BULL. AUSTRAL. MATH. SOC. MOS 42A28
VOL. 3 (1970), 179-184.

A theorem on absolute summability of Fourier series by Riesz means

Prem Chandra

In 1951 Mohanty established the following theorem.

If $\phi(t)\log\log\frac{k}{t}$ *is of bounded variation in* $(0, \pi)$, *where* $k \ge \pi e^2$ *and* $\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$, *then* $\sum_{n=1}^{\infty} A_n(x)$ *is summable* $|R, \exp\{(\log w)^{\Delta}\}, 1|$, *for however large positive* Δ.

In this present note we have generalised the above theorem by taking a more general type of Riesz means and under the condition, $\phi(t)\left(\log\log\frac{k}{t}\right)^c$ is of bounded variation in $(0, \pi)$, where c is finite, imposed upon the generating function of Fourier series.

1. Definitions and notations

Let $L = L(w)$ be a differentiable, monotonic increasing function of w tending to infinity with w. For a given infinite series $\sum a_n$, we write

$$A_r(w) = \sum_{n \le w} \{L(w)-L(n)\}^r a_n \qquad (r \ge 0) .$$

The series $\sum a_n$ is summable $|R, L, r|$ $(r > 0)$ or symbolically $\sum a_n \in |R, L, r|$ $(r > 0)$, if

Received 21 May 1970. Communicated by S. Izumi.

$$\int_h^\infty \left| \frac{L'(w)}{\{L(w)\}^{r+1}} \sum_{n\le w} \{L(w)-L(n)\}^{r-1} L(n) a_n \right| dw$$

is convergent, where h is a positive number. (Obrechkoff [2], [3].)

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. We can, without any loss of generality, write the Fourier series of $f(t)$ as

$$\sum_{n=1}^{\infty} \left(a_n \cos nt + b_n \sin nt\right) = \sum_{n=1}^{\infty} A_n(t) ,$$

assuming that the constant term is zero.

Throughout we use the following notations:

(1.1) $\phi(t) = \frac{1}{2}\{f(x+t)+f(x-t)\}$;

(1.2) $g(w, t) = \sum_{n\le w} L(n) \cos nt \left(\log \log(n+2)\right)^{c-1}$, (c is finite) ;

(1.3) $h(w, t) = \sum_{n\le w} L(n) \sin nt / n \left(\log \log(n+2)\right)^{1-c}$, (c is finite) .

2. Introduction

Concerning the absolute Riesz summability of Fourier series of order unity, Mohanty [1] proved the following:

THEOREM M. *If* $\phi(t) \log \log \frac{k}{t} \in BV(0, \pi)$,[1] *where* $k \ge \pi e^2$, *then* $\sum_{n=1}^{\infty} A_n(x) \in |R, \exp\{(\log w)^\Delta\}, 1|$, *for however large positive* Δ .

Generalising the above theorem, we prove the following:

THEOREM. *Let* c *be finite and* Δ *be positive however large. If the type of Riesz means* $L(w)$ *satisfies the following conditions:*

(2.1) $\{L(w)/w(\log \log w)^{1-c}\}$

[1] '$f(x) \in BV(a, b)$' means $f(x)$ is of bounded variation in (a, b) .

https://doi.org/10.1017/S0004972700045846 Published online by Cambridge University Press

is monotonic increasing with $w \geq w_0$ *,* [2]

(2.2) $$wL'(w) = O\{L(w)(\log w)^{\Delta-1}\} .$$

Then, if $\phi(t)\left(\log\log\frac{k}{t}\right)^c \in BV(0, \pi)$ *, where* $k \geq \pi e^2$ *, then*

$$\sum_{n=1}^{\infty} \frac{A_n(x)}{\{\log\log(n+2)\}^{1-c}} \in |R, L(w), 1| .$$

3. Proof of the theorem

We have

$$\begin{aligned} A_n(x) &= \frac{2}{\pi}\int_0^\pi \phi(t)\cos nt\,dt \\ &= \frac{2}{\pi}\int_0^\pi \phi(t)\left(\log\log\frac{k}{t}\right)^c \frac{\cos nt}{\left(\log\log\frac{k}{t}\right)^c}\,dt \\ &= \frac{2}{\pi}\,\phi(\pi)\left(\log\log\frac{k}{\pi}\right)^c \int_0^\pi \frac{\cos nu}{\left(\log\log\frac{k}{u}\right)^c}\,du \\ &\qquad - \frac{2}{\pi}\int_0^\pi d\left\{\phi(t)\left(\log\log\frac{k}{t}\right)^c\right\}\int_0^t \frac{\cos nu}{\left(\log\log\frac{k}{u}\right)^c}\,du . \end{aligned}$$

Since $\phi(t)\left(\log\log\frac{k}{t}\right)^c \in BV(0, \pi)$, the series

$$\sum_{n=1}^{\infty} \frac{A_n(x)}{\{\log\log(n+2)\}^{1-c}} \in |R, L(w), 1| \quad \text{if}$$

$$\begin{aligned} I &= \int_e^\infty \frac{L'(w)}{\{L(w)\}^2}\left|\int_0^t g(w, u)\left(\log\log\frac{k}{u}\right)^{-c} du\right| dw \\ &= O(1) , \end{aligned}$$

uniformly in $0 < t \leq \pi$.

Integrating by parts, we have,

---

[2] In the case that $\{L(w)/w(\log\log w)^{1-c}\}$ is monotonic decreasing, the result follows by using the second theorem of consistency for absolute Riesz summability.

https://doi.org/10.1017/S0004972700045846 Published online by Cambridge University Press

$$\int_0^t \frac{g(w,\,u)}{\left(\log\,\log\frac{k}{u}\right)^c}\,du = \frac{h(w,\,t)}{\left(\log\,\log\frac{k}{t}\right)^c} + c\int_0^t \frac{h(w,\,u)}{u\left(\log\frac{k}{u}\right)\left(\log\,\log\frac{k}{u}\right)^{1+c}}\,du\ .$$

Therefore

$$I \le \left(\log\,\log\frac{k}{t}\right)^{-c}\int_e^\infty \frac{L'(w)}{\{L(w)\}^2}\,|h(w,\,t)|\,dw$$

$$+\int_e^\infty \frac{L'(w)}{\{L(w)\}^2}\left|\int_0^t \frac{cu^{-1}h(w,\,u)}{\log\frac{k}{u}\left(\log\,\log\frac{k}{u}\right)^{1+c}}\,du\right|dw$$

$$= I_1 + I_2\ , \text{ say.}$$

Now

$$\int_0^t \frac{\sin nu}{u\log\frac{k}{u}\left(\log\,\log\frac{k}{u}\right)^{1+c}}\,du = O\left\{\left(\log(n+1)\right)^{-1}\left(\log\,\log(n+2)\right)^{-(1+c)}\right\},$$

we have

$$I_2 = O\left\{\int_e^\infty \frac{L'(w)}{\{L(w)\}^2}\left|\sum_{n\le w}\frac{n^{-1}L(n)}{\log(n+1)\left(\log\,\log(n+2)\right)^2}\right|dw\right\}$$
$$= O(1)\ ,$$

since

$$\sum_{n=1}^\infty \left(n\log(n+1)\right)^{-1}\left(\log\,\log(n+2)\right)^{-2} < \infty\ .$$

For $T_1 = k/t$ and $T_2 = (k/t)\left(\log\frac{k}{t}\right)^{\Delta-1}$, we write

$$I = \left(\log\,\log\frac{k}{t}\right)^{-c}\left(\int_e^{T_1} + \int_{T_1}^{T_2} + \int_{T_2}^\infty\right)\left(\frac{L'(w)}{\{L(w)\}^2}|h(w,\,t)|\,dw\right)$$
$$= I_{1,1} + I_{1,2} + I_{1,3}\ , \text{ say.}$$

By using the fact $|\sin nt| \le nt$, we have

https://doi.org/10.1017/S0004972700045846 Published online by Cambridge University Press

$$I_{1,1} \leq \frac{t}{\left(\log\, \log\frac{k}{t}\right)^{c}} \int_{e}^{T_1} \frac{L'(x)}{\{L(w)\}^2} \left| \sum_{n \leq w} \frac{L(n)}{\{\log\, \log(n+2)\}^{1-c}} \right| dw$$

$$= O\left\{t\left(\log\, \log\frac{k}{t}\right)^{-c} \int_{e}^{T_1} \frac{L'(w)}{\{L(w)\}^2}\, dw \int_{1}^{w} \frac{L(x)}{\{\log\, \log(x+2)\}^{1-c}}\, dx\right\} + O(1)$$

$$= O\left\{t\left(\log\, \log\frac{k}{t}\right)^{-c} \int_{1}^{T} \{\log\, \log(x+2)\}^{c-1} dx\right\} + O(1)$$

$$= O(1)\ ,$$

uniformly in $0 < t \leq \pi$; and using $\sin nt = O(1)$, we have

$$I_{1,2} = O\left\{\left(\log\, \log\frac{k}{t}\right)^{-c} \int_{T_1}^{T_2} \frac{L'(w)}{\{L(w)\}^2}\, dw \int_{1}^{w} \frac{x^{-1}L(x)}{\{\log\, \log(x+2)\}^{1-c}}\, dx\right\} + O(1)$$

$$= O\left\{\left(\log\, \log\frac{k}{t}\right)^{-c} \int_{T_1}^{T_2} \frac{x^{-1}}{\{\log\, \log(x+2)\}^{1-c}}\, dx\right\} + O(1)$$

$$= O\left\{\left(\log\, \log\frac{k}{t}\right)^{-1} (\log T_2 - \log T_1)\right\} + O(1)$$

$$= O(1)\ ,$$

uniformly in $0 < t \leq \pi$.

Since, by applying Abel's Lemma, in view of (2.1),

$$h(w,\ t) = O\{t^{-1}L(w)/w(\log\, \log w)^{1-c}\}\ ,$$

we have

$$I_{1,3} = O\left\{t^{-1}\left(\log\, \log\frac{k}{t}\right)^{-c} \int_{T_2}^{\infty} \frac{w^{-1}L'(w)}{L(w)(\log\, \log w)^{1-c}}\, dw\right\}$$

$$= O\left\{t^{-1}\left(\log\, \log\frac{k}{t}\right)^{-c} \int_{T_2}^{\infty} \frac{w^{-2}(\log w)^{\Delta-1}}{(\log\, \log w)^{1-c}}\, dw\right\} \quad \text{(by (2.2))}$$

$$= O(1)\ ,$$

uniformly $0 < t \leq \pi$.

This terminates the proof of the theorem.

https://doi.org/10.1017/S0004972700045846 Published online by Cambridge University Press

References

[1] R. Mohanty, "On the absolute Riesz summability of Fourier series and allied series", *Proc. London Math. Soc.* (2) **52** (1951), 295-320.

[2] Nicolas Obrechkoff, "Sur la sommation absolue des séries de Dirichlet", *C.R. Acad. Sci. Paris* **186** (1928), 215-217.

[3] Nikola Obreschkoff, "Über die absolute Summierung der Dirichletschen Reihen", *Math. Z.* **30** (1929), 375-385.

Government Science College,
Jabalpur, India.