, in particular, the local monodromy is generated by the transposition. As an example of this situation, one can take $X = x(\partial/\partial x) + y^2(\partial/\partial y)$ and $Y = \partial/\partial y$, where

$$\mathrm{sing}(X + tY) = (0, \pm\sqrt{-t}).$$

Returning to the global situation, suppose first that there exists $\mathcal{F} \in \mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})$ with one singularity with the 2-jet equal to the 2-jet of X and all other singularities with multiplicity one. Since $\Theta_M \otimes \mathcal{L}^{\otimes k}$ is generated by global sections, there exists $Y \in \mathrm{H}^0(M, \Theta_M \otimes \mathcal{L}^{\otimes k})$ such that $Y(p)$ is not in the image of $DX(p)$. The local discussion above shows that G, the monodromy group of π, contains a transposition.

Let p be a point of M and m_p its ideal sheaf. If we consider the inclusion of $\Theta_M \otimes m_p^3$ into Θ_M, then we will define $J_p^2\Theta_M$ as the cokernel of this inclusion. More succinctly, the sequence

$$0 \to \Theta_M \otimes m_p^3 \to \Theta_M \to J_p^2\Theta_M \to 0$$

is exact. It is clear from the definition that $J_p^2\Theta_M$ is supported on p and its sections are 2-jets of vector fields at p. Again from Serre's vanishing Theorem, when $k \gg 0$, $\mathrm{H}^1(M, \Theta_M \otimes m_p^3 \otimes \mathcal{L}^{\otimes k}) = 0$ and, consequently, the map

$$\mathrm{H}^0(M, \Theta_M \otimes \mathcal{L}^{\otimes k}) \to \mathrm{H}^0(M, J_p^2\Theta_M)$$

is surjective. Thus, there are foliations in $\mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})_p$ with arbitrary 2-jet. One can use the arguments applied in the first step to assure that there exists $\mathcal{F} \in \mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})$ with one singularity with the 2-jet equal at p to the 2-jet of X p and all other singularities with multiplicity one.

Conclusion

To conclude, the argument is well known. Let (p_1, q_1) and (p_2, q_2) be pairs of singularities in $\mathrm{sing}(\mathcal{F})$. Suppose that G contains the transposition $\tau = (p_1\ q_1)$. Since G is 2-transitive, there exists $\alpha \in G$ such that $\alpha(p_1) = p_2$ and $\alpha(q_1) = q_2$. Since $\alpha\tau\alpha^{-1} = (p_2\ q_2)$, we conclude that G contains all of the transpositions in the full symmetric group. This is sufficient to prove Theorem A1. □

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