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Remarks on a formula of Ramanujan

Andrés Chirre
Departamento de Ciencias - Sección Matemáticas, Pontificia Universidad Católica del Perú, Lima, Peru (cchirre@pucp.edu.pe)

Steven M. Gonek
Department of Mathematics, University of Rochester, Rochester, NY 14627, USA (gonek@math.rochester.edu)

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Assuming an averaged form of Mertens' conjecture and that the ordinates of the non-trivial zeros of the Riemann zeta function are linearly independent over the rationals, we analyse the finer structure of the terms in a well-known formula of Ramanujan.

Keywords: Riemann zeta function; nontrivial zeros; Riemann hypothesis; Möbius μ function

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1. The formula of Ramanujan

Let $\mu(n)$ be the Möbius function and set

$$F(b) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \, \mathrm{e}^{-(b/n)^2}.$$

In their paper 'Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes' [4], Hardy and Littlewood derived the formula

$$\sqrt{a}F(a) - \sqrt{b}F(b) = -\tfrac{1}{2} \sum_{\rho} \frac{\Gamma(\frac{1}{2} - \frac{\rho}{2})}{\zeta'(\rho)} b^{\rho - \frac{1}{2}}, \tag{1.1}$$

where $a, b > 0$ and $ab = \pi$. Here the sum runs over the nontrivial zeros $\rho = \beta + i\gamma$ of the zeta function and we have assumed they are all simple (the sum can be modified accordingly if they are not). The formula was suggested to them by some work of Ramanujan. Hardy and Littlewood mentioned that there is a way to bracket the terms in the sum over zeros to ensure convergence, but they were not explicit about how to do this. Titchmarsh [13] (see pp. 219–220), however, proved that the series converges provided any two zeros ρ_1, ρ_2 in the sum for which

$$|\gamma_1 - \gamma_2| \leqslant \mathrm{e}^{-A\gamma_1/\log\gamma_1} + \mathrm{e}^{-A\gamma_2/\log\gamma_2}, \tag{1.2}$$

with A a sufficiently small positive constant, are grouped together. In addition, Hardy and Littlewood proved that for any $\epsilon > 0$, the estimate

$$F(b) \ll_\epsilon b^{-\frac{1}{2}+\epsilon} \tag{1.3}$$

as $b \to \infty$ is equivalent to the Riemann hypothesis (RH), and they conjectured that, in fact, $F(b) \ll b^{-\frac{1}{2}}$.

Several mathematicians have studied various aspects and analogues of $F(b)$ and Ramanujan's formula. For instance, W. Staś [10–12] proved, under various hypotheses, results of the form

$$\max_{T^{1-o(1)} \leqslant b \leqslant T} |F(b)| \gg T^{-\frac{1}{2}-o(1)},$$

for T sufficiently large. A. Dixit [2, 3] proved analogues of (1.1) with Dirichlet characters and the insertion of other functions in the sums. Other results along similar lines may be found in [1, 5, 6, 9] to cite just a few examples.

Our purpose here is to record a few observations about the finer behaviour of $F(b)$ as well as the sum over zeros on the right-hand side of (1.1) under the assumption of two well-known and widely believed hypotheses. We will refer to our first hypothesis as the *weak Mertens hypothesis* (WMH).

Weak Mertens Hypothesis. Let $M(x) = \sum_{n \leqslant x} \mu(n)$. Then as $X \to \infty$,

$$\int_1^X \left(\frac{M(x)}{x}\right)^2 \mathrm{d}x \ll \log X. \tag{1.4}$$

We assume WMH throughout. It has the following consequences:

(A) RH,

(B) all the zeros ρ are simple,

(C) $\zeta'(\rho)^{-1} = o(|\rho|)$,

(D) there is a positive constant A such that if $\gamma < \gamma'$ are consecutive ordinates of nontrivial zeros of $\zeta(s)$, then

$$\gamma' - \gamma > \frac{A}{\gamma} \exp\left(-A \frac{\log\gamma}{\log\log\gamma}\right). \tag{1.5}$$

For proofs that WMH implies (B), (C), and (D), we refer the reader to Titchmarsh [13] (§ 14.29, 14.31). The proof that WMH implies RH is not in Titchmarsh, but

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it is short so we provide it here. Set

$$f(x) = \int_1^x \frac{M(u)}{u}\, \mathrm{d}u.$$

By the Cauchy–Schwarz inequality and (1.4),

$$f(x)^2 \leqslant x \int_1^x \left(\frac{M(u)}{u}\right)^2 \mathrm{d}u \ll x \log x.$$

Hence $f(x) \ll (x \log x)^{\frac{1}{2}}$. Thus, for $s = \sigma + it$ with $\sigma > 1$

$$\frac{1}{s\zeta(s)} = \int_1^\infty \frac{M(x)}{x^{s+1}}\, \mathrm{d}x = \int_1^\infty \frac{\mathrm{d}f(x)}{x^s}\, \mathrm{d}x = s \int_1^\infty \frac{f(x)}{x^{s+1}}\, \mathrm{d}x, \tag{1.6}$$

and it follows that the last integral in (1.6) is an analytic function for $\sigma > 1/2$. Thus, $\zeta(s)$ has no zeros in $\sigma > 1/2$. In other words, RH follows.

From (1.5) we see that there are no zeros with ordinates γ_1, γ_2 large such that (1.2) holds. Thus, assuming WMH, (1.1) holds with the sum interpreted as $\lim_{T_\nu \to \infty} \sum_{|\gamma| \leqslant T_\nu}$ for *any* increasing sequence $\{T_\nu\}$. However, on WMH even more is true – the series is in fact absolutely convergent. To see this, write

$$\sum_\rho \frac{\Gamma\left(\frac{1}{2} - \frac{\rho}{2}\right)}{\zeta'(\rho)} b^{\rho - \frac{1}{2}} = \sum_\gamma a(\gamma) b^{i\gamma}.$$

By Stirling's formula,

$$\log \Gamma(s) = (s - \tfrac{1}{2}) \log s - s + \tfrac{1}{2} \log 2\pi + O(|s|^{-1}),$$

where $|s| \to \infty$ in any angle $-\pi + \delta < \arg s < \pi - \delta$ with $\delta > 0$. Thus

$$\log \left|\Gamma\left(\tfrac{1}{2} - \tfrac{\rho}{2}\right)\right| = -\tfrac{\pi}{4}|\gamma| - \tfrac{1}{4} \log |\gamma| + O(1).$$

Using this and (C), we find that

$$a(\gamma) = o(|\gamma|^{3/4}\, \mathrm{e}^{-\pi|\gamma|/4}). \tag{1.7}$$

Hence, since $N(T) = \sum_{0 < \gamma \leqslant T} 1 \sim (T/2\pi) \log T$ and the zeros $\rho = \frac{1}{2} + i\gamma$ are symmetric about the real axis, we have

$$\sum_\gamma |a(\gamma) b^{i\gamma}| = O\left(\sum_\gamma |\gamma|^{3/4}\, \mathrm{e}^{-\pi|\gamma|/4}\right) \ll 1.$$

Returning to (1.1), we see that since the zeros $\rho = \frac{1}{2} + i\gamma$ are symmetric about the real axis and $\zeta'(s)$ and $\Gamma(s)$ are real on the real axis, we may rewrite (1.1) as

$$\sqrt{a} F(a) - \sqrt{b} F(b) = -\Re \sum_{\gamma > 0} a(\gamma) b^{i\gamma}.$$

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Then, since $ab = \pi$ with $a, b > 0$, we may replace a by π/b and write

$$F(b) = \frac{1}{\sqrt{b}} \Re \sum_{\gamma>0} a(\gamma) b^{i\gamma} + \frac{\sqrt{\pi}}{b} F\left(\frac{\pi}{b}\right), \tag{1.8}$$

where the sum over γ on the right-hand side is absolutely convergent under the assumption of WMH. Since $\sum_{n=1}^{\infty} \mu(n) n^{-1} = 0$, we have

$$\begin{aligned} F\left(\frac{\pi}{b}\right) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} (\mathrm{e}^{-(\pi/bn)^2} - 1) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{k=1}^{\infty} \frac{(-1)^k (\pi/bn)^{2k}}{k!} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k (\pi/b)^{2k}}{k!} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2k+1}} = \sum_{k=1}^{\infty} \frac{(-1)^k (\pi/b)^{2k}}{k! \zeta(2k+1)}, \end{aligned}$$

where the interchange of summations is justified by absolute convergence. For $b \geqslant \pi$ it is easily checked that the absolute value of the terms of this alternating series are decreasing, so for any integer $K \geqslant 1$, we see that

$$F\left(\frac{\pi}{b}\right) = \sum_{k=1}^{K} \frac{(-1)^k (\pi/b)^{2k}}{k! \zeta(2k+1)} + E_{K+1}(b),$$

where

$$|E_{K+1}(b)| \leqslant \frac{(\pi/b)^{2K+2}}{(K+1)!}.$$

Inserting this into (1.8), we now find that if WMH is true and $b \geqslant \pi$, then

$$F(b) = \frac{1}{\sqrt{b}} \Re \sum_{\gamma>0} a(\gamma) b^{i\gamma} + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{K} \frac{(-1)^k (\pi/b)^{2k+1}}{k! \zeta(2k+1)} + \frac{\sqrt{\pi}}{b} E_{K+1}(b). \tag{1.9}$$

We will use this for the calculations in § 3. However, even the cruder estimate

$$F(b) = \Re \frac{1}{\sqrt{b}} \sum_{\gamma>0} a(\gamma) b^{i\gamma} + O(b^{-3}) \tag{1.10}$$

immediately leads to the following theorem.

THEOREM 1.1. *Assume WMH. Then for $b \geqslant \pi$ we have*

$$|F(b)| \leqslant \frac{C}{\sqrt{b}} + O(b^{-3}),$$

where

$$C = \sum_{\gamma>0} |a(\gamma)| = \sum_{\gamma>0} \left| \frac{\Gamma\left(\frac{1}{4} - i\frac{\gamma}{2}\right)}{\zeta'(\frac{1}{2} + i\gamma)} \right|.$$

To analyse the sum over γ in (1.9) and (1.10), we assume, in addition to WMH, the following *linear independence hypothesis* (LI).

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Linear Independence Hypothesis. The positive ordinates γ of the zeros of the zeta function are linearly independent over the rationals.

To use this we first assume the $\gamma > 0$ have been ordered as $\gamma_1, \gamma_2, \gamma_3 \ldots$, in such a way that $|a(\gamma_1)| \geqslant |a(\gamma_2)| \geqslant |a(\gamma_3)| \geqslant \cdots$. Then

$$\sum_{\gamma>0} \frac{\Gamma(\frac{1}{4} - i\frac{\gamma}{2})}{\zeta'(\frac{1}{2} + i\gamma)} b^{i\gamma} = \sum_{\gamma>0} a(\gamma) b^{i\gamma} = \sum_{n=1}^{\infty} a(\gamma_n)\, \mathrm{e}^{i\gamma_n \log b}.$$

LI implies that as b varies over $[\pi, \infty)$, this sum is dense in the set of complex numbers

$$\mathscr{A} = \left\{ \sum_{n=1}^{\infty} |a(\gamma_n)| \mathrm{e}^{i\theta_n} \;:\; \theta_n \in [0,1), n = 1, 2, 3, \ldots \right\}.$$

This set, being a 'sum' of circles centred at the origin, is, as is well-known, either a closed annulus or a closed disk according to the following criteria:

(1) If $|a(\gamma_1)| > \sum_{n=2}^{\infty} |a(\gamma_n)|$, then $\mathscr{A}$ is a closed annulus centred at the origin with outer radius

$$C = \sum_{n=1}^{\infty} |a(\gamma_n)|$$

and inner radius

$$c = |a(\gamma_1)| - \sum_{n=2}^{\infty} |a(\gamma_n)|.$$

(2) If $|a(\gamma_1)| \leqslant \sum_{n=2}^{\infty} |a(\gamma_n)|$, then $\mathscr{A}$ is a closed disk centred at the origin of radius

$$C = \sum_{n=1}^{\infty} |a(\gamma_n)|.$$

In either of these two cases, the real parts of the complex numbers $\sum_{n=1}^{\infty} |a(\gamma_n)|\, \mathrm{e}^{i\theta_n}$ in $\mathscr{A}$ fill out the interval $[-C, C]$. As the sum $\sum_{n=1}^{\infty} a(\gamma_n)\, \mathrm{e}^{i\gamma_n \log b}$ is dense in $\mathscr{A}$ (assuming LI), this and (1.10) give the following result.

THEOREM 1.2. *Assume WMH and LI. Then $\sqrt{b}F(b)$ is dense in $[-C, C]$ and, in particular, we have*

$$\liminf_{b\to\infty} \sqrt{b}F(b) = -C \quad \text{and} \quad \limsup_{b\to\infty} \sqrt{b}F(b) = C.$$

For N a large positive integer, let

$$\mathscr{A}_N = \left\{ \sum_{n=1}^{N} |a(\gamma_n)|\, \mathrm{e}^{i\theta_n} \;:\; \theta_n \in [0,1), n = 1, 2, 3, \ldots, N \right\},$$

which again is either an annulus or disk centred at the origin. By the reasoning above, if one assumes LI, the curve $f_N(b) = \sum_{n=1}^{N} a(\gamma_n)\, \mathrm{e}^{i\gamma_n \log b}$ is dense in $\mathscr{A}_N$.

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By the Kronecker-Weyl theorem, it is also uniformly distributed in $\mathscr{A}_N$. Thus, the distribution function of the curve $\Re f_N(b)$ as $b \to \infty$ tends to the distribution function of the x coordinate of points (x, y) in the annulus or disk $\mathscr{A}_N$. Since $\sum_{n=1}^{\infty} a(\gamma_n)e^{i\gamma_n \log b}$ is absolutely convergent, the same is true for the real part of this series but with $\mathscr{A}$ in place of $\mathscr{A}_N$. Moreover, by (1.10),

$$\left| \sqrt{b}F(b) - \Re \sum_{n=1}^{\infty} a(\gamma_n)\,\mathrm{e}^{i\gamma_n \log b} \right| \ll b^{-\frac{5}{2}}.$$

Thus, as $b \to \infty$, the probability distribution function of $\sqrt{b}F(b)$ tends to the distribution function of the x coordinate of points (x, y) in either the annulus centred at the origin with inner radius c and outer radius C, or the disk centred at the origin of radius C. Depending on whether the set $\mathscr{A}$ is an annulus or a disk, we therefore have the following probability density function for $\sqrt{b}F(b)$.

THEOREM 1.3. *Assume WMH and LI. Let c and C be as above, let*

$$\mathscr{A} = \left\{ \sum_{n=1}^{\infty} |a(\gamma_n)|\,\mathrm{e}^{i\theta_n} \,:\, \theta_n \in [0,1), n = 1, 2, 3, \ldots \right\},$$

and let $p(x)$ be the probability density function of $\sqrt{b}F(b)$ for b large. If $\mathscr{A}$ is an annulus with inner radius c and outer radius C, then

$$p(x) = \begin{cases} 0 & \text{if } C \leqslant |x|, \\ \dfrac{2\sqrt{C^2 - x^2}}{\pi(C^2 - c^2)} & \text{if } c \leqslant |x| \leqslant C, \\ \dfrac{2(\sqrt{C^2 - x^2} - \sqrt{c^2 - x^2})}{\pi(C^2 - c^2)} & \text{if } |x| \leqslant c. \end{cases}$$

If $\mathscr{A}$ is a disk of radius C, then

$$p(x) = \begin{cases} 0 & \text{if } C \leqslant |x|, \\ \dfrac{2\sqrt{C^2 - x^2}}{\pi C^2} & \text{if } |x| \leqslant C. \end{cases}$$

It seems difficult to prove, even under the strong assumptions of WMH and LI, whether $\mathscr{A}$ is an annulus or disk, but we believe it to be an annulus. At issue is determining the relative size of the two quantities

$$|a(\gamma_1)| \quad \text{and} \quad \sum_{n=2}^{\infty} |a(\gamma_n)|,$$

where

$$a(\gamma) = \frac{\Gamma(\frac{1}{4} - i\frac{\gamma}{2})}{\zeta'(\frac{1}{2} + i\gamma)}.$$

There are two sources of difficulty in settling this question. One is that, although the size of Γ is well understood, the bound $\zeta'(\rho)^{-1} = o(|\rho|)$ from C) is not explicit

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enough; what would suffice is an estimate of the type $|\zeta'(\rho)^{-1}| \leqslant B|\rho|$ for all $\gamma > 0$ with B an explicit constant, or even $|\zeta'(\rho)^{-1}| \leqslant B|\gamma|^d$ with $d > 1$, and d and B both explicit. The other difficulty, which is related to the first, is that we do not know which γ should be γ_1, that is, which γ maximizes $|a(\gamma)|$. (Note that if $|a(\gamma)|$ is maximal for more than one γ, then $\mathscr{A}$ is a disk.) However, if a constant B as above exists that is not enormous, the fast exponential decay from the gamma function in $a(\gamma)$ suggests that the drop off between terms for successive γ's is large, and this suggests that $a(\gamma_1)$ (with $\gamma_1 = \gamma$) is much larger than $\sum_{n=2}^{\infty} |a(\gamma_n)|$. In § 3 we present the outcome of a limited number of calculations that suggest possible approximate values of c and C and we present several graphs of $\sqrt{b}F(b)$.

We next prove a formula for the second moment of F.

THEOREM 1.4. *Assume WMH. Then*

$$\int_1^X F(x)^2 \, \mathrm{d}x = A \log X + O(1) \tag{1.11}$$

as $X \to \infty$, where

$$A = \tfrac{1}{2} \sum_{\gamma > 0} |a(\gamma)|^2.$$

REMARK. *Note that $A > 0$.*

Proof. Writing

$$S = \sum_{\gamma > 0} a(\gamma) x^{i\gamma},$$

we find by (1.8) that

$$\begin{aligned}
\int_1^X F(x)^2 \, \mathrm{d}x &= \int_1^X \left(\frac{\Re S}{\sqrt{x}} + O(x^{-3}) \right)^2 \mathrm{d}x \\
&= \int_1^X \left(\Re S + O(x^{-5/2}) \right)^2 \frac{\mathrm{d}x}{x} \\
&= \int_1^X \left((\Re S)^2 + O(|S| x^{-5/2}) + O(x^{-5}) \right) \frac{\mathrm{d}x}{x}.
\end{aligned}$$

Since the series defining S is absolutely convergent, the last two terms of the integrand contribute $O(1)$. Thus,

$$\begin{aligned}
\int_1^X F(x)^2 \, \mathrm{d}x &= \int_1^X (\Re S)^2 \frac{\mathrm{d}x}{x} + O(1) \\
&= \frac{1}{4} \int_1^X (S^2 + 2|S|^2 + \overline{S}^2) \frac{\mathrm{d}x}{x} + O(1). \qquad (1.12)
\end{aligned}$$

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Again, by absolute convergence of the sum defining S, we have

$$\int_1^X S^2 \frac{\mathrm{d}x}{x} = \int_1^X \sum_{\gamma,\gamma'>0} a(\gamma)a(\gamma')x^{i(\gamma+\gamma')}\frac{\mathrm{d}x}{x} = \sum_{\gamma,\gamma'>0} a(\gamma)a(\gamma')\int_1^X x^{i(\gamma+\gamma')-1}\,\mathrm{d}x$$
$$= \sum_{\gamma,\gamma'>0} a(\gamma)a(\gamma')\frac{X^{i(\gamma+\gamma')}-1}{i(\gamma+\gamma')} \ll \left(\sum_{\gamma>0}|a(\gamma)|\right)^2 \ll 1.$$

Similarly, $\int_1^X \overline{S}^2 \frac{dx}{x} \ll 1$. Finally,

$$\int_1^X |S|^2\frac{\mathrm{d}x}{x} = \sum_{\gamma,\gamma'>0} a(\gamma)\overline{a(\gamma')}\int_1^X x^{i(\gamma-\gamma')-1}\,\mathrm{d}x$$
$$= \log X\sum_{\gamma>0}|a(\gamma)|^2 + \sum_{\substack{\gamma,\gamma'>0\\ \gamma\neq\gamma'}} a(\gamma)\overline{a(\gamma')}\frac{X^{i(\gamma+\gamma')}-1}{i(\gamma-\gamma')}$$
$$= \log X\sum_{\gamma>0}|a(\gamma)|^2 + O\left(\sum_{\substack{\gamma,\gamma'>0\\ \gamma\neq\gamma'}} |a(\gamma)\overline{a(\gamma')}|\min\left(\log X, \frac{1}{|\gamma-\gamma'|}\right)\right).$$

By (D), for any $\epsilon>0$ we have $|\gamma-\gamma'|^{-1}\ll\gamma^{1+\epsilon}$. Thus, by (1.7), we find that the O-term is

$$\ll \sum_{\gamma'>0}|a(\gamma')|\sum_{\gamma<\gamma'}|a(\gamma)|\gamma^{1+\epsilon} \ll \sum_{\gamma'>0}|a(\gamma')|\sum_{\gamma<\gamma'}\gamma^{7/4+\epsilon}\,\mathrm{e}^{-\pi|\gamma|/4}$$
$$\ll \sum_{\gamma'>0}|a(\gamma')| \ll 1.$$

Hence

$$\int_1^X |S|^2\frac{\mathrm{d}x}{x} = \log X\sum_{\gamma>0}|a(\gamma)|^2 + O(1).$$

Combining our estimates together in (1.12), we obtain

$$\int_1^X F(x)^2\,\mathrm{d}x = \log X\left(\tfrac{1}{2}\sum_{\gamma>0}|a(\gamma)|^2\right) + O(1).$$

□

Remark. *One can show that if a weak version of (1.11) holds, namely,*

$$\int_1^X F(x)^2 dx \ll \log X,$$

then (A) and (B) as well as the following analogue of (C) follow:

(C*) $\zeta'(\rho)^{-1} \ll \mathrm{e}^{c|\gamma|}$ *for some positive constant* c.

These can be proved along the lines of the proofs that (A)–(C) follow from WMH.

2. Riesz's function

Analogues of the results above may easily be extended to M. Riesz's function [8]

$$P(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}\, \mathrm{e}^{-x/n^2},$$

which is similar to $F(x)$ and was introduced around the same time as Hardy and Littlewood's work on Ramanujan's formula. Note that $P(x)$ has n^2 rather than n in the denominator and x rather than x^2 in the exponential. Agarwal, Garg, and Maji [1] recently generalized this to a one parameter family of functions

$$P_k(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^k}\, \mathrm{e}^{-x/n^2},$$

where $k \geqslant 1$ is a fixed real number. Note that $F(x) = P_1(x^2)$ and $P(x) = P_2(x)$. They then proved the following analogue of (1.1) (see their Theorem 1.1):

$$P_k(x) = \Gamma(\tfrac{k}{2})x^{-\frac{k}{2}} \sum_{n=1}^{\infty} \frac{\mu(n)}{n}\ {}_1F_1\left(\tfrac{k}{2}; \tfrac{1}{2}; -\tfrac{\pi^2}{n^2 x}\right) + \tfrac{1}{2}\sum_{\rho} a_k(\gamma) x^{-\frac{k-\rho}{2}}. \tag{2.1}$$

Here ${}_1F_1\left(\frac{k}{2}; \frac{1}{2}; z\right)$ is the generalized hypergeometric series,

$$a_k(\rho) = \frac{\Gamma\left(\frac{k-\rho}{2}\right)}{\zeta'(\rho)},$$

the zeros ρ are all assumed to be simple, and any two zeros ρ_1 and ρ_2 in the series on the right in (2.1) are grouped together if they satisfy the inequality (1.2). They used this to show that for any fixed real number $k \geqslant 1$ and any $\epsilon > 0$, the Riemann hypothesis is equivalent to

$$P_k(x) \ll_\epsilon x^{-\frac{k}{2}+\frac{1}{4}+\epsilon}$$

as $x \to \infty$ (similarly to (1.3)).

Assuming WMH and using (2.1), we may easily prove a version of (1.10) for $P_k(x)$. First note, as before, that from WMH it follows that RH holds, all the zeros ρ of $\zeta(s)$ are simple, and $|\zeta(\rho)^{-1}| = o(|\rho|)$. Also, by Stirling's formula, we have

$$\log\left|\Gamma\left(\tfrac{k}{2} - \tfrac{\rho}{2}\right)\right| = -\tfrac{\pi}{4}|\gamma| + (\tfrac{k}{2} - \tfrac{3}{4})\log|\gamma| + O(1).$$

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Thus,

$$\sum_{\rho} |a_k(\rho)| \ll \sum_{\gamma} |\gamma|^{\frac{k}{2}+\frac{1}{4}} \,\mathrm{e}^{-\pi|\gamma|/4} \ll 1.$$

Hence, the series

$$\tfrac{1}{2}\sum_{\rho} a_k(\rho) x^{-\frac{k-\rho}{2}} = \tfrac{1}{2} x^{-\frac{k}{2}+\frac{1}{4}} \sum_{\rho} a_k(\rho) x^{i\gamma/2} = x^{-\frac{k}{2}+\frac{1}{4}} \Re \sum_{\gamma>0} a_k(\rho) x^{i\gamma/2}$$

on the right-hand side of (2.1) converges absolutely. Next, for z complex and bounded, we have

$${}_1F_1(\tfrac{k}{2}, \tfrac{1}{2}, z) = \sum_{j=0}^{\infty} \frac{\Gamma(\frac{k}{2}+j)\Gamma(\frac{1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{1}{2}+j)j!} z^j = 1 + O(|z|).$$

Thus, the first term on the right-hand side of (2.1) equals

$$\Gamma(\tfrac{k}{2}) x^{-\frac{k}{2}} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} (1 + O(n^{-2}x^{-1})) \ll x^{-\frac{k}{2}-1},$$

since $\sum_n \mu(n) n^{-1} = 0$.

Using these estimates and observations with (2.1), we arrive at

$$P_k(x) = x^{-\frac{k}{2}+\frac{1}{4}} \; \Re \sum_{\gamma>0} a_k(\rho) x^{i\gamma/2} + O(x^{-\frac{k}{2}-1}).$$

With this formula as a starting point, we may easily prove analogues of Theorems 1.1–1.4 for $P_k(x)$. In the case of Theorem 1.4, we obtain an asymptotic formula for

$$\int_1^X P_k(x)^2 x^{k-\frac{3}{2}} \,\mathrm{d}x.$$

3. Calculations

We mentioned in § 1 that we believe $\mathscr{A}$ to be an annulus. In this final section we briefly report the results of calculations of a number of $|a(\gamma)|$'s, and use these to approximate the values of the inner and out radii, c and C, of the annulus $\mathscr{A}$. We also provide several graphs of $\sqrt{b}F(b)$. We have used Mathematica for these calculations and to generate our graphs.

For a table of values of $|a(\gamma)|$ for the first ten ordinates $\gamma > 0$, see Table 1.

Notice that, for the most part, these terms are quickly decreasing. If we sum them to approximate C, the outer radius of $\mathscr{A}$, we obtain the value $C \approx 0.0000293414$. To approximate c we subtract the sum of the last nine values from $|a(\gamma_1)|$ and obtain $c \approx 0.0000291702$. Interestingly, performing the same calculations with the first 500 ordinates γ gives exactly the same values for C and c up to ten significant figures. This suggests (but, of course, does not prove) that $\mathscr{A}$ really is an annulus rather than a disk.

We conclude with several graphs of $\sqrt{b}F(b)$ for various ranges of b from the formula (1.9) using the first 50 ordinates γ and the sum over k with $K = 50$ and

Table 1. *Some values of* $|a(\gamma)|$.

γ_n	$\lvert a(\gamma_n)\rvert$
γ_1	$2.9255\ldots \cdot 10^{-5}$
γ_2	$8.2702\ldots \cdot 10^{-8}$
γ_3	$2.8609\ldots \cdot 10^{-9}$
γ_4	$4.0789\ldots \cdot 10^{-11}$
γ_5	$5.2534\ldots \cdot 10^{-12}$
γ_6	$9.4006\ldots \cdot 10^{-14}$
γ_7	$8.7272\ldots \cdot 10^{-15}$
γ_8	$1.0550\ldots \cdot 10^{-15}$
γ_9	$3.0507\ldots \cdot 10^{-17}$
γ_{10}	$8.3287\ldots \cdot 10^{-18}$

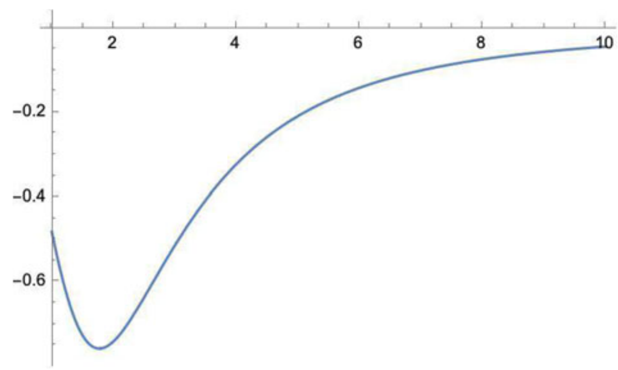

Figure 1. Graph of $\sqrt{b}F(b)$ for $1 \leqslant b \leqslant 10$.

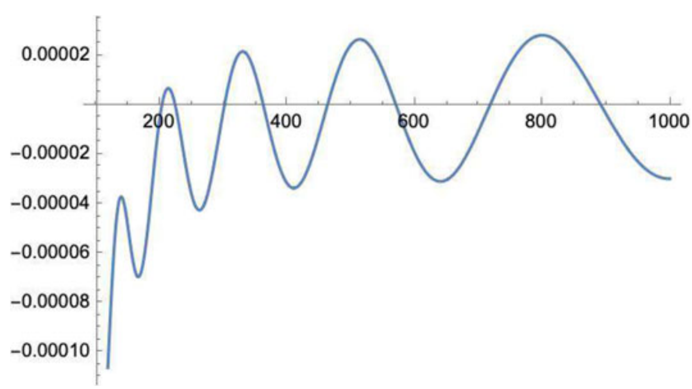

Figure 2. Graph of $\sqrt{b}F(b)$ for $100 \leqslant b \leqslant 1000$.

ignoring the error term $E_{51}(b)$. Although our estimate for E_{K+1} in (1.9) was for $b \geqslant \pi$, it is not difficult to check that $E_{51}(b)$ is quite small even when $1 \leqslant b \leqslant \pi$. Thus, figure 1 is accurate for this range of b as well. For other ranges, see figures 2 and 3.

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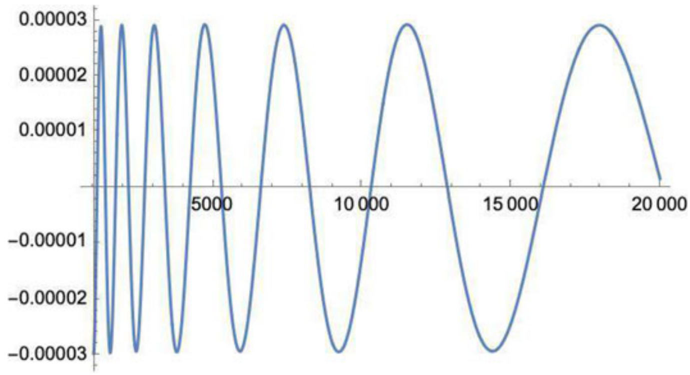

Figure 3. Graph of $\sqrt{b}F(b)$ for $1000 \leqslant b \leqslant 20,000$.

For some related graphs see Paris [7].

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