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A BETTER COMPARISON OF cdh- AND $l$dh-COHOMOLOGIES

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Warmly dedicated to Shuji Saito on his 60th birthday

Abstract. In order to work with non-Nagata rings which are Nagata "up-to-completely-decomposed-universal-homeomorphism," specifically finite rank Hensel valuation rings, we introduce the notions of *pseudo-integral closure*, *pseudo-normalization*, and *pseudo-Hensel valuation ring*. We use this notion to give a shorter and more direct proof that $H^n_{\mathrm{cdh}}(X, F_{\mathrm{cdh}}) = H^n_{l\mathrm{dh}}(X, F_{l\mathrm{dh}})$ for homotopy sheaves F of modules over the $\mathbb{Z}_{(l)}$-linear motivic Eilenberg–Maclane spectrum. This comparison is an alternative to the first half of the author's volume Astérisque 391 whose main theorem is a cdh-descent result for Voevodsky motives. The motivating new insight is really accepting that Voevodsky's motivic cohomology (with $\mathbb{Z}[\frac{1}{p}]$-coefficients) is invariant not just for nilpotent thickenings, but for all universal homeomorphisms.

§1. Introduction

Context—motives of singular schemes (with compact support). In [Voe00], Voevodsky constructed a triangulated category of motives $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ using smooth schemes. In order to

- extend the motive functor $M : \mathsf{Sm}_k \to DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ from smooth k-schemes to all separated finite type k-schemes, and
- have access to a well-defined theory of motives with compact support $M^c : \mathsf{Sch}^{\mathrm{prop}}_k \to DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$,

he proves a cdh-descent result. However, his proof only works in the presence of strong resolution of singularities (for example, in characteristic zero). In [Kel17], the resolution of singularities assumption was removed, at least if one works with $\mathbb{Z}[\frac{1}{p}]$-coefficients, where p is the exponential characteristic of the base field. The proof in [Kel17] has two main steps, namely [Kel17, Corollary 2.5.4] and [Kel17, Theorem 3.2.12]. This present article provides

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a shortcut to the first one. For more details about the strategy of [Kel17] we recommend consulting [Kel17, Chapter 1] and/or [Kel17, Chapter 4].

Main result. The main theorem of this article is:

THEOREM 1. (cf. Theorem 30) *Suppose S is a finite dimensional Noetherian separated scheme of positive characteristic $p \neq l$, Sch_S the category of separated finite type S-schemes, and F a presheaf of $\mathbb{Z}_{(l)}$-modules on Sch_S satisfying all of:*

(uh-invariance) *$F(X) \cong F(Y)$ for any universal homeomorphism $Y \to X$.*
(Traces) *F has covariant "trace" morphisms associated to finite flat surjective morphisms, see Definition 13.*
(G1) *$F(R) \subseteq F(\mathrm{Frac}(R))$ for every finite rank Hensel valuation ring[1] R.*
(G2) *$F(R) \to F(R/\mathfrak{p})$ is surjective for every finite rank Hensel valuation ring R and prime $\mathfrak{p} \subset R$.*

Then the canonical comparison morphism is an isomorphism:

$$H^n_{\mathrm{cdh}}(S, F_{\mathrm{cdh}}) \xrightarrow{\sim} H^n_{l\mathrm{dh}}(S, F_{l\mathrm{dh}}).$$

However, the main *result* of this article is its proof. The proof of [Kel17, Corollary 2.5.4] is a poorly structured collection of lemmas, which are difficult to arrange into some kind global narrative, and the hypotheses of [Kel17, Corollary 2.5.4] are an awkward list of very special properties that do not give much insight into why the cohomologies should agree.

The proof of Theorem 30 on the other hand, is short[2] and linear, and one can mostly explain how its hypotheses are used in the proof: $\mathbb{Z}_{(l)}$-linearity and traces are to give descent for finite-flat-surjective-prime-to-l morphisms (cf. Lemma 2), universal homeomorphism invariance is to correct non-Nagataness of Hensel valuation rings (cf. Lemma 3), and (G2) is to control $\underline{H}^1_{l\mathrm{dh}}F$ (cf. Equation (2) on page 204). Unfortunately, (G1) remains a little

[1] We implicitly extend F to all quasi-compact separated S-schemes using left Kan extension. See also Conventions on page 188.

[2] Even though the proof "finishes" on page 205, of course this introductory section does not form part of the proof, most if not all of Sections 2 and 4 is background scheme theory included for the convenience of the reader, and most of Section 5 is routine checking that pseudo-integral closures have the properties that we want. If one was writing in the more concise style preferred by some authors, one could fit the proof in 10 pages, probably less.

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mysterious. It is easy to say how it is used in the proof (it produces inclusions $F(X) \subseteq F_{l\mathrm{dh}}(X) \subseteq \prod_{x \in X} F(x)$, cf. Lemma 28, Proposition 29), but not why it should be a necessary ingredient. See also Remark 23 for more on this.

The class of presheaves covered by the hypotheses of Theorem 30 is reasonably large. Any cohomology theory representable in the motivic stable homotopy category is invariant under universal homeomorphisms (at least with $\mathbb{Z}[\frac{1}{p}]$-coefficients), [EK18], [CD15, Lemma 3.15]. It is quite common for cohomology theories in algebraic geometry to have some kind of trace or transfer morphisms, see Example 14 for a list of examples. Many cohomology theories in algebraic geometry satisfy (G1), see also Gersten's Conjecture for algebraic K-theory [Qui73, Theorem 7.5.11], or Cousin complexes in the Bloch–Ogus–Gabber theorem [CTHK]. In particular, the cohomology theories representable in the motivic stable homotopy category that we are interested in satisfy (G1), [KM18, Remark 3.2], [Kel17, Theorem 3.3.1].

Condition (G2) seems newer. It is true for algebraic K-theory, or more generally, for nilpotent invariant theories commuting with filtered colimits which satisfy "Milnor" excision, [KM18, Lemma 3.5]. The author is currently working with Elmanto, Hoyois, and Iwasa to prove "Milnor" excision for SH with $\mathbb{Z}[\frac{1}{p}]$-coefficients. See also Bhatt and Mathew's newly minted arc-topology [BM18, Theorem 1.6].

The problem—non-Nagataness. One of the first things one might try when comparing the cohomology of a finer topology λ with a coarser one σ is the change of topology spectral sequence

$$H^p_\sigma(X, (\underline{H}^q_\lambda F)_\sigma) \Rightarrow H^{p+q}_\lambda(X, F_\lambda).$$

If one can show that $H^p_\lambda(P, F_\lambda) = 0$ (for $p > 0$ and $F_\lambda(P) = F(P)$) for schemes P in a family inducing a conservative family of fiber functors of the σ-topos, it follows that $(\underline{H}^q_\lambda F)_\sigma = 0$ (for $p > 0$ and $F_\sigma = F_\lambda$), the spectral sequence collapses, and one is done. If $(\sigma, \lambda) = (\text{Zariski, étale})$ then this would be to show that $H^p_{\mathrm{et}}(-, F_{\mathrm{et}})$ vanishes on all local rings. In our setting where $(\sigma, \lambda) = (\mathrm{cdh}, l\mathrm{dh})$ it amounts to showing that $H^p_{l\mathrm{dh}}(-, F_{l\mathrm{dh}})$ vanishes on finite rank Hensel valuation rings, (Appendix B).

To prove this vanishing, one would like to use a structure of trace morphisms on F (as formalized in Definition 13) and the well-known fact that every $l$dh-covering of (the spectrum of) a Hensel valuation ring is refinable by a finite flat surjective morphism of degree prime to l, via something like the following well-known lemma:

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LEMMA 2. (Lemma 15, [Kel17, Lemma 2.1.8]) *Suppose that F is a $\mathbb{Z}_{(l)}$-linear presheaf with traces in the sense of Definition 13, and $f: Y \to X$ a finite flat surjective morphism of degree prime to l. Then the complex*

$$0 \to F(X) \to F(Y) \to F(Y \times_X Y) \to F(Y \times_X Y \times_X Y) \to \cdots$$

is exact.

The problem is that finite flat algebras over Hensel valuation rings are not necessarily products of Hensel valuation rings. We can try to return to Hensel valuation rings by normalizing (cf. Lemma 16), but then this takes us out of the category of finite algebras. To summarize:

Problem. Given a Hensel valuation ring R and a finite faithfully flat R-algebra $R \to A$, in general, there is no A-algebra A' such that $R \to A'$ is finite faithfully flat and A' is a product of Hensel valuation rings.

The normalization $R \to \widetilde{A}$ is a product of Hensel valuation rings, and faithfully flat, but unless $\mathrm{Frac}(\widetilde{A})/\mathrm{Frac}(R)$ is finite separable and R is discrete [Bou64, Chapter 6, Section 8, No. 5, Theorem 2, Corollary 1], the morphism $R \to \widetilde{A}$ is in general no longer finite, not even for a general discrete valuation ring R (cf. Example 18).

The solution—pseudo-normalizations. The observation which rescues us is that $\mathbb{Z}[\frac{1}{p}]$-motivic cohomology, and more generally the sheaves we are interested in, are invariant under universal homeomorphism. Since we can restrict our attention to finite rank Hensel valuation rings (cf. Corollary A.3), and all residue field extensions of a finite extension of valuation rings are finite, we do not have to normalize to catch all the information in the normalization that we need.

LEMMA 3. (See Lemma 20) *Suppose R is a finite rank Hensel valuation ring and $R \to A$ a finite faithfully flat algebra. Then there exists an A-algebra A' such that $R \to A'$ is finite faithfully flat and $\mathrm{Spec}(\widetilde{A'}) \to \mathrm{Spec}(A')$ induces an isomorphism on underlying topological spaces and all residue fields.*

Since our sheaves do not distinguish between A' and the (product of) valuation ring(s) $\widetilde{A'}$, we can use A' as though it were a valuation ring. To work with this lemma we introduce the notion of pseudo-normalization.

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DEFINITION 4. (See Definition 19) Let A be a ring and $A \to B$ an A-algebra. Define a *pseudo-integral closure*[3] of A in B to be a *finite* sub-A-algebra

$$A \to B^{\mathrm{pic}} \subseteq B^{\mathrm{ic}} \subseteq B$$

of the integral closure B^{ic} of A in B such that $\mathrm{Spec}(B^{\mathrm{pic}}) \to \mathrm{Spec}(B^{\mathrm{ic}})$ induces an isomorphism on topological spaces and residue fields. A *pseudo-normalization* of A is a pseudo-integral closure of A in its normalization

$$A \to \breve{A} \subseteq (A^{\mathrm{red}})^{\sim}.$$

So the lemma above now becomes:

LEMMA 5. (See Lemma 20) *Every finite faithfully flat algebra A over a finite rank Hensel valuation ring admits a pseudo-normalization $A \to \breve{A}$.*

Outline. In Section 2 (resp. 3, resp. 4), we recall some well-known material on *universal homeomorphisms* (resp. *presheaves with traces*, resp. *Hensel valuation rings*). An interesting observation is that a morphism of schemes becomes an isomorphism of $l$dh-sheaves under Yoneda if and only if it is a universal homeomorphism, Corollary 9, Remark 10, (valid for any $l \neq p$). The h-version of this statement is well-known and due to Voevodsky [Voe96] for excellent Noetherian schemes and Rydh [Ryd10] in general.

In Section 5, we introduce the notion of *pseudo-integral closure* and *pseudo-normalization*, and develop some basic properties.

Sections 6–8 (pages 199–205) contain our proof.

In Section 6, the condition (G1) appears, and we use it to show that for Hensel valuation rings R we have $F(R) \cong F_{l\mathrm{dh}}(R)$, Proposition 24.

In Section 7, we continue using (G1) to show that traces on F induce traces on $F_{l\mathrm{dh}}$, Proposition 29.

Section 8 (pages 203–205) contains our main theorem (Theorem 30) and the rest of its proof.

In Appendix A, we recall the definitions of the cdh- and $l$dh-topologies, and observe that for finite dimensional Noetherian schemes, the class of *finite dimensional* Hensel valuation rings induces a conservative family of fiber functors.

[3] If one prefers names that explain the meaning, we find *finite cduh-local integral closure* and *finite cduh-local normalization* to be the most accurate.

In Appendix B, we confirm that everything we need passes from finite type separated S-schemes to all quasi-compact separated S-schemes just as one would expect.

Convention. We work with a base scheme S which will almost always be separated and Noetherian, and often of finite dimension. We write:

Sch_S for the category of separated finite type S-schemes. Unless otherwise indicated, *presheaf* means a presheaf on Sch_S which is extended to the category

SCH_S of all quasi-compact separated S-schemes by left Kan extension. That is $F(T) = \varinjlim_{T\to X\to S} F(X)$ where the colimit is over factorizations through $X \in \mathsf{Sch}_S$. Another way of saying this, (when S is quasi-compact and quasi-separated), is that

presheaf means an additive functor $F : \mathsf{SCH}_S^{\mathrm{op}} \to \mathrm{Ab}$ that commutes with filtered colimits. For more on this see Appendix B.
We use

$h_{-} : C \to \mathsf{PreShv}(C); X \mapsto h_X$ for the Yoneda functor, and

$(-)_\tau : \mathsf{PreShv}(C) \to \mathsf{Shv}_\tau(C); F \mapsto F_\tau$ for the sheafification functor. We write

$\underline{H}^n_\tau F$ for the presheaf $H^n_\tau(-, F_\tau)$ associated to a presheaf F. This paper's topologies (principally cdh, $l$dh, fpsl') are classically defined on Sch_S for S Noetherian. We extend them to SCH_S in the canonical way (cf. Appendix B).

$Q(A)$ is the total ring of fractions of a ring A. We will only ever apply this to reduced rings.

l and p are always distinct primes.

§2. Universal homeomorphisms

In this section, we recall some basic material on universal homeomorphisms (uh), and show that $l$dh-sheafification preserves uh-invariances. For some background on the $l$dh-topology for Noetherian, and non-Noetherian schemes, we direct the reader to appendices A and B, respectively.

Definition 6. A morphism of schemes $f : Y \to X$ is a *universal homeomorphism* or uh if it satisfies the following equivalent conditions.

(1) For every X-scheme $T \to X$, the morphism $T \times_X Y \to T$ induces a homeomorphism on the underlying topological spaces.
(2) [Stacks, Tag 01S4, Tag 01S3] f is a homeomorphism, and for every $y \in Y$, the field extension $k(y)/k(f(y))$ is purely inseparable.

(3) [EGAIV4, Corollary 18.12.11] f is integral, surjective and universally injective.
(4) [Stacks, Tag 01WM] f is affine, surjective, and universally injective.

Note there are no finiteness conditions. There are universal homeomorphisms that are not of finite type, pertinently Example 18, and finite universal homeomorphisms that are not finitely presented. For example, $A = \varinjlim_{n\to\infty} \mathbb{F}_p[x_1, \dots, x_n]/\langle x_i x_j : i \neq j \rangle$ and the map $\phi : A \to A; x_1 \mapsto x_1^p$. To generate the kernel of $A[y] \to A; \sum a_m y^m \mapsto \sum \phi(a_m) x_1^m$, in addition to $y^p - x_1$ one needs all the yx_n for $n > 1$.

LEMMA 7. *Suppose that $Y \to X$ is a universal homeomorphism, and $\mathcal{A} \subseteq \mathcal{O}_Y$ is a sub-$\mathcal{O}_X$-algebra. Then both of $Y \to \underline{\mathrm{Spec}}(\mathcal{A}) \to X$ are universal homeomorphisms.*

Proof. By Definition 6(3) it suffices to prove that $Y \to \underline{\mathrm{Spec}}(\mathcal{A})$ is surjective. But this follows from integrality and injectivity of $\mathcal{A} \subseteq \mathcal{O}_Y$, [Stacks, Tag 00GQ]. ∎

Recall that a morphism of schemes $Y \to X$ is *completely decomposed* if it induces a surjection $Y(K) \to X(K)$ of K-points for every field K.

LEMMA 8. *Let S be a Noetherian separated scheme of positive characteristic p. Every universal homeomorphism $Y \to X$ in Sch_S is refinable by a composition $Y_1 \to Y_0 \to X$ such that $Y_0 \to X$ is proper and completely decomposed, and $Y_1 \to Y_0$ is finite flat surjective and locally of degree a power of p.*

Proof. We reproduce the standard proof using Raynaud–Gruson flatification and induction on the dimension. The initial case is dimension -1, that is, the empty scheme. Suppose that $Y \to X$ is a universal homeomorphism with $\dim X = n \geqslant 0$, and suppose that the statement is true for a target scheme of dimension $< n$. Replacing X and Y with their reductions, we can assume that $Y \to X$ is generically flat. In this case, by Raynaud–Gruson flatification [RG71, Theorem 5.2.2], there exists a nowhere dense closed subscheme $Z \subset X$ such that the strict transform $Y' \to \mathrm{Bl}_Z X$ is globally flat. On the other hand, as $Y \to X$ is an integral morphism of finite type, it is finite. Therefore, $Y' \to \mathrm{Bl}_Z X$ is flat *and* finite. Moreover, as $Y \to X$ is a universal homeomorphism, for each generic point $\xi \in X$ with corresponding point $\eta \in Y$, the finite field extension $k(\eta)/k(\xi)$ is purely inseparable. In particular, as Y' and $\mathrm{Bl}_Z X$ are reduced, $Y' \to \mathrm{Bl}_Z X$ is locally of degree a

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power of p. Using the Noetherian inductive hypothesis to choose a refinement $W_1 \to W_0 \to Z$ of the universal homeomorphism $Y \times_X Z \to Z$, we have produced the desired refinement $W_1 \sqcup Y' \to W_0 \sqcup \mathrm{Bl}_Z X \to X$ of $Y \to X$. ∎

COROLLARY 9. *Let S be a Noetherian scheme of positive characteristic $p \neq l$ and $f : Y \to X$ a universal homeomorphism in Sch_S. Then the image $(h_f)_{l\mathrm{dh}} \in \mathsf{Shv}_{l\mathrm{dh}}(\mathsf{Sch}_S)$ of f is an isomorphism.*

REMARK 10. In fact, the converse is also true: [Voe96, Theorem 3.2.9] and [Ryd10, Theorem 8.16] say that the image of $(h)_{\mathrm{h}} : \mathsf{Sch}_S \to \mathsf{Shv}_{\mathrm{h}}(\mathsf{Sch}_S)$ is equivalent to the localization of Sch_S at the class uh. As uh is a right multiplicative system and satisfies the 2-out-of-6 property, it follows that a morphism in Sch_S is in uh if and only if it becomes an isomorphism in $\mathsf{Sch}_{\mathrm{h}}(\mathsf{Sch}_S)$, [KS06, 7.1.20]. Since $\mathsf{Sch}_S \to \mathsf{Shv}_{\mathrm{h}}(\mathsf{Sch}_S)$ factors through $\mathsf{Shv}_{l\mathrm{dh}}(\mathsf{Sch}_S)$, the converse of Corollary 9 follows.

Proof. Surjectivity is a result of f being (refinable by) an $l$dh-cover, Lemma 8. For injectivity, consider some $s, s' \in (h_Y)_{l\mathrm{dh}}(T)$ sent to the same element of $(h_X)_{l\mathrm{dh}}(T)$. Let $T' \to T$ be an $l$dh-cover such that s, s' can be represented by some morphisms $t, t' : T' \to Y$ in Sch_S with T' reduced. Possibly refining T', we can assume that $ft = ft'$. Let $\eta_1, \ldots, \eta_n$ be the generic points of T'. As $f : Y \to X$ is a universal homeomorphism, it is injective on the underlying topological space, and all residue field extensions are purely inseparable. Consequently, $\hom(\amalg\eta_i, Y) \to \hom(\amalg\eta_i, X)$ is injective, so $t|_{\amalg\eta_i} = t'|_{\amalg\eta_i}$. But $Y \to X$ is separated, and T' reduced, so $t = t'$. ∎

COROLLARY 11. *Let S be a Noetherian scheme of positive characteristic $p \neq l$. For any presheaf F and $n \geqslant 0$, the associated presheaf $(\underline{H}^n_{l\mathrm{dh}} F)(-) = H^n_{l\mathrm{dh}}(-, F_{l\mathrm{dh}})$ is* uh*-invariant.*

Proof. Let $F_{l\mathrm{dh}} \to I^\bullet$ be an injective resolution in $\mathsf{Shv}_{l\mathrm{dh}}(\mathsf{Sch}_S)$. Then we have $(\underline{H}^n_{l\mathrm{dh}} F)(-) = H^n(\hom_{\mathsf{Shv}_{l\mathrm{dh}}}((h_-)_{l\mathrm{dh}}, I^\bullet))$. But Corollary 9 says that $(h_-)_{l\mathrm{dh}}$ sends universal homeomorphisms to isomorphisms. Hence, the same is true of $\underline{H}^n_{l\mathrm{dh}} F$. ∎

§3. Traces and fpsl'-descent

This section contains material on presheaves with traces from [Kel17] included for the convenience of the reader.

DEFINITION 12. We abbreviate finite flat surjective to fps, and finite flat surjective of degree prime to l to fpsl'.

DEFINITION 13. [Kel17, Definition 2.1.3] Let $\mathcal{S}$ be a category of schemes admitting fiber products. A *structure of traces* on a presheaf $F : \mathcal{S}^{op} \to Ab$ is a morphism $\mathrm{Tr}_f : F(Y) \to F(X)$ for every fps morphism $f : Y \to X$, satisfying the following axioms.

(1) (Add) We have $\mathrm{Tr}_{f \amalg f'} = \mathrm{Tr}_f \oplus \mathrm{Tr}_{f'}$ for every pair $Y \xrightarrow{f} X$, $Y' \xrightarrow{f'} X'$ of fps morphisms.
(2) (Fon) We have $\mathrm{Tr}_f \circ \mathrm{Tr}_g = \mathrm{Tr}_{f \circ g}$ for every composable pair $W \xrightarrow{g} Y \xrightarrow{f} X$ of fps morphisms.
(3) (CdB) We have $F(p) \circ \mathrm{Tr}_f = \mathrm{Tr}_g \circ F(q)$ for every fps morphism $f : Y \to X$ and every cartesian square

$$\begin{array}{ccc} W \times_X Y & \xrightarrow{q} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow[p]{} & X. \end{array}$$

(4) (Deg) We have $\mathrm{Tr}_f \circ F(f) = d \cdot \mathrm{id}_{F(X)}$ for every fps morphism $f : Y \to X$ of constant degree d.

A presheaf equipped with a structure of traces is called a *presheaf with traces*. A presheaf with traces taking values in the category of R-modules for some ring R, is called a *presheaf of R-modules with traces* or an *R-linear presheaf with traces*.

EXAMPLE 14. (For these and more examples see [Kel17, Example 2.1.4])

(1) Every constant additive presheaf has a unique structure of traces determined by the axioms (Add) and (Deg).
(2) The trace and determinant equip $(\mathcal{O}, +)$ and $(\mathcal{O}^*, *)$ respectively with a structure of traces, [Kel17, Example 2.1.4(vii)].
(3) If F is a presheaf with traces, then the associated Nisnevich sheaf F_{Nis} inherits a unique structure of traces compatible with the canonical morphism $F \to F_{\mathsf{Nis}}$, [Kel17, Corollary 2.1.13]. This is essentially because a finite algebra over a Hensel local ring is a product of Hensel local rings.
(4) Pushforward of vector bundles would induce a structure of traces on higher K-theory K_n and homotopy invariant K-theory KH_n for every $n \in \mathbb{Z}$, except (Deg) is only satisfied Zariski locally. The Nisnevich sheafifications of these sheaves $(K_n)_{\mathsf{Nis}}$ and $(KH_n)_{\mathsf{Nis}}$ have canonical structures of traces (cf. [Kel14, Proof of Lemma 3.1]).

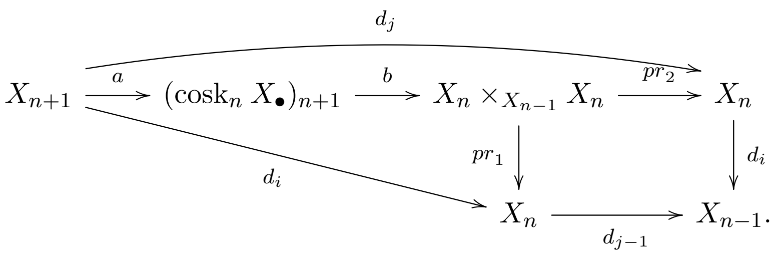
LEMMA 15. (cf. [Kel17, Lemma 2.1.8]) *Suppose that F is a $\mathbb{Z}_{(l)}$-linear presheaf with traces, and $X_\bullet \to X_{-1}$ is a simplicial X_{-1}-scheme such that each $X_{n+1} \to (\operatorname{cosk}_n X_\bullet)_{n+1}$ is an* fpsl$'$*-morphism of constant degree, for example, $X_{n-1} = Y^{\times_X n}$ for some* fpsl$'$*-morphism of constant degree $Y \to X$. Then the complex*

$$0 \to F(X_{-1}) \to F(X_0) \to F(X_1) \to F(X_2) \to \cdots$$

is exact. Here the morphisms are $\sum(-1)^i d_i$.

Consequently, F is a fpsl$'$*-sheaf, and we have both $\check{H}^n_{\mathrm{fpsl}'}(-, F) = 0$, and $H^n_{\mathrm{fpsl}'}(-, F_{\mathrm{fpsl}'}) = 0$ for $n > 0$.*

Proof. For each $0 \leqslant i < j \leqslant n$ we have the commutative diagram



All morphisms are fpsl$'$-morphisms of constant degree, [Stacks, Tag 01GN]. Setting, $D_m = \deg(X_m \to X_{-1}) / \deg(X_{m-1} \to X_{-1}) = \deg(X_m \xrightarrow{d_i} X_{m-1})$, the composition ba is of degree D_{n+1}/D_n. Now, it follows from the above diagram, that

$$\frac{1}{D_{n+1}} Tr_{d_i} F(d_j) \overset{(Fon)}{=} \frac{1}{D_{n+1}} Tr_{pr_1} Tr_{ba} F(ba) F(pr_2) \overset{(\mathrm{Deg})}{=} \frac{1}{D_n} Tr_{pr_1} F(pr_2)$$
$$\overset{(CdB)}{=} \frac{1}{D_n} F(d_{j-1}) Tr_{d_i}.$$

In particular, $(1/D_\bullet)Tr_{d_0}$ is a chain homotopy between the zero morphism and the identity morphism of the chain complex in the statement.

For the second statement, notice that any fpsl$'$-morphism is refinable by an fpsl$'$-morphism of constant degree. Hence, the Čech cohomologies of these two classes of morphisms are the same, and moreover, they generate the same topology. Since the colimit over all hypercovers calculates sheaf cohomology, [SGA42, Theorem 7.4.1(2)], vanishing of sheaf cohomology also follows from exactness of the sequence. ∎

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§4. Hensel valuation rings

Recall that an integral domain R is a *valuation ring* if for all nonzero $a \in \mathrm{Frac}(R)$, we have $a \in R$ or $a^{-1} \in R$. Equivalently, the set of ideals of R is totally ordered by inclusion, [Bou64, Chapitre VI, §1.2, Théorème 1].

A valuation ring R is a *Hensel valuation ring* or *hvr* if it extends uniquely to every finite field extension, [EP05, §4.1]. That is, for every finite field extension $L/\mathrm{Frac}(R)$, there exists a unique valuation ring $R' \subseteq L$ such that $L = \mathrm{Frac}(R')$ and $R = L \cap R'$. A valuation ring is a hvr if and only if it satisfies Hensel's Lemma, [EP05, Theorem 4.1.3].

We will frequently use the following lemma. Recall that $Q(-)$ denotes the total ring of fractions.

LEMMA 16. *Let R be a hvr, let $R \to A$ be a finite R-algebra, and let $Q(A^{\mathrm{red}}) \to L$ be a finite morphism with L reduced. Then the integral closure A^{ic} of A^{red} in L is a product of valuation rings. If A and L are integral domains the induced morphism* $\mathrm{Spec}(A^{\mathrm{ic}}) \to \mathrm{Spec}(A)$ *is a homeomorphism.*

Proof. The integral closure A^{ic} is the product of the integral closures of the images of A^{red} in the residue fields of L, so we can assume A is an integral domain and L a field. Replacing R with its image[4] in A, we can assume $R \to A$ is injective. Note that since A is finite, the integral closure of A in L is equal to the integral closure of R in L. Now the first claim follows from the facts that the integral closure of a valuation ring is the intersection of the extensions [EP05, Corollary 3.1.4], and since R is Henselian, by definition there is a unique extension to L. Now, $R \subseteq A \subseteq A^{\mathrm{ic}}$ are integral extensions of rings so $\mathrm{Spec}(A^{\mathrm{ic}}) \to \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ are surjective [Stacks, Tag 00GQ]. Since $R \subseteq A \subseteq A^{\mathrm{ic}}$ are integral extensions, the incomparability property implies that $\mathrm{Spec}(A^{\mathrm{ic}}) \to \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ are injective. Finally, since the prime ideals of A^{ic} and R are totally ordered, the bijection $\mathrm{Spec}(A^{\mathrm{ic}}) \to \mathrm{Spec}(A)$ is a homeomorphism. ∎

LEMMA 17. *Suppose that R is a hvr of characteristic p with fraction field K, and suppose that $K \to K'$ is a purely inseparable extension, and R' the integral closure of R in K'. Then* $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ *is a universal homeomorphism.*

[4] The image is an integral domain because A is, and its ideals are the ideals of R containing the kernel of $R \to A$, so they are totally ordered. That is, the image of R in A is certainly a valuation ring.

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Proof. The extension $R \to R'$ is integral by definition, and $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$ is surjective as it is dominant and satisfies the going up property. So it remains to see that it is injective, and each extension of residue fields is purely inseparable. As R is Henselian, R' is a valuation ring, and so its poset of primes is totally ordered. By the incomparability property, it follows that there is exactly one prime of R' lying over any prime of R. Let $\mathfrak{p} \subset R$ be a prime and $\mathfrak{p}' \subset R'$ the prime lying over it. Localizing at $\mathfrak{p}$ we can assume both are maximal ideals. Then a given $a \in k(\mathfrak{p}') = R'/\mathfrak{p}'$, lifts to some $b \in R'$, and since K'/K is purely inseparable, $b^{p^i} \in K$ for some i. Now R' is the integral closure of R in K', so b satisfies some monic $f(T) \in R[T]$. But then b^{p^i} satisfies the monic $f(T)^{p^i}$. Since valuation rings are normal, it follows that $b^{p^i} \in R$. Consequently, $a^{p^i} \in k(\mathfrak{p}) = R/\mathfrak{p}$. So $k(\mathfrak{p}')/k(\mathfrak{p})$ is purely inseparable. ∎

§5. Pseudo-normalization

One of the obstacles to using valuation rings to study finite type morphisms of Noetherian schemes is non-Nagataness: Suppose R is a hvr and $R \subset A$ a finite extension with A an integral domain. Then the normalization $\widetilde{A}$ of A is a valuation ring, Lemma 16. If $\operatorname{Frac}(A)/\operatorname{Frac}(R)$ is finite separable and R is discrete then the morphism $R \to \widetilde{A}$ is also finite [Bou64, Chapter 6, Section 8, No. 5, Theorem 2, Corollary 1]. However, in general, the morphism $R \to \widetilde{A}$ may not be finite.

Example 18. [Bou64, Chapter 6, Section 8, Exercise 3b] Let $k = \mathbb{F}_p(X_n)_{n\in\mathbb{N}}$, and equip $K = k(U, V)$, with the (discrete) valuation induced by $\phi : k(U, V) \to k((U)); V \mapsto \sum_{i=0}^{\infty} X_i^p U^{ip}$. Let $K' = K(V^{1/p})$. Then, (exercise), the unique extension of valued fields K'/K induces an isomorphism on both value groups and residue fields. Consequently, the corresponding morphism $R \to R'$ of discrete valuation rings is not finite, and so the normalization R' of $R[V^{1/p}]$ is not finite over $R[V^{1/p}]$.

On the other hand, at least if the rank of R is finite, if we take a large enough member $\breve{A}$ of the filtered poset of finitely generated sub-A-algebras of $\widetilde{A}$, the induced morphism $\operatorname{Spec}(\widetilde{A}) \to \operatorname{Spec}(\breve{A})$ will be a universal homeomorphism (and even completely decomposed). Hence, as far as uh-sheaves (and even cdh-sheaves, cf. [HK18, Lemma 2.9]) are concerned, the normalization might as well be finite.

To formalize this phenomenon, we introduce the notions of pseudo-integral closure, pseudo-normalization, and pseudo-hvr.

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CONVENTION. [EGAII, Corollary 6.3.8], [Stacks, Tag. 035N, 035P] By *normalization* $\widetilde{A}$ of a ring A with finitely many minimal primes, we mean the integral closure of its associated reduced ring A^{red} in the total ring of fractions of $Q(A^{\mathrm{red}})$ of A^{red}.

Note, that usually one only talks of normalizations of reduced rings.

DEFINITION 19. Let A be a ring and $A \to B$ an A-algebra. Define a *pseudo-integral closure* of A in B to be a *finite* sub-A-algebra $A \to B^{\mathrm{pic}} \subseteq B^{\mathrm{ic}}$ of the integral closure $B^{ic} \subseteq B$ of A in B such that $\mathrm{Spec}(B^{\mathrm{pic}}) \to \mathrm{Spec}(B^{\mathrm{ic}})$ is a completely decomposed universal homeomorphism.

A *pseudo-normalization* of a ring A with finitely many minimal primes is a pseudo-integral closure of A in its normalization $A \to \breve{A} \subseteq (A^{\mathrm{red}})^{\sim}$. We will write $\mathsf{PseIntClo}(B/A)$ and $\mathsf{PseNor}(A)$ for the set of pseudo-integral closures, and pseudo-normalizations respectively.

A *pseudo-hvr* is an integral domain A such that $\widetilde{A}$ is an hvr, and A is a pseudo-normalization of itself, $A = \breve{A}$. That is, $\mathrm{Spec}(\widetilde{A}) \to \mathrm{Spec}(A)$ is a completely decomposed universal homeomorphism.

Even if pseudo-integral closures and pseudo-normalizations exist they are certainly not unique in general. If the normalization, respectively, integral closure is finite, then it is the final pseudo-normalization, respectively, pseudo-integral closure.

The following lemma contains the basic facts we need about pseudo-normalizations $\breve{A}$, at least for finite rings A over a finite rank hvr R: (3) they exist, (2), (4) they are (ind-)functorial for dominant morphisms, and (5) they preserve flat morphisms.

LEMMA 20. *Let R be a finite rank hvr.*

(1) *Let $A \to B$ be a ring homomorphism. If $B^{\mathrm{pic}} \subseteq B^{\mathrm{ic}}$ is a pseudo-integral closure, then any finitely generated subalgebra $B^{\mathrm{pic}} \subseteq B' \subseteq B^{\mathrm{ic}}$ is also a pseudo-integral closure.*

(2) *If the collection of pseudo-integral closures of a ring homomorphism $A \to B$ is nonempty, then it is a filtered poset and $B^{\mathrm{ic}} = \bigcup_{B^{\mathrm{pic}} \in \mathsf{PseIntClo}(B/A)} B^{\mathrm{pic}}$.*

(3) *If A is a finite R-algebra, and $Q(A^{\mathrm{red}}) \to K$ finite with K reduced, the poset $\mathsf{PseIntClo}(K/A)$ of pseudo-integral closures is nonempty. In particular, the conclusion of part (2) holds.*

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(4) *If* $\phi : A \to B$ *is a morphism of finite* R*-algebras such that* $\operatorname{Spec}(\phi)$ *sends generic points to generic points, and*

$$\begin{array}{ccc} K & \longrightarrow & L \\ \uparrow & & \uparrow \\ Q(A^{\mathrm{red}}) & \longrightarrow & Q(B^{\mathrm{red}}) \end{array}$$

a square of finite morphisms of reduced rings, then for any $A^{\mathrm{pic}} \in \mathsf{PseIntClo}(K/A)$ *there is a* $B^{\mathrm{pic}} \in \mathsf{PseIntClo}(L/B)$ *compatible with the above square. In other words, the canonical morphism of integral closures* $A^{\mathrm{ic}} \to B^{\mathrm{ic}}$ *of* $A \to B$ *in* $K \to L$ *induces a morphism of ind-objects*

$$\text{“}\varinjlim\text{''}_{\mathsf{PseIntClo}(K/A)} A^{\mathrm{pic}} \to \text{“}\varinjlim\text{''}_{\mathsf{PseIntClo}(L/B)} B^{\mathrm{pic}}.$$

(5) *If* $\phi : A \to B$ *is a finite flat morphism of finite* R*-algebras, then there exists an inclusion* $\breve{A} \subseteq \breve{B}$ *of pseudo-normalizations compatible with* ϕ*, which is flat.*

Proof.

(1) Since $B^{\mathrm{ic}} \supseteq B^{\mathrm{pic}}$ is integral so is $B^{\mathrm{ic}} \supseteq B'$, and consequently, $\operatorname{Spec}(B^{\mathrm{ic}}) \to \operatorname{Spec}(B')$ is surjective, [Stacks, Tag00GQ]. But $\operatorname{Spec}(B^{\mathrm{ic}}) \to \operatorname{Spec}(B^{\mathrm{pic}})$ is a homeomorphism, so $\operatorname{Spec}(B^{\mathrm{ic}}) \to \operatorname{Spec}(B')$ is also injective. Finally, for any point $x \in \operatorname{Spec}(B^{\mathrm{ic}})$ with images y, z in $\operatorname{Spec}(B'), \operatorname{Spec}(B^{\mathrm{pic}})$, The isomorphism $k(x) \cong k(z)$ implies an isomorphism $k(x) \cong k(y)$.

(2) Follows from part (1).

(3) Let A^{ic} be the integral closure of A^{red} in K. Since A^{red} is reduced with finitely many minimal primes, $Q(A^{\mathrm{red}})$ is a finite product of fields, [Stacks, Tag 02LV], so K is also a finite product of fields. Hence A^{ic} is the product of the integral closures in the residue fields of K, and therefore it suffices to consider the case A and K are both integral domains. Replacing R with its image in A, we can also assume $R \to A$ is injective. Now $R \subseteq A^{\mathrm{ic}}$ is an extension of valuation rings, Lemma 16. As $R \subseteq A^{\mathrm{ic}}$ is generically finite, for every prime $\mathfrak{p} \subset R$, the extension $k(\mathfrak{p}) \subset k(\mathfrak{q})$ is finite, where $\mathfrak{q} \subset A^{\mathrm{ic}}$ is the prime lying over $\mathfrak{p}$, [Stacks, Tag 0ASH]. For each prime of R, choose a set of generators of the corresponding finite field extension, lift them to A, and let A^{pic} be the sub-R-algebra that they generate. Certainly, A^{pic} is a finite R-algebra. The morphism $A^{\mathrm{pic}} \subseteq A^{\mathrm{ic}}$ is an integral ring extension so $\operatorname{Spec}(A^{\mathrm{ic}}) \to \operatorname{Spec}(A^{\mathrm{pic}})$ is

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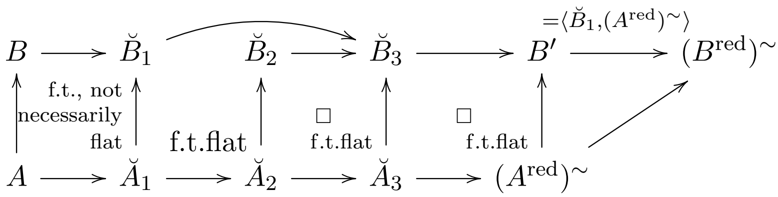
surjective, [Stacks, Tag00GQ]. As R is Henselian, $\mathrm{Spec}(A^{\mathrm{ic}}) \to \mathrm{Spec}(R)$ is a homeomorphism, and we conclude that $\mathrm{Spec}(A^{\mathrm{ic}}) \to \mathrm{Spec}(A^{\mathrm{pic}})$ is also a homeomorphism. By construction it induces isomorphisms on all field extensions. Hence, it is a completely decomposed universal homeomorphism. So $\mathsf{PseIntClo}(K/A)$ is nonempty.

(4) This follows from Part (3)—Existence, and Part (1)—Closure under subextension: Choose any $B_0^{\mathrm{pic}} \in \mathsf{PseIntClo}(B)$ and take B^{pic} to be the sub-B-algebra of L generated by the image of A^{pic} and B_0^{pic}.

(5) We will construct the following diagram. Note all morphisms except possibly the ones with source A and B are inclusions.



Given some $\breve{A}_1 \to \breve{B}_1$ extending $A \to B$, consider the sub-$(A^{\mathrm{red}})^\sim$-algebra $\breve{B}_1 \subseteq B' \subseteq (B^{\mathrm{red}})^\sim$ generated by the image of $\breve{B}_1$. As $(B^{\mathrm{red}})^\sim$ is $(A^{\mathrm{red}})^\sim$-torsion-free, so is B', and as $(A^{\mathrm{red}})^\sim$ is a (product of) valuation rings, Lemma 16, we deduce that B' is $(A^{\mathrm{red}})^\sim$-flat. As $\breve{A}_1 \to \breve{B}_1$ is finite type, $(A^{\mathrm{red}})^\sim \to B'$ is also finite type. Since $(A^{\mathrm{red}})^\sim$ is a product of local rings, the finitely generated flat module B' is a free module (Noetherianness is not needed, [Mat89, Theorem 7.10]). As $(A^{\mathrm{red}})^\sim = \bigcup_{\mathsf{PseNor}(A)} \breve{A}$, we have[5] $B' \cong (A^{\mathrm{red}})^\sim \otimes_{\breve{A}_2} \breve{B}_2$ for some finite free $\breve{A}_2 \to \breve{B}_2$ with $\breve{A}_2$ in $\mathsf{PseNor}(A)$ (the ring $\breve{B}_2$ is not necessarily in $\mathsf{PseNor}(B)$ yet). Since $(\bigcup_{\breve{A} \supseteq \breve{A}_2 \in \mathsf{PseNor}(A)} \breve{A}) \otimes_{\breve{A}_2} \breve{B}_2 \cong B' \supseteq \breve{B}_1$, tensoring with some large enough $\breve{A}_3 \supseteq \breve{A}_2$ in $\mathsf{PseNor}(A)$, we get our $\breve{B}_3 \cong \breve{A}_3 \otimes_{\breve{A}_2} \breve{B}_2 \supseteq \breve{B}_1 \supseteq \breve{B}_0$, with $\breve{A}_3 \to \breve{B}_3$ in $\mathsf{PseNor}(\phi)$. □

[5]This is true for any globally free finite flat morphism $A := \varinjlim A_\lambda \to B$ from the colimit of a filtered system: Choose an A-basis $1 = e_1, \ldots, e_n$ for B whose first element is the unit. Then the multiplication of B is determined by the $n(n-1)^2$ coefficients of the products $e_i e_j = \sum_k a_k^{ij} e_k \in B \cong A^n$ for $2 \leqslant i, j \leqslant n, 1 \leqslant k \leqslant n$ subject to linear and quadratic conditions imposed by the commutativity and associativity axioms. These $n(n-1)^2$ elements are in the image of some A_λ, and the conditions become satisfied in some, possibly higher, $A_{\lambda'}$. They then define a finite free $A_{\lambda'}$-algebra of rank n whose pullback to A is B.

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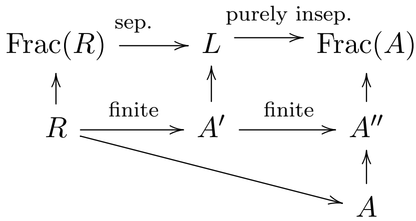
Lemma 21. *Let R be a hvr of characteristic $p \neq l$. Then any* fpsl'-*morphism $R \to A$ is corefinable by the composition of a generically étale* fpsl'-*morphism $R \to A'$ and a* uh-*morphism $A' \to A''$ such that $R \to A''$ is also* fpsl'.

Proof. First consider the case when A is an integral domain. Consider the separable closure L of $\mathrm{Frac}(R)$ in $\mathrm{Frac}(A)$, and choose pseudo-integral closures A', A'' of R, A in L, $\mathrm{Frac}(A)$ respectively, Lemma 20(4).



By definition, $A' \to \widetilde{A'}$ and $A'' \to \widetilde{A''}$ induce completely decomposed uh-morphisms, and $\widetilde{A'} \to \widetilde{A''}$ induces a universal homeomorphism by Lemma 17. Hence $A' \to A''$ is a universal homeomorphism.

So now it suffices to show that every fpsl'-morphism $R \to A$ is corefinable by an fpsl'-morphism $R \to A''$ with A'' an integral domain. Suppose first that R is a field $K = R$. Then one of the residue fields of A is of degree prime to l: Indeed, write $A = \prod A_{\mathfrak{p}_i}$ as the product of its local rings. As $l \nmid \dim_K A$ we have $l \nmid \dim_K A_{\mathfrak{p}_i}$ for some i. As $\dim_K A_{\mathfrak{p}_i} = \sum_j \dim_K \mathfrak{p}_i^j/\mathfrak{p}_i^{j+1}$ and each $\mathfrak{p}_i^j/\mathfrak{p}_i^{j+1}$ is a $k(\mathfrak{p}_j)$-vector space, we see that $l \nmid [k(\mathfrak{p}_i) : K]$. For a general R, the previous case applied to $\mathrm{Frac}(R) \to \mathrm{Frac}(R) \otimes_R A$ produces a minimal prime $\mathfrak{p}$ of A such that $l \nmid [A_{\mathfrak{p}}^{\mathrm{red}} : \mathrm{Frac}(R)]$. But then since flat = torsion-free over valuation rings, $R \to A/\mathfrak{p}$ is fpsl'. ∎

Lemma 22. *If R is a finite rank hvr and $R \to A$ a finite R-algebra, then every l*dh-*cover of* $\mathrm{Spec}(A)$ *is refinable by a finite one.*

Proof. Since every $l$dh-covering is refinable by the composition of an fpsl'-covering and a cdh-covering (see the proof of B.5(4)) it suffices to prove the statement for cdh-coverings. Replacing A with $\prod_{\substack{\mathfrak{p} \subset A \\ \mathfrak{p}\text{prime}}} (A/\mathfrak{p})^{\smile}$ we can assume that A is a pseudo-hvr. That is, $\widetilde{A}$ is a hvr and $\mathrm{Spec}(\widetilde{A}) \to \mathrm{Spec}(A)$ is a completely decomposed universal homeomorphism. Let $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the cdh-covering in question, which we assume is affine, and since our cdh-topology is pulled back from a Noetherian

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scheme (cf. Proposition B.4), we also assume its of finite presentation $B \cong A[x_1, \dots, x_n]/\langle f_1, \dots, f_m \rangle$. Every cdh-covering of a hvr admits a section, so there is a factorization $A \to B \to \widetilde{A}$. As $A \to \widetilde{A}$ is an integral extension, the images of the x_i in $\widetilde{A}$ satisfy some monic polynomials $g_i(T) \in A[T]$. Then the composition $A \to B \to B' := A[x_1, \dots, x_n]\langle f_1, \dots, f_m, g_1(x_1), \dots, g_n(x_n)\rangle$ is a finite morphism, and its Spec is completely decomposed because Spec of the composition $A \to B' \to \widetilde{A}$ is completely decomposed. □

§6. $l$dh-descent for pseudo-hvrs

From this point we start working with the following "Gestern" condition. A presheaf F satisfies (G1) if:

(G1) For every hvr R, the canonical morphism $F(R) \to F(\mathrm{Frac}(R))$ is a monomorphism.

Remark 23. We find it disappointing that we do not know a proof avoiding this condition, as its not really clear heuristically why it should be involved in passing traces from F to $F_{l\mathrm{dh}}$.

Voevodsky shows in [Voe00b, Corollary 4.18] that for a homotopy invariant presheaf with transfers F and a smooth semilocal k-scheme X, the morphism $F(X) \to F(\eta)$ to the generic scheme η is a monomorphism. So, at least in the homotopy invariant setting over a field, traces imply a version of (G1). However, in our setting we do not have this: if S is a Noetherian base scheme of dimension > 0, s a nongeneric point, and F the constant additive sheaf of some nontrivial abelian group A, then $F(- \times_S s)$ is clearly uh-invariant, and has traces by Example 14(1), however, will not satisfy (G1).

Proposition 24. *Suppose that F is a* uh*-invariant $\mathbb{Z}_{(l)}$-linear presheaf with traces satisfying (G1). Then for any pseudo-hvr R of positive characteristic $p \neq l$, we have $F(R) = F_{l\mathrm{dh}}(R)$.*

Proof. Since F is uh-invariant, so is $F_{l\mathrm{dh}}$, Corollary 11, so we can assume R is a hvr.

Injectivity: $s \in \ker\big(F(R) \to F_{l\mathrm{dh}}(R)\big)$ if and only if there is some $l$dh-covering $f : Y \to \mathrm{Spec}(R)$ such that $F(f)s = 0$. Since R is a hvr we can assume that f is an fpsl'-morphism, Proposition B.5(4). But then $s \overset{(\mathrm{Deg})}{=} (1/\deg f)\, \mathrm{Tr}_f\, F(f)s = 0$.

Surjectivity: For every $s \in F_{l\mathrm{dh}}(R)$ there is an $l$dh-covering $f : Y \to \mathrm{Spec}(R) =: X$ such that $s|_Y \in \mathrm{im}(F(Y) \to F_{l\mathrm{dh}}(Y))$. Since R is a hvr we can assume that f is an fpsl'-morphism, Proposition B.5(4). In fact, we can assume f factors as $Z_0 \overset{f_1}{\to} Y_0 \overset{f_0}{\to} X$ with f_1 a uh-morphism, and f_0 a generically étale fpsl'-morphism, Lemma 21. Since F is uh-invariant, so is $F_{l\mathrm{dh}}$, Corollary 11, so we can forget f_1 and just work with f_0. Since f_0 is generically étale, $Y_0 \times_X Y_0$ is generically reduced. Choose $Y_1 = (Y_0 \times_X Y_0)^{\smile}$ such that the composition $\pi_1 : Y_1 \to Y_0 \times_X Y_0 \overset{pr_1}{\to} Y_0$ is still flat, Lemma 20(5), and therefore fpsl'. We claim that

$$0 \to F(X) \overset{F(f_0)}{\longrightarrow} F(Y_0) \overset{F(\pi_1)-F(\pi_2)}{\longrightarrow} F(Y_1)$$

is exact where π_2 is the composition $Y_1 \to Y_0 \times_X Y_0 \overset{pr_2}{\to} Y_0$. Indeed, by id $\overset{\text{(Deg)}}{=}$ $(1/\deg f_0)\,\mathrm{Tr}_{f_0} F(f_0)$ it is exact at $F(X)$. For exactness at $F(Y_0)$, we claim that $\mathrm{Tr}_{\pi_1} F(\pi_2) = F(f_0)\,\mathrm{Tr}_{f_0}$. Indeed, since Y_1 is Spec of a (product of) pseudo-hvrs, and F is uh-invariant, by (G1) it suffices to check this after pulling back to the generic points of Y_1. But by (CdB), it suffices to show $\mathrm{Tr}_{\pi_1} F(\pi_2) = F(f_0)\,\mathrm{Tr}_{f_0}$ holds generically, that is, over the generic point η of X. But

$$\begin{array}{ccc} \eta \times_X Y_1 & \longrightarrow & \eta \times_X Y_0 \\ \downarrow & & \downarrow \\ \eta \times_X Y_0 & \longrightarrow & \eta \end{array}$$

is cartesian, so the claim follows from (CdB). Hence, the above sequence is exact at $F(Y_0)$ since if $F(\pi_2)s' = F(\pi_1)s'$, then $s' \overset{\text{(Deg)}}{=} (1/\deg \pi_1)\,\mathrm{Tr}_{\pi_1} F(\pi_1)s' \overset{\text{cycle}}{=} (1/\deg \pi_1)\,\mathrm{Tr}_{\pi_1} F(\pi_2)s' \overset{\text{("CdB")}}{=}$ $(1/\deg \pi_1)F(f_0)\,\mathrm{Tr}_{f_0} s'$.

So we have established that the top row in the following diagram is exact. Injectivity for pseudo-hvrs says the right vertical morphism is injective. Hence, by diagram chase, we find a preimage of s in $F(X)$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & F(X) & \longrightarrow & F(Y_0) & \longrightarrow & F(Y_1) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F_{l\mathrm{dh}}(X) & \longrightarrow & F_{l\mathrm{dh}}(Y_0) & \longrightarrow & F_{l\mathrm{dh}}(Y_1) \end{array}$$ ∎

§7. Traces on $F_{l\mathrm{dh}}$

In this section, we show that for a nice presheaf with traces F, the associated $l$dh-sheaf also has traces. We essentially transplant the method of [Kel12, Theorem 3.5.5], which becomes much easier in the context of this paper. For another approach to putting trace morphisms on $F_{l\mathrm{dh}}$ (with more restrictive hypotheses than we use here) see [Kel17].

Recall that in [Kel12, § 3.5] we defined

$$F_{\mathrm{cdd}}(X) = \prod_{x\in X} F(x)$$

for F a presheaf and X a scheme.

REMARK 25. One sees directly that F_{cdd} is a cdh-sheaf. More specifically, for any completely decomposed morphism $Y \to X$, the sequence

$$0 \to F_{\mathrm{cdd}}(X) \to F_{\mathrm{cdd}}(Y) \to F_{\mathrm{cdd}}(Y \times_X Y)$$

is exact (showing this by hand using a splitting of $\amalg_{y\in Y} y \to \amalg_{x\in X} x$ is an easy exercise).

What takes a little bit more work is the following theorem.

PROPOSITION 26. [Kel12, Theorem 3.5.5, Lemma 3.3.6(2), Definition 3.3.4, Proposition 3.5.7] *Let F be a* uh*-invariant $\mathbb{Z}_{(l)}$-linear presheaf with traces, where $l \neq \mathrm{char}\, s$ for all points $s \in S$ of the base scheme. Then there is a unique structure of traces on F_{cdd} such that $F \to F_{\mathrm{cdd}}$ is a morphism of presheaves with traces. Moreover, the trace morphisms on F_{cdd} satisfy:*

(Tr) *If A is a pseudo-hvr, $\phi : A \to B$ an* fps *morphism, $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are the minimal ideals of B, and $\eta_i : B \to (B/\mathfrak{p}_i)^{\smile}$ the canonical morphisms, then*

$$\mathrm{Tr}_\phi = \sum m_i \,\mathrm{Tr}_{\eta_i\circ\phi}\, F(\eta_i) \qquad \begin{array}{ccc} F(B) & \xrightarrow{\sum F(\eta_i)} & \oplus F((B/\mathfrak{p}_i)^{\smile}) \\ & {}_{\mathrm{Tr}_\phi}\searrow \quad \swarrow_{\sum m_i \mathrm{Tr}_{\eta_i\circ\phi}} & \\ & F(A) & \end{array}$$

where $m_i = \mathrm{length}\, B_{\mathfrak{p}_i}$, and of course, $(B/\mathfrak{p}_i)^{\smile}$ are chosen to be flat over A, (or some cduh*-extension of it, and we implicitly use* uh*-invariance), Lemma 20(5).*

REMARK 27. We do not need the following description, but in case the reader is interested, we recall that the trace morphisms on F_{cdd} are defined as follows. Given an fps morphism $f: Y \to X$, and $y \in Y$, the yth component of the trace morphism $F_{\mathrm{cdd}}(Y) = \prod_{y \in Y} F(y) \to \prod_{x \in X} F(x) = F_{\mathrm{cdd}}(X)$ is given by (length $\mathcal{O}_{x \times_X Y, y}$) $\mathrm{Tr}_{f|_{y/x}}$.

LEMMA 28. *Suppose that F is a* uh*-invariant $\mathbb{Z}_{(l)}$-linear presheaf with traces satisfying (G1), and R a finite rank hvr of positive characteristic $p \neq l$. Then on the category of finite R-algebras, there is a factorization $F \to F_{l\mathrm{dh}} \overset{\iota}{\hookrightarrow} F_{\mathrm{cdd}}$ with ι a monomorphism.*

Proof. On hvrs, and in particular on fields, $F \cong F_{l\mathrm{dh}}$, Proposition 24, so $F_{\mathrm{cdd}} \cong (F_{l\mathrm{dh}})_{\mathrm{cdd}}$. Our injection is $\iota: F_{l\mathrm{dh}} \to (F_{l\mathrm{dh}})_{\mathrm{cdd}} \overset{\sim}{\leftarrow} F_{\mathrm{cdd}}$.

Let A be a finite R-algebra. Spec of the morphism $A \to \prod_{\substack{\mathfrak{p} \subset A \\ \mathfrak{p} \text{ prime}}} (A/\mathfrak{p})^{\smile}$ is a cdh-covering. Since $F_{l\mathrm{dh}}$ is uh-invariant, Corollary 11, each of the morphisms $F_{l\mathrm{dh}}((A/\mathfrak{p})^{\smile}) \to F_{l\mathrm{dh}}((A/\mathfrak{p})^{\sim})$ is an isomorphism. But $F \cong F_{l\mathrm{dh}}$ on the hvrs $(A/\mathfrak{p})^{\sim}$ and $k(\mathfrak{p})$, Proposition 29, and F satisfies (G1), so each $F_{l\mathrm{dh}}((A/\mathfrak{p})^{\sim}) \to F_{l\mathrm{dh}}(k(\mathfrak{p}))$ is injective. Hence, $F_{l\mathrm{dh}}(A) \to \prod_{\substack{\mathfrak{p} \subset A \\ \mathfrak{p} \text{ prime}}} F_{l\mathrm{dh}}(k(\mathfrak{p})) = (F_{l\mathrm{dh}})_{\mathrm{cdd}}(A) \cong F_{\mathrm{cdd}}(A)$ is injective. ∎

PROPOSITION 29. *Suppose that F is a* uh*-invariant $\mathbb{Z}_{(l)}$-linear presheaf with traces satisfying (G1), and R is a finite rank hvr of positive characteristic $p \neq l$. Then on the category of finite R-algebras, the trace morphisms on F_{cdd} descend to the subpresheaf $F_{l\mathrm{dh}}$. In particular, $F_{l\mathrm{dh}}$ has a structure of traces on the category of finite R-algebras.*

Proof. Explicitly, we want to show that for every fps-morphism of R-algebras $\phi: A \to B$, the trace morphism $\mathrm{Tr}_\phi: F_{\mathrm{cdd}}(B) \to F_{\mathrm{cdd}}(A)$ sends the image of $F_{l\mathrm{dh}}(B) \hookrightarrow F_{\mathrm{cdd}}(B)$ inside the image of $F_{l\mathrm{dh}}(A) \hookrightarrow F_{\mathrm{cdd}}(A)$. First note that if $A \to A'$ is a morphism whose Spec is a completely decomposed proper (=finite) morphism, then a diagram chase using the short exact sequences, Remark 25, associated to $A \to A' \rightrightarrows A' \otimes_A A'$ by $F_{l\mathrm{dh}} \hookrightarrow F_{\mathrm{cdd}}$ shows that $F_{l\mathrm{dh}}(A) = F_{l\mathrm{dh}}(A') \cap F_{\mathrm{cdd}}(A)$. So replacing A with $A' = \prod_{\substack{\mathfrak{p} \subset A \\ \mathfrak{p} \text{ prime}}} (A/\mathfrak{p})^{\smile}$ and using (Add) and (CdB), we can assume that A is a pseudo-hvr. Now, using the morphism $B \to \prod_{\substack{\mathfrak{q} \subset B \\ \mathfrak{q} \text{ prime}}} (B/\mathfrak{q})^{\smile}$ and the property (Tr) stated in Proposition 26, we can assume that B is also a pseudo-hvr. So now $A \to B$ is a fps-morphism between pseudo-hvrs. But then $F(A) \overset{\sim}{\to} F_{l\mathrm{dh}}(A)$, $F(B) \overset{\sim}{\to} F_{l\mathrm{dh}}(B)$ are isomorphisms, Proposition 24. So the result follows from the fact that $F \to F_{\mathrm{cdd}}$ is a morphism of presheaves with traces, Proposition 26. ∎

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§8. Comparison of cdh- and $l$dh-descent

THEOREM 30. *Suppose S is a finite dimensional Noetherian separated scheme of positive characteristic $p \neq l$ and F is a* uh*-invariant $\mathbb{Z}_{(l)}$-linear presheaf with traces satisfying both conditions:*

(G1) *$F(R) \to F(\mathrm{Frac}(R))$ is injective for every finite rank hvr R.*
(G2) *$F(R) \to F(R/\mathfrak{p})$ is surjective for every finite rank hvr R and prime $\mathfrak{p}$ of codimension one.*

Then the canonical comparison morphism is an isomorphism:

$$H^n_{\mathrm{cdh}}(S, F_{\mathrm{cdh}}) \xrightarrow{\sim} H^n_{l\mathrm{dh}}(S, F_{l\mathrm{dh}}).$$

Proof. By the change of topology spectral sequence

$$H^i_{\mathrm{cdh}}(S, (\underline{H}^j_{l\mathrm{dh}}F)_{\mathrm{cdh}}) \Rightarrow H^{i+j}_{l\mathrm{dh}}(S, F_{l\mathrm{dh}})$$

it suffices to show that $F_{\mathrm{cdh}} = F_{l\mathrm{dh}}$, and $(\underline{H}^j_{l\mathrm{dh}}F)_{\mathrm{cdh}} = 0$ for $j > 0$. Since finite rank hvrs form a conservative family of fiber functors for the cdh-site Sch_S, Corollary A.3, and cohomology commutes with filtered limits of schemes with affine transition morphisms, Proposition B.5, it suffices to show that for every finite rank hvr R we have $F(R) = F_{l\mathrm{dh}}(R)$, and $H^j_{l\mathrm{dh}}(R, F_{l\mathrm{dh}}) = 0$ for $j > 0$.

It was already shown in Proposition 24 that we have $F(R) = F_{l\mathrm{dh}}(R)$. We now show that for any finite R-algebra A, we have

$$H^j_{l\mathrm{dh}}(A, F_{l\mathrm{dh}}) = 0 \tag{1}$$

for $j > 0$. We work by induction on $(\dim \mathrm{Spec}(A), j)$ where $\mathbb{N} \times \mathbb{N}_{>0}$ has the lexicographical ordering. Explicitly, we suppose that (1) is true for $\dim \mathrm{Spec}(A) < \dim \mathrm{Spec}(R)$, and all $0 < j$ and suppose also that (1) is true when $\dim \mathrm{Spec}(A) = \dim \mathrm{Spec}(R)$ and $0 < j < J$. We will show that it is true for $\dim \mathrm{Spec}(A) = \dim \mathrm{Spec}(R)$ and $j = J$.

Here is a plan of what we will prove, where $A' = A^{\mathrm{pic}}$ is a pseudo-integral closure of A^{red} in $(Q(R) \otimes_R A)^{\mathrm{red}}$, and $R' = \widetilde{A'}$:

$$H^J_{l\mathrm{dh}}(A, F_{l\mathrm{dh}}) \overset{(\alpha)}{\cong} H^J_{l\mathrm{dh}}(A', F_{l\mathrm{dh}}) \qquad \text{blowup l.e.s, induction via (2), uh-inv., (G2)}$$

$$\overset{(\beta)}{\cong} H^J_{l\mathrm{dh}}(R', F_{l\mathrm{dh}}) \qquad \text{uh-inv., Corollary 11}$$

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$$\begin{aligned} &\overset{(\gamma)}{\cong} \check{H}^J_{l\mathrm{dh}}(R', F_{l\mathrm{dh}}) && \text{induction via (7)} \\ &\overset{(\delta)}{\cong} \check{H}^J_{\mathrm{fpsl}'}(R', F_{l\mathrm{dh}}) && R' \text{ is a product of hvrs,} \\ & && \text{Proposition 38(4)} \\ &= 0 && \text{traces, Lemma 15, Proposition 29} \end{aligned}$$

Step α. Let $A' = A^{\mathrm{pic}}$ be a pseudo-integral closure of A^{red} inside $(Q(R) \otimes_R A)^{\mathrm{red}}$, let $\mathfrak{p} \subset R$ be the prime of height one (if $\dim \mathrm{Spec}(R) = 0$ set $\mathfrak{p}$ to be the unit ideal $\mathfrak{p} = R$), and consider the blowup sequence, Proposition B.5(5),

$$\begin{aligned} \cdots &\to H^{j-1}_{l\mathrm{dh}}(A'/\mathfrak{p}, F_{l\mathrm{dh}}) \\ &\to H^{j}_{l\mathrm{dh}}(A, F_{l\mathrm{dh}}) \to H^{j}_{l\mathrm{dh}}(A', F_{l\mathrm{dh}}) \oplus H^{j}_{l\mathrm{dh}}(A/\mathfrak{p}, F_{l\mathrm{dh}}) \\ &\to H^{j}_{l\mathrm{dh}}(A'/\mathfrak{p}, F_{l\mathrm{dh}}) \to \cdots \end{aligned} \tag{2}$$

By induction on $\dim \mathrm{Spec}(A)$, (1) is true for $A/\mathfrak{p}$ and $A'/\mathfrak{p}$, so we obtain an isomorphism

$$H^J_{l\mathrm{dh}}(A, F_{l\mathrm{dh}}) \cong H^J_{l\mathrm{dh}}(A', F_{l\mathrm{dh}}). \tag{3}$$

Here, (G2) and uh-invariance of $F_{l\mathrm{dh}}$, Corollary 11, is used in the case $J = 1$ to obtain surjectivity of the morphism $F_{l\mathrm{dh}}(A') \to F_{l\mathrm{dh}}(A'/\mathfrak{p})$.

Step β. Note Spec of $A' \to R' := \widetilde{A'}$ is a uh. Since $H^J_{l\mathrm{dh}}(-, F_{l\mathrm{dh}})$ is uh-invariant, Corollary 11, we are reduced to showing that $H^J_{l\mathrm{dh}}(R', F_{l\mathrm{dh}})$ vanishes.

Step γ. Consider the Čech spectral sequence

$$E_2^{i,j} = \check{H}^i_{l\mathrm{dh}}(R', \underline{H}^j_{l\mathrm{dh}}F) \Rightarrow H^{i+j}_{l\mathrm{dh}}(R', F_{l\mathrm{dh}}). \tag{4}$$

Note that the $i = 0, j > 0$ part vanishes automatically, since $\check{H}^0_\tau(-, \underline{H}^j_\tau F) = 0$ vanishes for any topology τ and $j > 0$. In particular,

$$E_2^{0,J} = \check{H}^0_{l\mathrm{dh}}(R', \underline{H}^J_{l\mathrm{dh}}F) \cong 0. \tag{5}$$

Since every $l$dh-covering of R' is refinable by a finite one, Lemma 22, and $\underline{H}^j_{l\mathrm{dh}}F(-) = H^j_{l\mathrm{dh}}(-, F_{l\mathrm{dh}})$ vanishes on finite R'-algebras for $0 < j < J$ by induction, it follows that

$$E_2^{J-j,j} = \check{H}^{J-j}_{l\mathrm{dh}}(R', \underline{H}^j_{l\mathrm{dh}}F) \cong 0; \quad \text{for } 0 < j < J. \tag{6}$$

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The vanishing so far, (5) and (6), with the spectral sequence (4) shows that

$$E_2^{J,0} = \check{H}^J_{l\mathrm{dh}}(R', F_{l\mathrm{dh}}) \cong H^J_{l\mathrm{dh}}(R', F_{l\mathrm{dh}}). \tag{7}$$

Step δ. Since R' is a product of hvrs, every $l$dh-covering is refinable by an fpsl'-covering, Proposition B.5(4). So it suffices to show that the isomorphic groups

$$\check{H}^J_{l\mathrm{dh}}(R', F_{l\mathrm{dh}}) \cong \check{H}^J_{\mathrm{fps}l'}(R', F_{l\mathrm{dh}}) \tag{8}$$

are zero.

Step ϵ. Since $F_{l\mathrm{dh}}$ has a structure of traces, Proposition 29, this vanishing follows directly from $\mathbb{Z}_{(l)}$-linearity and the structure of traces, Lemma 15.

□

Acknowledgments. The idea for this strategy is from 2013, but it took five years to realize how to get around the non-Nagataness of valuation rings. This proof owes its existence to Veronika Ertl. It was during discussions with her about de Rham–Witt differentials of valuation rings that I had the idea to use pseudo-normalizations. I thank Giuseppe Ancona, Matthew Morrow, and Simon Pepin-Lehalleur for comments that improved the writing style. I also thank an anonymous referee for helping catch a mathematical error in a previous version.

Appendix A. The cdh- and $l$dh-topologies

In [GK15] it was observed that Hensel valuation rings, or hvrs, form a conservative family of fiber functors for the cdh-site of a Noetherian scheme (see Section 4 for some facts about hvrs). Here, we observe that if dim S is finite, then in fact, it suffices to consider hvrs of finite rank.

DEFINITION A.1. Let S be a separated Noetherian scheme, Sch_S the category of separated finite type S-schemes, and $l \in \mathbb{Z}$ a prime. We quickly recall the following definitions.

(1) A morphism $f: Y \to X$ is *completely decomposed* if for all $x \in X$ there exists $y \in Y$ with $f(y) = x$ and $k(y) = k(x)$.
(2) The cdh-topology is generated by families of étale morphisms $\{Y_i \to X\}_{i \in I}$ such that $\amalg Y_i \to X$ is completely decomposed, and families of proper morphisms $\{Y_i \to X\}_{i \in I}$ such that $\amalg Y_i \to X$ is completely decomposed.

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(3) The $l$dh-topology is generated by the cdh-topology, and finite flat surjective morphisms of degree prime to l.

LEMMA A.2. *Let A be a Noetherian ring, R a hvr, and $A \to R$ a morphism. Then R is a filtered colimit A-algebras which are finite rank hvrs.*

Proof. Note $A \to R$ factors through a localization of A, and local Noetherian rings have finite Krull dimension, so we can assume A has finite Krull dimension. Certainly, R is the filtered union of its finitely generated sub-A-algebras. For each such A-algebra $A \to A_\lambda \subseteq R$, define $R_\lambda = \mathrm{Frac}(A_\lambda) \cap R \subseteq \mathrm{Frac}(R)$ to be the valuation ring induced on the fraction field of A_λ by R. This is finite rank: Certainly, R_λ is the union of its finitely generated sub-A_λ-algebras $A_\lambda \subseteq A_{\lambda\mu} \subseteq R_\lambda$. By $A_\lambda \subseteq A_{\lambda\mu} \subseteq \mathrm{Frac}(A_\lambda)$ we have $\mathrm{Frac}(A_\lambda) = \mathrm{Frac}(A_{\lambda\mu})$ so $\dim A_\lambda \geqslant \dim A_{\lambda\mu}$ [EGAIV2, Theorem 5.5.8], and therefore[6] $\dim \mathrm{Spec}(A_\lambda) \geqslant \dim \mathrm{Spec}(R_\lambda)$.

Consider the Henselizations R^h_λ of the R_λ. Henselizations of valuation rings are valuation rings of the same rank, [Stacks, Tag 0ASK], so the R^h_λ are also of finite rank. The inclusion $R_\lambda \subseteq R$ extends uniquely to an inclusion[7] $R^h_\lambda \subseteq R$ as $R_\lambda \subseteq R$ local morphism of local rings (indeed $R^*_\lambda = R^*$) toward a Hensel local ring. Moreover, any inclusion of finitely generated sub-A-algebras $A_\lambda \subseteq A_{\lambda'} \subseteq R$ induces a unique factorization $R^h_\lambda \subseteq R^h_{\lambda'} \subseteq R$ for the same reason. For every $a \in R$, there is a finitely generated sub-A-algebra A_λ with $a \in A_\lambda$. Clearly, this implies $a \in R_\lambda$, so $a \in R^h_\lambda$, and it follows that R is the union of the finite rank hvrs R_λ, and this is a filtered union because the poset $\{A_\lambda\}$ is filtered. □

COROLLARY A.3. *Let S be a Noetherian separated scheme, and* Sch_S *the category of finite type separated S-schemes equipped with the* cdh*-topology.*

[6]Suppose that $\mathfrak{p}_0 \supsetneq \cdots \supsetneq \mathfrak{p}_n$ is a sequence of prime ideals of R_λ with $n > \dim A_\lambda$. For each i choose $a_i \in \mathfrak{p}_i \setminus \mathfrak{p}_{i+1}$, and consider the finitely generated sub-A_λ-algebra $A'_\lambda = A_\lambda[a_0, \ldots, a_n] \subseteq R_\lambda$. Then $\mathfrak{p}_i \cap A'_\lambda \neq \mathfrak{p}_{i+1} \cap A'_\lambda$ for each i, but by [EGAIV2, Theorem 5.5.8] we have $\dim A'_\lambda \leqslant \dim A_\lambda$, so there is a contradiction and we conclude that $\mathfrak{p}_0 \supsetneq \cdots \supsetneq \mathfrak{p}_n$ cannot exist.

[7]The map $R^h_\lambda \subseteq R$ is indeed injective: Henselizations of valuation rings are valuation rings of the same rank, [Stacks, Tag 0ASK]. In particular, R^h_λ is an integral domain, and $\mathrm{Spec}(R^h_\lambda) \to \mathrm{Spec}(R_\lambda)$ is an isomorphism of topological spaces. As $\mathrm{Spec}(R) \to \mathrm{Spec}(R_\lambda)$ sends the generic point to the generic point, $\mathrm{Spec}(R) \to \mathrm{Spec}(R^h_\lambda)$ must also send the generic point to the generic point. In other words, $R^h_\lambda \to R$ is injective.

For any S-scheme $P \to S$ define

$$F(P) = \varinjlim_{P \to X \to S} F(X),$$

where the colimit is over factorizations with $X \to S$ in Sch_S. Then the family of functors

$$\left\{ \begin{array}{ccc} \mathsf{Shv}_{\mathrm{cdh}}(\mathsf{Sch}_S) & \to & Set \\ F & \mapsto & F(P) \end{array} \middle| \begin{array}{c} \mathrm{Spec}(R) \to S \\ R \textit{ is a finite rank hvr} \end{array} \right\}$$

is a conservative family of fiber functors.

Proof. It was proven in [GK15, Theorems 2.3 and 2.6] that the family of all hvrs induces a conservative family of fiber functors. But Lemma A.2 says that any hvr is a filtered colimit of finite rank hvrs. So given a cdh-sheaf F, if $F(R) = 0$ for every finite rank hvr, we have $F(R) = 0$ for all hvrs, and therefore $F = 0$. ∎

Appendix B. Sites of non-Noetherian schemes

DEFINITION B.1. We write SCH_S for the category of *all*[8] quasi-compact separated (and therefore quasi-separated) S-schemes.

REMARK B.2. Since our base scheme S will always be a Noetherian separated scheme, and in particular quasi-compact quasi-separated, SCH_S is nothing more than the category of those S-schemes $T \to S$ of the form $T = \varprojlim_{\lambda \in \Lambda} T_\lambda$ for some filtered system $T_- : \Lambda \to \mathsf{Sch}_S$ with affine transition morphisms, [Tem11, Theorem 1.1.2].

REMARK B.3. Following Suslin and Voevodsky, we use the term *covering family* in the sense of [SGA4, Exp. II. Definition 1.2]. That is, in addition to satisfying the axioms of a pretopology, any family refinable by a covering family is a covering family.

PROPOSITION B.4. *Let S be a Noetherian separated scheme, let τ be a topology on Sch_S such that every covering family is refinable by one indexed by a finite set, and let τ' be the coarsest topology on SCH_S making $\mathsf{Sch}_S \to \mathsf{SCH}_S$ continuous (cf. [SGA4, Exp. III. Proposition 1.6]). Then the covering*

[8] This is a bit of overkill, since we just really only want to enlarge Sch_S to include schemes of the form $\mathrm{Spec}(A) \to \mathrm{Spec}(R) \to S$ with R a finite rank valuation ring and $R \to A$ finite, but whatever.

families for τ' are those families which are refinable by pullbacks of covering families in Sch_S.

Proof. Certainly any τ-covering family in Sch_S must be a τ'-covering family in SCH_S, and therefore the pullback of any τ-covering family in Sch_S must also be a τ'-covering family in SCH_S, so it suffices to show that the collection of such families (i) contains the identity family, (ii) is closed under pullback, and (iii) is closed under "composition" in the sense that if $\{U'_i \to X'\}_{i\in I}$ and $\{V'_{ij} \to U'_i\}_{j\in J_i}$ are such families, then so is $\{V'_{ij} \to X'\}_{i\in I, i\in J_i}$. The first two are clear, so consider the third. By hypothesis, without loss of generality we can assume that I is finite. Suppose $U'_i \to Y_i$, and $X' \to X$ are morphisms with $Y_i, X \in \mathsf{Sch}_S$, and $\{V_{ij} \to Y_i\}, \{U_i \to X\}$ are τ-coverings such that $U'_i = U_i \times_X X'$ and $V'_{ij} = V_{ij} \times_{Y_i} U'_i$.

$$\begin{array}{ccccccc}
V'_{ij} & \to & U'_i & \to & X' & \in \mathsf{SCH}_S \\
\big\downarrow & & \big\downarrow\ \big\downarrow & & \big\downarrow & \\
 & & U_i & \to & X & \in \mathsf{Sch}_S \\
 & & \big\downarrow & & & \\
V_{ij} & \to & Y_i & & &
\end{array}$$

Without loss of generality we can assume that X' is the limit $\varprojlim_\Lambda X_\lambda$ of a filtered system $\{X_\lambda\}$ in Sch_S with affine transition morphisms (cf. Remark B.2), and since $\hom(\varprojlim X_\lambda, X) = \varinjlim \hom(X_\lambda, X)$ [EGAIV3, Corollary 8.13.2] that $X = X_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. In particular, now $U'_i = \varprojlim_{\lambda\leqslant\lambda_0}(X_\lambda \times_{X_{\lambda_0}} U_i)$. Now since $\hom(\varprojlim_{\lambda\leqslant\lambda_0}(X_\lambda \times_{X_{\lambda_0}} U_i), Y_i) = \varinjlim_{\lambda\leqslant\lambda_0} \hom(X_\lambda \times_{X_{\lambda_0}} U_i, Y_i)$ [EGAIV3, Corollary 8.13.2], for each i we can assume that $Y_i = X_{\lambda_i} \times_{X_{\lambda_0}} U_i$ for some $\lambda_i \leqslant \lambda_0$. Choosing a $\mu \leqslant \lambda_i$ small enough (this is where we use finiteness of I) and pulling back everything to X_μ, we can assume that $Y_i = U_i$. In this case, $\{V'_{ij} \to U'_i \to X'\}$ is the pullback of the τ-covering family $\{V_{ij} \to U_i \to X_\mu\}$ in Sch_S. ∎

In light of Proposition B.4, the τ'-covers in Sch_S are refinable by the τ-covers, so for such topologies (e.g., cdh, $l$dh, fpsl') we use the same symbol to denote the induced topology on SCH_S. Another consequence of this observation is that the adjunction

$$\iota^s : \mathsf{Shv}_\tau(\mathsf{Sch}_S) \rightleftarrows \mathsf{Shv}_\tau(\mathsf{SCH}_S) : \iota_s$$

induced by the continuous functor $\mathsf{Sch}_S \to \mathsf{SCH}_S$ satisfies $\iota_s\iota^s = \mathrm{id}$. See [SGA4, Exposé 3] for some material about this basic adjunction.

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We will write

$$\iota^* : \mathsf{PreShv}(\mathsf{Sch}_S) \rightleftarrows \mathsf{PreShv}(\mathsf{SCH}_S) : \iota_*$$

for the presheaf adjunction.

PROPOSITION B.5. *Let S be a Noetherian separated scheme, and $T \to S$ in SCH_S. Write T as the limit*

$$T = \varprojlim_{\lambda \in \Lambda} T_\lambda$$

of some filtered system $\{T_\lambda\}$ in Sch_S with affine transition morphisms (cf. Remark B.2). Let τ be both a topology on Sch_S such that every covering family is refinable by one indexed by a finite set, and also the induced topology on SCH_S, for example, $\tau = \mathrm{cdh}, l\mathrm{dh}, \mathrm{fpsl}'$.

(1) $\varinjlim_{\lambda \in \Lambda} \check{H}^n_\tau(T_\lambda, F) \xrightarrow{\sim} \check{H}^n_\tau(T, \iota^* F)$ *for any presheaf $F \in \mathsf{PreShv}(\mathsf{Sch}_S)$.*
(2) $H^n_\tau(T, \iota^s F) = \varinjlim_{\lambda \in \Lambda} H^n_\tau(T_\lambda, F)$ *for any sheaf $F \in \mathsf{Shv}_\tau(\mathsf{Sch}_S)$.*
(3) *Every $l\mathrm{dh}$-cover of T is refinable by the composition of a cdh-cover $\{V_i \to T\}_{i=1}^n$ and fpsl' morphisms $W_i \to V_i$.*
(4) *Every $l\mathrm{dh}$-cover of the spectrum of an hvr is refinable by an fpsl'-morphism.*
(5) *If $Z \to T$ a closed immersion, and $Y \to T$ a proper morphism, such that $Y \setminus Z \times_T Y \to T \setminus Z$ is an isomorphism, then*

$$0 \to \mathbb{Z}(Z \times_T Y) \to \mathbb{Z}(Z) \oplus \mathbb{Z}(Y) \to \mathbb{Z}(T) \to 0$$

becomes a short exact sequence after cdh-sheafification, where $\mathbb{Z}(W) := \mathbb{Z}\hom_{\mathsf{SCH}_S}(-, W)$.
(6) *If $F \in \mathsf{PreShv}(\mathsf{Sch}_S)$ has a structure of traces, then $\iota^* F \in \mathsf{PreShv}(\mathsf{SCH}_S)$ inherits a canonical structure of traces extending that of F.*
(7) *If $F \in \mathsf{PreShv}(\mathsf{Sch}_S)$ is uh-invariant, then $\iota^* F \in \mathsf{PreShv}(\mathsf{SCH}_S)$ is invariant for finitely presented uh-morphisms, and uh-morphisms between affine schemes with finitely many points.*

REMARK B.6. In part (7) we can actually prove that $\iota^* F$ is invariant for all uh-morphisms, assuming only that S is quasi-compact and quasi-separated, but as we do not need this stronger more general statement in this present work, we do not include its proof. In fact, David Rydh explained to us that the proof below works more or less unchanged, with quasi-compactness of the constructible topology in place of the hypothesis that the schemes have finitely many points.

Proof.

(1) Both surjectivity and injectivity follows directly from Proposition B.4 together with [EGAIV3, Theorem 8.8.2] saying that morphisms over T lift through the filtered system $\{T_\lambda\}$.

(2) As i^s is exact, the functor i_s preserves injective resolutions.[9] Then any injective resolution of $\iota^s F \to \mathcal{I}^\bullet$ restricts to an injective resolution $F = \iota_s \iota^s F \to \iota_s \mathcal{I}^\bullet$, and we have $H^n_\tau(T, \iota^s F) = (H^n \mathcal{I}^\bullet)(T) = (H^n \mathcal{I}^\bullet)(\varprojlim T_\lambda) \stackrel{(*)}{=} \varinjlim (H^n \mathcal{I}^\bullet)(T_\lambda) = \varinjlim (H^n \iota_s \mathcal{I}^\bullet)(T_\lambda) = \varinjlim H^n_\tau(T_\lambda, F)$. For (*) note for any injective sheaf I, $\hom(-, I)$ is exact by definition, and Yoneda, and sheafification are both left exact.

(3) By Proposition B.4, this follows from the case where $T \in \mathsf{Sch}_S$ which is well-known, [Kel17, Proposition 2.1.12(iii)]. This latter is proved using Raynaud–Gruson flatification, see also the proof of Lemma 8.

(4) By part (3), every $l$dh-cover is refinable by a fpsl'-cover followed by a cdh-cover. But every cdh-cover of a hvr has a section: for completely decomposed proper morphisms this follows from the valuative criterion for properness, and for completely decomposed étale morphisms, this follows from Hensel's Lemma.

(5) To check exactness, it suffices to check exactness after evaluating the sequence of presheaves on an hvr. But in this case one readily checks exactness using the valuative criterion for properness.

(6) Given a finite flat surjective morphism $f : Y \to X$ in SCH_S, there is a filtered system (X_λ) in Sch_S with affine transition morphisms, Remark B.2, such that $X = \varprojlim X_\lambda$, there is some λ and $f_\alpha : Y_\alpha \to X_\alpha$ in Sch_S such that $f = X \times_{X_\alpha} f_\alpha$, [EGAIV3, Theorem 8.8.2(ii)]. We can assume f_α is surjective and finite, [EGAIV3, Theorem 8.10.5]. By restricting to finitely many affine opens $U \subseteq X_\alpha$, and choosing isomorphism inducing global sections $\mathcal{O}_X^d \to f_* \mathcal{O}_Y$, we see that we can also assume f_α is flat. Note that Noetherianness was used here to kill $\ker(\mathcal{O}_X^d \to f_* \mathcal{O}_Y)$.

[9] If the reader is worried SCH_S is too big for injective resolutions, then just choose some large enough regular cardinal κ and instead work with the category $\mathsf{SCH}_S^{\leqslant \kappa}$ of quasi-compact separated S-schemes which are filtered limits of filtered systems in Sch_S with affine transition morphisms indexed by a category Λ with $< \kappa$ morphisms. Then $\mathsf{SCH}_S^{\leqslant \kappa}$ will be essentially small.

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Now define trace morphisms on ι^*F by choosing such presentations and define Tr_f to be $\varinjlim \mathrm{Tr}_{f_\lambda}$. Note that this is well defined. Let $a \in F(Y)$ be represented by some $a_\lambda \in F(X_\lambda \times_{X_\alpha} Y_\alpha)$, and let (X'_λ), $f'_\alpha : Y'_\alpha \to X'_\alpha$, $a'_{\lambda'} \in F(X'_\lambda \times_{X'_{\alpha'}} Y'_{\alpha'})$ be some other choice of representative. By [EGAIV3, Proposition 8.13.1] the canonical morphism $X \to X'_{\lambda'}$ factors as some $X \to X_\mu \to X'_{\lambda'}$. Since $X \times_{X_\alpha} Y_\alpha \cong Y \cong X \times_{X'_{\alpha'}} Y'_{\alpha'}$, there exists some possibly smaller μ and an isomorphism $X_\mu \times_{X_\alpha} Y_\alpha \cong X_\mu \times_{X'_{\alpha'}} Y'_{\alpha'}$ compatible with the former [EGAIV3, Theorem 8.8.2(i)]. As a_λ and $a'_{\lambda'}$ agree in the colimit $F(Y)$, possibly making μ smaller again, we can assume a_λ and $a'_{\lambda'}$ already agree in $F(X_\mu \times_{X_\alpha} Y_\alpha) \cong F(X_\mu \times_{X'_{\alpha'}} Y'_{\alpha'})$. Then it follows from (CdB) that $\mathrm{Tr}_{f_\lambda}(a_\lambda)$ agrees with $\mathrm{Tr}_{f'_{\lambda'}}(a'_{\lambda'})$ in $F(X_\mu)$, and therefore also in the colimit $F(X)$.

(Add) Given $f : Y \to X$ and $f' : Y' \to X'$, choose representatives for $Y \to X \sqcup X'$ and $Y \to X \sqcup X'$ separately. Then (Add) follows.

(Fon) Given $g : W \to Y$ and $f : Y \to X$ choose a representative for f, and then use the system $\{X_\lambda \times_{X_\alpha} Y_\alpha\}$ to choose a representative for g. Then (Fon) follows.

(CdB) Choose a representative for f, then descend W to (X_λ) using [EGAIV3, Theorem 8.8.2(ii)].

(Deg) is clear.

(7) If $f : T' \to T$ is a finitely presented universal homeomorphism in SCH_S, then there exists some $\alpha \in \Lambda$, and a universal homeomorphism $f_\alpha : T'_\alpha \to T_\alpha$ in Sch_S, such that $f = T \times_{T_\alpha} f_\alpha$, [EGAIV3, Theorem 8.8.2(ii), Theorem 8.10.5(vi)(vii)(viii)], so $\iota^*F(f) = \varinjlim_{\lambda \leqslant \alpha} F(T \times_{T_\alpha} f_\alpha)$ is an isomorphism. Note that EGA's "radiciel" is equivalent to universally injective.

Suppose that $f : T' = \mathrm{Spec}(\mathcal{O}_{T'}) \to \mathrm{Spec}(\mathcal{O}_T) = T$ is a universal homeomorphism between affine schemes with finitely many points. Write $\mathcal{O}_{T'} = \varinjlim_{\mathcal{O}_T \to A \xrightarrow{\phi} \mathcal{O}_{T'}} A$ as a filtered colimit of finitely presented $\mathcal{O}_T$-algebras.

As $T' \to T$ is a universal homeomorphism each $\mathrm{Spec}(\phi(A)) \to T$ is a finite universal homeomorphism, Lemma 7. Also, $\phi(A) = \varinjlim_{I \subseteq \ker(A \to \phi(A))} A/I$ is the filtered colimit of the quotients of A by the finitely generated ideals of the kernel $K = \ker(A \to \phi(A))$. Since $\mathrm{Spec}(A)$ has finitely many points, we can always find some finitely generated ideal J such that for each $J \subseteq I \subseteq K \subseteq A$ the closed immersion $\mathrm{Spec}(\phi(A)) \to \mathrm{Spec}(A/J)$ is surjective. Hence, $f : T' \to T$ is a filtered limit of universal homeomorphisms of finite presentation. We have already seen that ι^*F sends such morphisms to isomorphisms, so $\iota^*F(f) = \varinjlim F(\mathrm{Spec}(A) \to T)$ is an isomorphism. □

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