

Equivariant Heegaard genus of reducible 3-manifolds

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Abstract

The equivariant Heegaard genus of a 3-manifold W with the action of a finite group G of diffeomorphisms is the smallest genus of an equivariant Heegaard splitting for W . Although a Heegaard splitting of a reducible manifold is reducible and although if W is reducible, there is an equivariant essential sphere, we show that equivariant Heegaard genus may be super-additive, additive, or sub-additive under equivariant connected sum. Using a thin position theory for 3-dimensional orbifolds, we establish sharp bounds on the equivariant Heegaard genus of reducible manifolds, similar to those known for tunnel number. Along the way, we make use of a new invariant for W which is much better behaved under equivariant sums.

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1. Introduction

Throughout, let W be a compact, connected, oriented 3-manifold having no spherical boundary components. Let G be a finite group of orientation-preserving diffeomorphisms of W . A subset $X \subset W$ is **equivariant** if for all $g \in G$, either $g(X) \cap X = \emptyset$ or $g(X) = X$. It is **invariant** if $g(X) = X$ for all $g \in G$. Famously, the Equivariant Sphere Theorem [16] says that if W is reducible, then there exists an equivariant essential sphere. Consequently, if W is reducible, then there exists an equivariant system of essential spheres $S \subset W$ giving a factorisation of W into irreducible 3-manifolds. The (disconnected) manifold $W|_S$ created by cutting open along S and gluing in 3-balls is said to be obtained by **equivariant surgery** on S . We extend the action of G across the 3-balls.

Closed 3-manifolds also have equivariant Heegaard splittings [45]. (A Heegaard splitting for a disconnected manifold is the union of Heegaard splittings of its components.) The **equivariant Heegaard genus** $g(W;G)$ is the minimum genus of an *equivariant Heegaard splitting* for W . (This is the same as the minimal genus of an *invariant Heegaard surface* for W .) If G is trivial, then $g(W) = g(W;G)$ is the **Heegaard genus** of W . In general, $g(W;G) \geq g(W)$. A **reducing sphere** for a Heegaard surface $H \subset W$ is a sphere intersecting W in a single essential simple closed curve. If W is reducible, Haken's lemma states that every Heegaard surface for W admits a reducing sphere [7]. In particular, if W is reducible, every invariant Heegaard surface for W admits a reducing sphere.

Question 1: (Equivariant Haken's Lemma) If W is reducible and if H is an invariant Heegaard surface for W , must there exist an *equivariant* essential sphere in W that is a reducing sphere for H ?

One important consequence of Haken's lemma is that Heegaard genus of 3-manifolds is additive under connected sum. That is, $g(W) = g(W|_S)$. We could also ask:

Question 2: (Additivity of Equivariant Heegaard Genus) If W is reducible and if $S \subset W$ is an invariant system of reducing spheres such that each component of $W|_S$ is irreducible, is $g(W;G) = g(W|_S;G)$?

A positive answer to Question 1 implies a positive answer to Question 2. Both questions have a positive answer in the case when G acts freely on W (Corollary 2.8 below). Another reason to think that the answer to both questions might be "yes," is that recently Scharlemann [27] showed that given a collection of pairwise disjoint essential spheres $S \subset W$, there is an isotopy of any Heegaard surface H for W so that each component of S is a reducing sphere for H . In particular, if S is as in Question 2 and if $H \subset W$ is an invariant Heegaard surface, then H can be isotoped so that each component of S is a reducing sphere for H . Of course, the isotopy may destroy the fact that H is invariant.

In fact, the answers to Questions 1 and 2 are negative. Throughout, $|X|$ is either the cardinality or the number of components of X .

THEOREM. *Equivariant Heegaard genus can be sub-additive, additive, or super-additive. In particular:*

- (i) *if $k \geq 2$ is large enough, then there exists a reducible W , an invariant essential sphere S for W dividing W into two irreducible factors, and a cyclic group G of diffeomorphisms of W having order k such that*

$$g(W;G) < g(W|_S;G);$$

- (ii) *for $k \geq 2$, there exists a reducible W , an invariant essential sphere S for W dividing W into two irreducible factors, and a cyclic group G of diffeomorphisms of W having order k such that*

$$g(W;G) = g(W|_S;G);$$

- (iii) *for $k \geq 2$, there exists a reducible W having a finite cyclic group of diffeomorphisms G of order k , and an invariant essential sphere S dividing W into two irreducible manifolds, such that*

$$g(W;G) = g(W|_S;G) + k - 1.$$

Furthermore, $g(W;G)$ can be arbitrarily high for fixed k .

Conclusions (i) and (ii) are the content of Theorems 3.1 and 3.3. Conclusion (iii) is the content of Theorem 6.4. Consequently, no invariant Heegaard surface H for the manifolds W in Conclusions (i) or (iii) admits an equivariant reducing sphere. Perhaps not surprisingly we use facts about the non-additivity of tunnel number to prove those theorems. And so we are left with:

Question 3: What can we say about the relationship between $g(W;G)$ and $g(W|_S;G)$?

We establish an upper bound for $g(W;G)$ relative to a system of separating summing spheres. We also note that Rieck and Rubinstein [25] have produced an upper bound on

$g(W;G)$ in the case when $G = \{1, \tau\}$, with τ an involution. However, their upper bound does not apply in the case when W is reducible. Theorem 6.4 shows that the following inequality for $c = 1$ is sharp; the other inequality is likely also sharp.

THEOREM 6.10. *For any invariant system of summing spheres $S \subset W$ such that each component of S is separating,*

$$g(W;G) \leq g(W|_S;G) + (c(|G| + 1) - 2)(|W|_S| - 1)$$

where $c = 1$ if every point of W has cyclic stabiliser and $c = 2$ otherwise.

Turning to lower bounds, we show:

THEOREM 5.5. *Suppose that W is closed and connected and that G does not act freely. If $S \subset W$ is an invariant system of summing spheres, then*

$$g(W;G) \geq |G|(v/12 - \mu/2) + |S| + 1,$$

where v is the number of orbits of the components of $(W|_S)$ that are not S^3 or lens spaces and μ is the number of orbits of components of $W|_S$ that are homeomorphic to S^3 .

This bound is somewhat weak since $g(W;G) \geq g(W) = g(W|_S) \geq 2v$. However, it does give some indication of how the order of G affects the equivariant Heegaard genus. Additionally, Examples 5.6 and 5.7 suggest we are unlikely to be able to improve the lower bound without additional hypotheses. In a different direction, we show that equivariant Heegaard genus is additive or super-additive when the factors are equivariantly comparatively small. A 3-manifold W is **equivariantly comparatively small** relative to the action of a finite group of diffeomorphisms G if every equivariant essential surface $F \subset W$ has $g(F) > g(W;G)$.

THEOREM 5.10. *Suppose that when $S \subset W$ is an invariant system of summing spheres, then every component of $W|_S$ is equivariantly comparatively small. Then:*

$$g(W;G) \geq g(W|_S;G).$$

As a by-product of our methods, we produce a new invariant which we call **equivariant net Heegaard characteristic** $\text{net } x_\omega(W;G)$. It is similar to equivariant Heegaard genus, but its additivity properties are entirely understood, at least when every component of S is separating:

THEOREM 4.10. *There exists an invariant system of summing spheres $S \subset W$ such that*

$$\text{net } x_\omega(W;G) \geq \text{net } x_\omega(W|_S;G) - 2|S|,$$

and $W|_S$ is irreducible. If each component of S is separating, then equality holds.

Like equivariant Heegaard genus, $\text{net } x_\omega(W;G)$ is defined by minimising a certain quantity over a surface. In this case, however, it is not an equivariant Heegaard splitting but an equivariant generalised Heegaard splitting. Theorem 4.10 arises from the fact that equivariant spheres arise as thin surfaces. We explain this in Section 4 using orbifolds. Part of the point of this paper is to advertise the very useful properties of the invariant $\text{net } x_\omega$, which is closely related to the invariant “net extent” of [37, 38].

Orbifolds and Outline.

Three-dimensional orbifolds are the natural quotient objects resulting from finite diffeomorphism group actions on 3-manifolds. At least since Thurston's work on geometrisation [39, chapter 13], 3-dimensional orbifolds have been the essential tool for understanding finite group actions on manifolds; they are also important and interesting in their own right. A closed orientable 3-orbifold is locally modelled on the quotient of a 3-ball by a finite group of orientation preserving diffeomorphisms. The **singular set** is the set of points which are the images of those points with nontrivial stabiliser group. A closed, orientable 3-orbifold can thus be considered as a pair (M, T) where M is a closed orientable 3-manifold and $T \subset M$ is a properly embedded trivalent spatial graph with integer edge weights $\omega \geq 2$. If e is an edge of T with $\omega(e) = w$, then we say that e has weight w . If $v \in T$ is a vertex with incident edges having weights a, b, c , we let $x_\omega(v) = 1 - (1/a + 1/b + 1/c)$.

When (M, T) is a closed, 3-dimensional orbifold, for each vertex v we have $x_\omega(v) < 0$. This occurs if and only if, up to permutation, (a, b, c) is one of $(2, 2, k)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$ for some $k \geq 2$. Conversely, every trivalent graph T properly embedded in a compact (possibly with boundary) 3-manifold and having edge weights satisfying $x_\omega(v) < 0$ at each trivalent vertex v describes a compact, orientable 3-orbifold. We work somewhat more generally by taking edge weights in the set $\mathbb{N}_2^\infty = \{n \in \mathbb{N} : n \geq 2\} \cup \{\infty\}$. We define $1/\infty = 0$. This is similar to what is allowed by the software Orb [11] for studying the geometry of 3-orbifolds. We consider the edges of weight ∞ as consisting of points belonging to a knot, link, or spatial graph in M on which G acts freely. Allowing such weights does not create any additional difficulties and means that our methods may be useful for studying symmetries of knots, links, and spatial graphs. Additionally, some of our bounds are achieved only when there are edges of infinite weight; as these bounds are asymptotically achieved when edges have only finite weights, it is conceptually clearer to allow edges of infinite weight. Henceforth, we define "3-orbifold" as follows.

Definition 1.1. A **3-orbifold** is a pair (M, T) where M is a compact, orientable 3-manifold, possibly with boundary and $T \subset M$ is a properly embedded trivalent graph such that each edge e has a weight $\omega(e) \in \mathbb{N}_2^\infty$. We also require that at every trivalent vertex v , $x_\omega(v) < 0$.

When studying symmetries of knots or spatial graphs in a 3-manifold, vertices of degree 4 or more may arise, as may vertices incident to edges of infinite weight. Our framework can handle such vertices by considering them as boundary components. Our approach should be contrasted with the usual methods of proving Haken's lemma which involve intricate methods of controlling the intersections between a Heegaard surface and an essential sphere. We replace those arguments with appeals to machinery that guarantee that essential spheres show up as thin surfaces in a certain type of thin position. The basic structure of our arguments is as follows.

In Section 2, we define orbifold Heegaard splittings; show that they are the quotients of invariant Heegaard surfaces of W by G ; and show that each orbifold Heegaard splitting of the quotient orbifold lifts to a Heegaard splitting of W . We also review the correspondence between equivariant essential spheres and orbifold reducing spheres. Section 3 uses the correspondence to prove that equivariant Heegaard genus can be sub-additive or additive. Section 4 adapts Taylor–Tomova's version of thin position to orbifolds. There we define the "net Heegaard characteristic" $\text{net } x_\omega$ of an orbifold and establish its correspondence with

equivariant net Heegaard characteristic. The equivariant net Heegaard characteristic of W is bounded above by $2g(W;G) - 2$. We also prove Theorem 4.10. The key idea is to use Taylor–Tomova’s partial order on the orbifold version of Scharlemann–Thompson’s generalised Heegaard splittings for 3-manifolds. The minimal elements of this partial order are called “locally thin.” We show that $\text{net } x_\omega$ is non-increasing under the partial order and that orbifold reducing spheres show up as thin surfaces in the generalised Heegaard splittings. These two properties make $\text{net } x_\omega$ additive under orbifold-sums. Understanding the net Heegaard characteristic of the result of decomposing the orbifold by essential spheres allows us to produce lower bounds on $\text{net } x_\omega$, and thus on $g(W;G)$. Section 6 establishes upper bounds on $g(W;G)$ by adapting the amalgamation of generalised Heegaard splittings to orbifolds. Although, in principle, this should be straightforward to those who understand amalgamation of generalised Heegaard splittings, significant issues arise; issues which in some sense seem to characterize the non-additivity of equivariant Heegaard genus.

2. General Notions, Orbifold Sums, and Orbifold Heegaard Surfaces

Throughout all manifolds and orbifolds are orientable. If X is a topological space, we let $Y \sqsubset X$ mean that Y is a path component of X . We let $X \setminus Y$ denote the complement of an open regular neighbourhood of Y in X .

We refer the reader to [1, 2] for more on orbifolds. Suppose that (M, T) is an orbifold. A properly embedded orientable surface $S \subset M$, transverse to T , naturally inherits the structure of a 2-suborbifold; that is a surface locally modelled on the quotient of a disc or half disc by a finite group of orientation-preserving isometries. We call the points $S \cap T$ the **punctures** of S . If $p \in S \cap T$ is a puncture its **weight** $\omega(p)$ is the weight of the edge intersecting it. The **orbifold characteristic** of S is defined to be:

$$x_\omega(S) = -\chi(S) + \sum_{p \in S \cap T} \left(1 - \frac{1}{\omega(p)}\right).$$

This is the negative of the orbifold Euler characteristic of S . Observe that if $\pi : R \rightarrow S$ is an orbifold covering map of finite degree d , then $x_\omega(R) = dx_\omega(S)$. An orbifold is **bad** if it is not covered by a manifold and **good** if it is. The bad 2-dimensional orbifolds are spheres that either have a single puncture of finite weight or have two punctures of different weight. 2-orbifolds that are spheres with three punctures are **turnovers**. A good connected 2-orbifold S is **spherical** if $x_\omega(S) < 0$; **euclidean** if $x_\omega(S) = 0$; and **hyperbolic** if $x_\omega(S) > 0$. A compact, orientable 3-orbifold is good if it does not contain a bad 2-orbifold [2, theorem 2.5].

Definition 2.1. A 3-orbifold (M, T) is **nice** if:

- (i) for each $S \sqsubset \partial M$, $x_\omega(S) \geq 0$ and if S is a sphere then $|S \cap T| \geq 3$;
- (ii) the pair (M, T) has no bad 2-suborbifolds and no spheres that are once-punctured with infinite weight puncture;

The reason for forbidding bad 2-suborbifolds is that we cannot surger along them to create a valid 3-orbifold. The requirement that $x_\omega(S) \geq 0$ for $S \sqsubset \partial M$ helps with some of our calculations. The other requirements help with our use of the material from [36, 37]. We note that (W, \emptyset) is a nice orbifold. (Recall ∂W has no spheres by our initial definition of W .)

2.1. Factorisations.

Petronio [24] proved a unique factorisation theorem for closed good 3-orbifolds (M, T) without nonseparating spherical 2-suborbifolds. We are working in a slightly more general context (because our orbifolds may not be closed, we allow good nonseparating spherical 2-suborbifolds, and we allow infinite weight edges). Nevertheless, we adopt Petronio's terminology and his results carry over to our setting as we now describe.

Suppose that (M_1, T_1) and (M_2, T_2) are distinct nice orbifolds. Let $p_1 \in M_1$ and $p_2 \in M_2$. We can perform a connected sum of M_1 and M_2 by removing a regular neighbourhood of p_1 and p_2 and gluing the resulting 3-manifolds together along the newly created spherical boundary components; after gluing the corresponding sphere is a **summing sphere**. To extend the sum to the pairs (M_1, T_1) and (M_2, T_2) we place conditions on the points. We require that they are either in the interiors of edges of the same weight or that they are on vertices with incident edges having matching weights. The gluing map is then required to match punctures to punctures of the same weight.

When p_1 and p_2 are disjoint from T_1 and T_2 , the sum is a **distant sum** and we write $(M, T) = (M_1, T_1) \#_0 (M_2, T_2)$. When each p_i is in the interior of an edge of T_i , it is a **connected sum** and we write $(M, T) = (M_1, T_1) \#_2 (M_2, T_2)$. When each p_i is a trivalent vertex, it is a **trivalent vertex sum** and we write $(M, T) = (M_1, T_1) \#_3 (M_2, T_2)$. This sum is well-defined, for a particular choice of vertices p_1, p_2 and bijection from the ends of edges incident to p_1 to those incident to p_2 . See [42, section 4]. The pair $\mathbb{S}(0) = (S^3, \emptyset)$ is the identity for $\#_0$. The pair $\mathbb{S}(2) = (S^3, T)$ where T is the unknot is the identity for $\#_2$. The pair $\mathbb{S}(3) = (S^3, T)$ where T is a planar (i.e. trivial) θ -graph, is the identity for $\#_3$. In each case, we allow T to have whatever weights make sense in context. Note that when $k < i \leq j$, both factors of $\mathbb{S}(i) \#_k \mathbb{S}(j)$ are trivial. This means that some care is needed when discussing prime factorisations.

Conversely, suppose that (M, T) is a nice orbifold. Given a connected spherical 2-suborbifold $S \subset (M, T)$ we may split (M, T) open along S and glue in two 3-balls each containing a graph that is the cone on the points $S \cap T$. This operation is called **surgery** along S . Observe that the result is still an orbifold and that if M is connected, each component of $(M, T)|_S$ is incident to one or more **scars** from the surgery (i.e. the boundaries of the 3-balls we glued in). If S is such a sphere or the disjoint union of such spheres, we denote the result of surgery along (all components of) S by $(M, T)|_S$. Each component of S produces two scars in $(M, T)|_S$; we say those scars are **matching**. If S separates M , then surgery is the inverse operation to distant sum, connected sum, or trivalent vertex sum.

Since we will be dealing with reducible orbifolds, we need to work with slight generalizations of compressing discs. Suppose that $S \subset (M, T)$ is a surface. An **sc-disc** for S is a zero or once-punctured disc D with interior disjoint from $T \cup S$, with boundary in $S \setminus T$, and which is not isotopic (by an isotopy everywhere transverse to T) into S . If ∂D does not bound a zero or once-punctured disc in S , then D is a **compressing disc** or **cut disc** corresponding to whether D is zero or once-punctured. A **c-disc** is a compressing disc or cut disc. Otherwise, D is a **semi-compressing disc** or **semi-cut disc** respectively. In other words, a semi-compressing disc or semi-cut disc is a zero or once-punctured disc with inessential boundary in the surface but which is not parallel into the surface. The **weight** $\omega(D)$ of an sc-disc D is equal to the weight of the edge of T intersecting it and 1 otherwise. Compressing a surface using an sc-disc D decreases x_ω by $2/\omega(D)$. If ∂M admits an sc-disc D , then $(M, T) \setminus D$ is the result of ∂ -**reducing** (M, T) .

If S does not admit a c -disc, it is **c-incompressible**. A **c-essential surface** is a surface $S \subset (M, T)$ where each component is:

- (1) a c -incompressible surface
- (2) not parallel in $M \setminus T$ into $\partial(M \setminus T)$ (i.e. not ∂ -parallel); and
- (3) not an unpunctured sphere bounding a 3-ball in $M \setminus T$.

Observe that for a surface, being parallel in (M, T) to a component of ∂M is more restrictive than being parallel into $\partial(M \setminus T)$.

Definition 2.2. An orbifold (M, T) is **orbifold-reducible** if there exists a c -essential spherical 2-suborbifold $S \subset (M, T)$.

Definition 2.3 (c.f. [24]). Suppose that (M, T) is a nice orbifold and that $S \subset (M, T)$ is a closed spherical essential 2-orbifold. Then S is an **system of summing spheres** if $(M, T)|_S$ is orbifold-irreducible. A system of summing spheres S is **efficient** if, for $i \in \{0, 2, 3\}$, whenever an $\mathbb{S}(i)$ component of $(M, T)|_S$ contains an i -punctured scar, then it contains the matching scar.

For a system of summing spheres $S \subset (M, T)$, we call $(M, T)|_S$, a **factorisation** of (M, T) using S . The main difference between a factorisation using an efficient system of summing spheres and a prime factorisation is that a factor in an efficient factorisation may contain a nonseparating sphere. By work of Petronio and Hog-Angeloni–Matveev, in the absence of nonseparating spherical 2-orbifolds, up to orbifold homeomorphism for an efficient system of summing spheres S , both S and its factorisation $(M, T)|_S$ are unique; however, these homeomorphisms are not necessarily realisable by an isotopy in (M, T) . We note that neither the Petronio, Hog-Angeloni–Matveev results nor the following theorem (which is based on those) is trivial. For instance, as observed in [24, example 1.3], the connected sum of any knot in S^3 with $(S^1 \times S^2, S^1 \times (\text{point}))$ is pairwise homeomorphic to $(S^1 \times S^2, S^1 \times (\text{point}))$. Consequently, $(S^1 \times S^2, S^1 \times (\text{point}))$ does not have a prime factorization. The crux of the existence of an efficient splitting system is deferred to Corollary 4.8 below and is a consequence of Taylor–Tomova’s work on thin position. Based on the discussion in [24, section 3], the ability of thin position to handle nonseparating spherical 2-suborbifolds seems to be an instance where the thin position techniques have an advantage over normal surface techniques. We relegate the proof to the appendix, since it is a slightly more elaborate version of Petronio’s and Hog-Angeloni–Matveev’s proofs.

THEOREM 2.4 (after Petronio, Hog-Angeloni–Matveev). *Suppose that (M, T) is a nice orbifold that is orbifold-reducible. Then there exists an efficient system of summing spheres $S \subset (M, T)$; indeed, any system of summing spheres contains an efficient subset. Furthermore, any two such systems S, S' are orbifold-homeomorphic, as are $(M, T)|_S$ and $(M, T)|_{S'}$.*

The Equivariant sphere theorem is an important tool for studying group actions on 3-manifolds. The statement we use is inspired by [1, theorem 3.23] and the subsequent remark.

THEOREM 2.5 (Equivariant sphere theorem [16] (c.f. [4])). *Suppose that $\rho: (W, T') \rightarrow (M, T)$ is a regular orbifold covering, with (W, T') and (M, T) nice. Then (W, T') is orbifold-reducible if and only if (M, T) is orbifold-reducible.*

2.2 Orbifold Heegaard Splittings.

As is well known, a Heegaard splitting of a closed 3-manifold is a decomposition of the 3-manifold into the union of two handlebodies glued along their common boundary. Every closed 3-manifold has such a decomposition. 3-manifolds with boundary have a similar decomposition into two compressionbodies glued along their positive boundaries. Zimmermann (e.g. [44–46]) defined Heegaard splittings of closed 3-orbifolds and used them to study equivariant Heegaard genus. In his decompositions, an orbifold handlebody is an orbifold that is the quotient of a handlebody under a finite group of diffeomorphisms. He gives an alternative description in terms of certain kinds of handle structures [45, proposition 1]. We adapt this latter definition to define orbifold compressionbodies. Petronio [23] also defines handle structures for 3-orbifolds; the definitions differ only on the definition of 2-handles. Zimmermann’s definition has the advantage that handle structures can be turned upside down. Most of our definitions apply to graphs in 3-manifolds more generally, so we state them in that generality if possible.

Definition 2.6 (Handle structures). A **ball 0-handle** or **ball 3-handle** is a pair (W, T_W) where W is a 3-ball and T_W is the cone on a finite (possibly empty) set of points in ∂W . We also call ball 0-handles and 3-handles, **trivial ball compressionbodies**. We set $\partial_+ W = \partial W$ and $\partial_- W = \emptyset$. A **product 0-handle** or **product 3-handle** is a pair (W, T_W) pairwise homeomorphic to $(F \times I, p \times I)$ where F is a closed orientable surface, $p \subset F$ is finitely (possibly zero) many points. We also call product 0-handles and product 3-handles, **trivial product compressionbodies**. We set $\partial_{\pm} W$ to be the preimage of $F \times \{\pm 1\}$. The **attaching region** for a 0-handle is the empty set and the **attaching region** for a 3-handle (W, T_W) is $(\partial_+ W, T_W \cap \partial_+ W)$. A **trivial compressionbody** is either a trivial ball compressionbody or a trivial product compressionbody.

A **1-handle** or **2-handle** is a pair (H, T_H) pairwise homeomorphic to $(D^2 \times I, p \times I)$ where p is either empty or is the center of D^2 . The **attaching region** of a 1-handle is the preimage of $(D^2 \times \partial I, p \times \partial I)$. The attaching region of a 2-handle is $((\partial D^2) \times I, \emptyset)$.

A **vp-compressionbody** (C, T_C) is the union of finitely many 0-handles and 1-handles so that the following hold:

- (1) the 0-handles are pairwise disjoint, as are the 1-handles;
- (2) 1-handles are glued along their attaching regions to the positive boundary of the 0-handles, and are otherwise disjoint from the 0-handles;
- (3) if (H, T_H) is a 1-handle such that one component (D, p) of its attaching region is glued to a 0-handle (W, T_W) , and if $T_H \neq \emptyset$, then $p \in T_H \cap \partial_+ W$;
- (4) C is connected.

See Figure 1 for an example. The “vp” stands for “vertex punctured” and is used since drilling out vertices changes trivial ball compressionbodies with vertices into trivial product compressionbodies. If (C, T_C) is a vp-compressionbody, we let $\partial_- C$ be the union of $\partial_- W$ over the 0-handles (W, T_W) and we let $\partial_+ C = \partial C \setminus \partial_- C$. Edges of T_C disjoint from $\partial_+ C$ are called **ghost arcs**. Closed loops disjoint from ∂C are **core loops**. Edges with exactly one endpoint on $\partial_+ C$ are **vertical arcs** and edges with both endpoints on $\partial_+ C$ are **bridge**

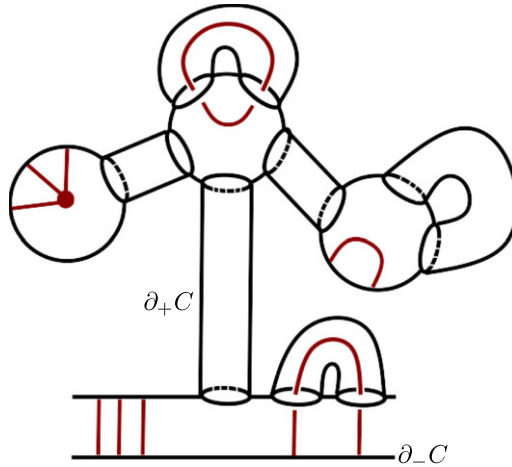


Fig. 1. An example of a vp-compressionbody (C, T_C) . It has one ghost arc, one core loop, one bridge arc, and three vertical arcs. The horizontal lines represent a closed, possibly disconnected surface F .

arcs. Dually, vp-compressionbodies may be defined as the union of 2-handles and 3-handles. Equivalently, if (C, T_C) is a connected pair with one component of ∂C designated as ∂_+C , then (C, T_C) is a vp-compressionbody if and only if there is a collection of pairwise disjoint sc-discs Δ for ∂_+C such that the result of ∂ -reducing (C, T_C) along Δ is the union of trivial ball compressionbodies and trivial product compressionbodies. The collection Δ is a **complete collection of sc-discs** for (C, T_C) if it is pairwise nonparallel. If T_C has weights such that (C, T_C) is both a vp-compressionbody and an orbifold, then we call (C, T_C) an **orbifold compressionbody**.

For a nice orbifold (M, T) , an **orbifold Heegaard surface** is a transversally oriented separating connected surface $H \subset (M, T)$ such that H cuts (M, T) into two distinct orbifold compressionbodies, glued along their positive boundaries. We define the **Heegaard characteristic** of (M, T) to be:

$$x_\omega(M, T) = \min_H x_\omega(H),$$

where the minimum is over all orbifold Heegaard surfaces H for (M, T) . The invariant $x_\omega(M, T)$ is twice the negative of the ‘‘Heegaard number’’ defined by Mecchia–Zimmerman [15]. Dividing by 2 would also make the comparison with [37, 38] easier. However, since orbifold Euler characteristic need not be integral, making that normalisation would unpleasantly complicate some of the calculations in this paper.

The **equivariant Heegaard characteristic** of W is

$$x_\omega(W; G) = \min_H x_\omega(H),$$

where the minimum is taken over all *invariant* Heegaard surfaces for H . Zimmermann proved the following when W is closed; the proof extends to the case when W has boundary as we explain. See also [5] for a similar result related to strong involutions on tunnel number 1 knots. The statement is deceptively simple as it implies (and its proof relies on) the Smith

Conjecture as well as a thorough understanding of group actions on 3-balls and products. See the remark on page 52 of [1].

LEMMA 2.7 (after Zimmermann [45]). *Suppose that W has orbifold quotient (M, T) . Every invariant Heegaard surface for W descends to an orbifold Heegaard surface for (M, T) and every orbifold Heegaard surface for (M, T) lifts to an invariant Heegaard surface for W . Consequently,*

$$x_\omega(W; G) = |G|x_\omega(M, T).$$

Proof. Suppose that Y is a compressionbody (i.e. orbifold compressionbody with empty graph) and that G is a finite group of orientation-preserving diffeomorphisms of Y . We show the quotient orbifold is an orbifold compressionbody. If $\partial_- Y = \emptyset$ (i.e. Y is a handlebody), then the quotient orbifold is an orbifold compressionbody by [45]. In particular, the quotient of a 3-ball is an orbifold trivial ball compressionbody. Consider the case when $Y = F \times I$ for a closed, connected, oriented surface F . Since each element of G is orientation-preserving, no element interchanges the components of ∂Y . If $Y = S^2 \times I$, we cap off $\partial_- Y$ with a 3-ball, extend the G -action across the 3-ball, and appeal to the 3-ball case to observe that the quotient orbifold is a trivial product orbifold compressionbody. Suppose that $F \neq S^2$. Let DY be its double and observe that we can extend the action of G to DY . The 3-manifold DY is a Seifert fiber space with no exceptional fibers, namely $F \times S^1$. Consider an embedded torus DQ that is the double of an essential spanning annulus Q in Y . By [9, theorem 1.5], DQ can be isotoped to DQ' so that for each $g \in G$, gDQ' is vertical in DQ and is isotopic to gDQ . Since each $g \in G$ preserves each component of ∂Y , there is an annulus $Q' \subset Y$, vertical in Y , such that for each $g \in G$, gQ' is isotopic to gQ by a proper isotopy in Y . Applying this observation to each annulus in a collection of spanning annuli in Y cutting Y into 3-balls, we can then conclude that the quotient orbifold of Y by the action of G is a trivial product orbifold, as desired. Finally, suppose that $\partial_+ Y$ is compressible. By the Equivariant Disc Theorem [17], Y admits an equivariant essential disc. Boundary-reducing along this disc and inducting on $x_\omega(\partial_+ Y) - x_\omega(\partial_- Y)$ shows that the quotient orbifold is again an orbifold compressionbody. Our lemma then follows from the definition of invariant Heegaard surfaces, orbifold Heegaard surfaces, and the multiplicativity of orbifold characteristic under finite covers.

COROLLARY 2.8. *If G acts freely on W , then there exists an invariant system of summing spheres $S \subset W$ such that $W|_S$ is irreducible and $\mathfrak{g}(W; G) = \mathfrak{g}(W|_S; G)$.*

Proof. Let (M, T) be the quotient orbifold of the action of G on W . Assume the action is free so that $T = \emptyset$. There exists an efficient system of summing spheres $\bar{S} \subset M$. By Haken's Lemma, $\mathfrak{g}(M) = \mathfrak{g}(M|\bar{S})$. Let S be the pre-image of \bar{S} in W . By Lemma 2.7 (after converting from Euler characteristic to genus), $\mathfrak{g}(W; G) = \mathfrak{g}(W|_S; G)$.

3. Examples of Additivity and Sub-Additivity

As an example of how to work with orbifold Heegaard surfaces, in this section we show that equivariant Heegaard genus can be both sub-additive and additive. The examples of super-additivity require different techniques. Our examples arise from cyclic branched covers of knots in S^3 . We begin with a preliminary calculation.

Suppose that $K \subset S^3$ is a knot with weight $k \geq 2$. When we ignore the weight k , an orbifold Heegaard surface H for (S^3, K) of genus g is a (g, b) -bridge surface where $b = |H \cap K|/2$ [3]. Conversely, remembering the weight k , makes each (g, b) -bridge surface H for K into an orbifold Heegaard surface for (S^3, K) . If H is such a surface, observe that

$$x_\omega(H) = 2(g + b - 1) - 2b/k.$$

If H is disjoint from K , then K lies as a core loop of the handlebody to one side of H . The **tunnel number** $t(K)$ of K is the minimum of $g(H) - 1$ over all such H . If H is any (g, b) -bridge surface for K , then by attaching tubes along the arcs of $K \setminus H$, we can convert H into a $(g + b, 0)$ -bridge surface for K . Consequently, $t(K) \leq g + b - 1$ and $x_\omega(S^3, K) \leq 2t(K)$. The same technique shows that if, for fixed g , H minimises b , then $b \leq b(K)$, where, $b(K)$ is the bridge number of K . (Recall $b(K)$ is the minimal b' such that K admits a $(0, b')$ -bridge surface.)

For $i = 1, 2$, let $K_i \subset S^3$ be a knot of weight k and let H_i be a (g_i, b_i) -bridge surface for K_i . Assume that $b_1, b_2 > 0$. If we perform the connected sum of K_1 and K_2 using points of $K_1 \cap H_1$ and $K_2 \cap H_2$, then $H = H_1 \# H_2$ is a $(g_1 + g_2, b_1 + b_2 - 1)$ -bridge surface for $K = K_1 \# K_2$. Note that if S is the summing sphere for (S^3, K) arising from our choice of connected sum, then $x_\omega(S) = -2/k$. Consequently,

$$x_\omega(H) = x_\omega(H_1) + x_\omega(H_2) - x_\omega(\bar{S}).$$

THEOREM 3.1. *If $k \geq 2$ is large enough, then there exists a reducible W , an invariant essential sphere S for W dividing W into two irreducible factors, and a cyclic group G of diffeomorphisms of W having order k such that*

$$g(W; G) < g(W|_S; G).$$

Proof. We recall that there are many examples (e.g. [21, 22, 33]) of prime knots K_1, K_2 for which tunnel number is sub-additive. That is, $t(K_1) + t(K_2) - t(K_1 \# K_2) \geq 1$. Recall $K = K_1 \# K_2$. Let \bar{S} be the summing sphere. By [34], $b(K) = b(K_1) + b(K_2) - 1$. Suppose that $k > b(K)$. Let H_i be a (g_i, b_i) -bridge surface for K_i such that $x_\omega(H_i) = x_\omega(S^3, K_i)$. According to the preamble, $x_\omega(H_i) \geq 2t(K_i) - 2b_i/k$. Thus,

$$\begin{aligned} x_\omega(S^3, K_1) + x_\omega(S^3, K_2) - x_\omega(\bar{S}) &\geq 2(t(K_1) + t(K_2)) - 2(b_1 + b_2 - 1)/k \\ &\geq 2t(K) + 2 - 2(b_1 + b_2 - 1)/k \\ &\geq 2(t(K) + 1 - b(K)/k) \\ &\geq 2t(K) \\ &\geq x_\omega(S^3, K). \end{aligned}$$

Passing to the k -fold cyclic branched cover W over K , with G the deck group and S the preimage of \bar{S} , produces the desired examples.

Remark 3.2. From Morimoto’s examples of tunnel number degeneration [21] we can see that in the proof of Theorem 3.1, we can choose the desired K_1 and K_2 so that $b(K) \leq 6$. Thus, $k \geq 7$ suffices in the statement of Theorem 3.1.

We now turn to examples of additivity.

THEOREM 3.3. *For $k \geq 2$, there exists a reducible W , an equivariant essential sphere S for W dividing W into two irreducible factors, and a cyclic group G of diffeomorphisms of W having order k such that*

$$\mathfrak{g}(W;G) = \mathfrak{g}(W|_S;G).$$

Proof. Fix $k \geq 2$. For $i = 1, 2$, let K_i be a torus knot of type (p, q) with p, q relatively prime and $|p| > |q| \geq 5$. By [34], each $\mathfrak{b}(K_i) = |q|$. Also, each K_i admits a $(1, 1)$ -bridge surface H_i . Thus, $x_\omega(S^3, K_i) = x_\omega(H_i) = 2(1 - 1/k)$. As in the preamble, let $K = K_1 \# K_2$ and $H = H_1 \# H_2$ so that H is a $(2, 1)$ -bridge surface for K . Let \bar{S} be the summing sphere. By the calculations in the preamble,

$$4 - \frac{2}{k} = x_\omega(H) = x_\omega(S^3, K_1) + x_\omega(S^3, K_2) - x_\omega(\bar{S}).$$

Let H' be a (g, b) -bridge surface for K such that $x_\omega(H') = x_\omega(S^3, K)$. In particular, for that genus g , the number of punctures $2b$ is minimal. Also, since $x_\omega(H') \leq x_\omega(H)$:

$$g \leq 2 - \frac{(k - 1)(b - 1)}{k} < 3.$$

If $g = 0$, then by Schubert’s result [34] that bridge number is (-1) -additive, we have $b = 2|q| - 1$. In which case,

$$2 - 1/k = x_\omega(H)/2 \geq x_\omega(H')/2 \geq (2|q| - 1)(1 - 1/k) - 1$$

a contradiction to the fact that $|q| \geq 5$ and $k \geq 2$.

If $g = 1$, our inequality shows that $b \leq 3$. Since K is not the unknot, $b \geq 1$. Doll studied the situation when $g = 1$ and showed that (for our choice of K_1 and K_2), $b \geq |q| - 1$. (The proof for arbitrary K_1 and K_2 can be found in the solution [3, section 5] of his Conjecture (1.1’) for the case $g = 1$.) In which case,

$$4 - 2/k = x_\omega(H) \geq x_\omega(H') = 2b(1 - 1/k) \geq 2(|q| - 1)(1 - 1/k) \geq 8(1 - 1/k).$$

But this contradicts the assumption that $k \geq 2$.

If $g = 2$, and $b = 0$, then the tunnel number of K would be 1, contradicting the fact that K is composite. Thus, $g = 2$ and $b \geq 1$. Since $x_\omega(H') \leq x_\omega(H)$, we have $b = 1$ and so, $x_\omega(H') = x_\omega(H)$.

Thus, $x_\omega(S^3, K) = x_\omega(H') = x_\omega(H) = x_\omega(S^3, K_1) + x_\omega(S^3, K_2) - x_\omega(S)$. Passing to the k -fold cyclic branched cover W over K , with G the deck group and S the preimage of \bar{S} , produces the desired examples.

4. Orbifold Thin Position

Building on a long line of work concerning thin position, beginning with Gabai [6] and particularly including Scharlemann–Thompson [28, 29], Hayashi–Shimokawa [10]

and Tomova [40], Taylor–Tomova created a thin position theory for spatial graphs in 3-manifolds. In this section, we explain the minor adaptations needed to make it work for 3-orbifolds.

Definition 4.1 (Taylor–Tomova [36]). Let M be a compact, orientable 3-manifold and $T \subset M$ a spatial graph. A properly embedded closed surface $\mathcal{H} \subset (M, T)$ is a **multiple vp-bridge surface** if the following hold:

- (i) \mathcal{H} is the disjoint union of \mathcal{H}^+ and \mathcal{H}^- where each of \mathcal{H}^\pm are the union of components of \mathcal{H} ;
- (ii) each component of $(M, T) \setminus \mathcal{H}$ is a vp-compressionbody;
- (iii) $\mathcal{H}^+ = \bigcup \partial_+ C$ and $\mathcal{H}^- \cup \partial M = \bigcup \partial_- C$ where each union is over all components $(C, T_C) \sqsubset (M, T) \setminus \mathcal{H}$.

When \mathcal{H} has a transverse orientation, we can consider the **dual digraph**; this is the digraph with a vertex for each vp-compressionbody and an oriented edge corresponding to each component of \mathcal{H} . We consider such \mathcal{H} equipped with a transverse orientation such that the dual digraph is acyclic and each $(C, T_C) \sqsubset (M, T) \setminus \mathcal{H}$ is a cobordism from $\partial_- C$ to $\partial_+ C$. Equipped in this way, \mathcal{H} is an **oriented multiple vp-bridge surface**. We let $\mathbb{H}(M, T)$ be the set of oriented multiple vp-bridge surfaces up to isotopy transverse to T . When (M, T) is an orbifold, we call the elements of $\mathbb{H}(M, T)$ **multiple orbifold Heegaard surfaces**.

After assigning an orientation, every orbifold Heegaard surface for a nice orbifold (M, T) is an element of $\mathbb{H}(M, T)$. Since every spatial graph in a 3-manifold can be put into bridge position with respect to any Heegaard surface for the 3-manifold, $\mathbb{H}(M, T) \neq \emptyset$. For $\mathcal{H} \in \mathbb{H}(M, T)$, observe that $\mathcal{H}^- = \emptyset$ if and only if $\mathcal{H} = \mathcal{H}^+$ is connected. If $T = \emptyset$, multiple vp-bridge surfaces induce the generalised Heegaard splittings of [31]. The following lemma is a straightforward extension of Lemma 2.7.

LEMMA 4.2. *Suppose that $(M, T) \rightarrow (M', T')$ is a finite-sheeted orbifold cover and that $\mathcal{H} \in \mathbb{H}(M', T')$. Then the preimage of \mathcal{H} is a multiple orbifold Heegaard surface for (M, T) .*

When (M, T) does not contain any once-punctured spheres, Taylor and Tomova [37] define an invariant called “net extent” on elements of $\mathbb{H}(M, T)$. We now adapt that invariant to the orbifold context.

Definition 4.3. Suppose that (M, T) is a nice orbifold. For $\mathcal{H} \in \mathbb{H}(M, T)$, the **net Heegaard characteristic** is

$$\text{net } x_\omega(\mathcal{H}) = x_\omega(\mathcal{H}^+) - x_\omega(\mathcal{H}^-).$$

We define the **net Heegaard characteristic** of (M, T) to be:

$$\text{net } x_\omega(M, T) = \min \{ \text{net } x_\omega(\mathcal{H}) : \mathcal{H} \in \mathbb{H}(M, T) \}.$$

We define $\text{net } x_\omega(W; G)$ similarly, but minimise only over invariant $\mathcal{H} \in \mathbb{H}(W, \emptyset)$.

Proposition 5.3 below ensures that $\text{net } x_\omega(M, T)$ is well-defined and that there exists $\mathcal{H} \in \mathbb{H}(M, T)$ with $\text{net } x_\omega(\mathcal{H}) = \text{net } x_\omega(M, T)$. The proof of the next lemma follows easily from Lemma 4.2.

LEMMA 4.4. *Suppose that W has quotient orbifold (M, T) . Then*

$$\text{net } x_\omega(W; G) = |G| \text{net } x_\omega(M, T).$$

In [36], Taylor and Tomova defined a set of operations on elements of $\mathbb{H}(M, T)$ and used them to define a partial order called **things to** and denoted $\mathcal{J} \rightarrow \mathcal{H}$. A minimal element in the partial order is said to be **locally thin**. The operations involved in the definition of the partial order are all versions of the traditional “destabilisation” and “weak reduction” of Heegaard splittings of 3-manifolds and “unperturbing” of bridge surfaces for links. We do not need the precise definitions of the operations in this text, but we do need the following information. For our purposes, we group the operations into four categories (deferring to [36] for precise definitions):

- (I) destabilisation, meridional destabilisation, ∂ -destabilisation, meridional ∂ -destabilisation, ghost ∂ -destabilisation, meridional ghost ∂ -destabilisation;
- (II) unperturbing, undoing a removable arc;
- (III) consolidation;
- (IV) untelescoping.

The next lemma summarises the key aspects of these operations. Pay particular attention to the disc D used in the operations (I). In Lemma 4.6, we will need to analyse the effect of compressing along this disc on net Heegaard characteristic.

LEMMA 4.5 (*Taylor–Tomova*). *The following hold:*

- (i) *all of the operations listed in (I) involve replacing a thick surface $J \sqsubset \mathcal{J}^+$ with a new thick surface H such that H is obtained from J by compressing along a compressing disc or cut disc D and, if ∂D separates J , discarding a component. The component that is discarded is parallel to a surface obtained by tubing together some components of ∂M and vertices of T along edges of T disjoint from \mathcal{H} ;*
- (ii) *all of the operations in (II) remove two punctures from a thick surface $J \sqsubset \mathcal{J}^+$;*
- (iii) *consolidation removes a thick surface and a thin surface from \mathcal{J} that together bound a product vp -compressionbody with interior disjoint from \mathcal{J} ;*
- (iv) *untelescoping replaces a thick surface $J \sqsubset \mathcal{J}^+$ with two new thick surfaces H_1 and H_2 and creates additional thin surfaces F . These surfaces arise from a pair of disjoint sc -discs on opposite sides of J . H_1 and H_2 are each obtained (up to isotopy) by compressing along one of the discs and F is obtained by compressing along both.*

LEMMA 4.6. *Suppose that (M, T) is an orbifold. As an invariant on $\mathbb{H}(M, T)$, $\text{net } x_\omega$ is non-increasing under the partial order \rightarrow .*

Proof. This lemma follows fairly directly from Lemma 4.5. Suppose that one of the moves in a thinning sequence replaces a thick surface J with another thick surface H . Consider, first, the possibility that the move was of type (I). Let D be the disc we compress along, as in Lemma 4.5. If ∂D is non-separating on J , we have

$$x_\omega(H) = x_\omega(J) - 2 + 2(1 - 1/\omega(D)) \leq x_\omega(J).$$

Thus, in such a case, the move does not increase net x_ω . If ∂D separates J , recall that after compressing we discard one component J' of the resulting surface. That is, $H \cup J'$ is the result of the compression of J . We have

$$x_\omega(H) = (x_\omega(J) - 1 + (1 - 1/\omega(D))) + (-x_\omega(J') - 1 + (1 - 1/\omega(D))).$$

Suppose, in order to obtain a contradiction, that $x_\omega(H) > x_\omega(J)$. Then

$$0 \geq -1 + (1 - 1/\omega(D)) > x_\omega(J') \geq -\chi(J') + (1 - 1/\omega(D))$$

where the last inequality follows from the fact that J' contains a scar from the compression by D . Since J' is a closed surface, it must be a sphere. We recall from Lemma 4.5 that it is parallel to a certain surface S obtained by tubing together components of ∂M and vertices of T along edges of T disjoint from \mathcal{H} . Let Γ be the graph with a vertex for each component of ∂M and each vertex of T that goes into the creation of S and with edges the edges we tube along. Since J' is a sphere and is parallel to S , Γ must be a tree and each component of ∂M that is a vertex of T is a sphere. If Γ has an edge, there are at least two leaves and, by our definition of orbifold, each must be incident to at least two vertical arcs, giving S at least 4 punctures. But in that case $x_\omega(J') \geq 0$, a contradiction. Thus, Γ is an isolated vertex; that is, J' is parallel to either a component of ∂M or to a vertex of T . If it is a component of ∂M , then $x_\omega(J') \geq 0$, by hypothesis. Thus, J' is parallel to a vertex v of T . The vertex v is trivalent with incident edges having weights a, b, c and

$$0 < -x_\omega(J') - 1 + (1 - 1/\omega(D)) = -1 + 1/a + 1/b + 1/c - 1/\omega(D).$$

If $\omega(D) = 1$, then we have a contradiction. If $\omega(D) \neq 1$, then D was a cut disc and so one of a, b, c is equal to $\omega(D)$. Thus, in this case also, we have a contradiction. We conclude that $x_\omega(H) \leq x_\omega(J)$ and that none of the moves of type (I) increase net x_ω .

The moves of type (II) remove two punctures from a thick surface and so cannot increase net x_ω . If $H \sqsubset \mathcal{H}^+$ and $F \sqsubset \mathcal{H}^-$ are parallel, then $x_\omega(H) = x_\omega(F)$ and so consolidation does not change net x_ω . Finally, consider the operation of untelescoping. The three new surfaces H_1, H_2 , and F (as in the statement of Lemma 4.5) are all obtained by compressions along sc-discs for J . An easy computation shows that $x_\omega(J) = x_\omega(H_1) + x_\omega(H_2) - x_\omega(F)$ and so untelescoping also leaves net x_ω unchanged.

The next theorem is key to our endeavors. We have stated only what we need for this paper. We say that \mathcal{H}^+ is **sc-strongly irreducible** if it is not possible to untelescope it (i.e. use move (IV) above).

THEOREM 4.7 (Taylor–Tomova). *Suppose that T is a spatial graph in an orientable 3-manifold M such that no component of ∂M is a sphere with two or fewer punctures and there is no once-punctured sphere in (M, T) . Then for every $\mathcal{J} \in \mathbb{H}(M, T)$ there exists a locally thin $\mathcal{H} \in \mathbb{H}(M, T)$ such that $\mathcal{J} \rightarrow \mathcal{H}$. Furthermore, for any locally thin \mathcal{H} the following hold:*

- (i) Each component of \mathcal{H}^+ is sc-strongly irreducible in $(M, T) \setminus \mathcal{H}^-$;
- (ii) \mathcal{H}^- is c-essential in (M, T) ;
- (iii) If (M, T) contains a c-essential sphere with 3 or fewer punctures, then so does \mathcal{H}^- ;
- (iv) If (M, T) is irreducible and if a component of \mathcal{H}^+ is a sphere with three or fewer punctures then $\mathcal{H} = \mathcal{H}^+$;
- (v) If $(C, T_C) \sqsubset (M, T) \setminus \mathcal{H}$ is a trivial product compressionbody, then $\partial_- C \subset \partial M$.

Proof. We note that in [36] saying that T is irreducible, by definition, means that (M, T) does not contain a once-punctured sphere. The existence of \mathcal{H} given \mathcal{J} is [36, theorem 6.17]. Conclusions (i), (ii), and (v) can be found as conclusions (i), (iv) and (iii) of [36, theorem 7.6], respectively. Conclusion (iii) follows from [36, theorem 8.2]. Conclusion (iv) is similar to the proof of conclusion (v) of [36, theorem 7.6]. The details are similar to some of the arguments that follow, so we omit them here.

Examining the proof of conclusion (iii) in Theorem 4.7 provides us with more information about orbifolds. In particular, it produces another proof of the existence of systems of summing spheres. By Theorem 2.4, this also implies that efficient systems of summing spheres exist. The proof is nearly identical to that of [36, theorem 8.2] and [37, proposition 5.1]. Since the argument is completely topological, it is also the case that if \mathcal{H} is locally thin, then \mathcal{H}^- contains a collection of turnovers cutting (M, T) into suborbifolds that contain no essential turnovers. For convenience, we provide the proof in the Appendix.

COROLLARY 4.8. *Suppose that (M, T) is a nice orbifold and let $\mathcal{H} \in \mathbb{H}(M, T)$ be locally thin. Then \mathcal{H}^- contains a system of summing spheres for (M, T) .*

THEOREM 4.9. *Suppose that (M, T) is a nice orbifold. Let $S \subset (M, T)$ be an efficient system of summing spheres. Then*

$$\text{net } x_\omega(M, T) \geq \text{net } x_\omega((M, T)|_S) - x_\omega(S).$$

If each component of S is separating, then equality holds.

Proof. We first show that $\text{net } x_\omega(M, T) \geq \text{net } x_\omega((M, T)|_S) - x_\omega(S)$. Choose $\mathcal{J} \in \mathbb{H}(M, T)$ such that $\text{net } x_\omega(\mathcal{J}) = \text{net } x_\omega(M, T)$. This is possible by Proposition 5.3 below. By Theorem 4.7 and Lemma 4.6, there exists a locally thin $\mathcal{H} \in \mathbb{H}(M, T)$ with $\mathcal{J} \rightarrow \mathcal{H}$ and $\text{net } x_\omega(\mathcal{H}) = \text{net } x_\omega(M, T)$. By Corollary 4.8, there exists an efficient set of summing spheres $S \subset \mathcal{H}^-$. As we remarked, S is unique up to orbifold homeomorphism, as is $(M, T)|_S$. When we split (M, T) open along S , each component of S is converted to two boundary components of $(M, T) \setminus S$. Boundary components are not included in the sum in the definition of $\text{net } x_\omega$ and capping them off with trivial ball compressionbodies does not change that. The result follows.

When each component of S is separating, the proof that $\text{net } x_\omega(M, T) \leq \text{net } x_\omega((M, T)|_S) - x_\omega(S)$ is nearly identical to that of [37, theorem 5.5]. In each component of $(M, T)|_S$ mark the points where sums will be performed. Since each component of S is separating, the dual graph to S is a tree. In each component (M_i, T_i) of $(M, T)|_S$ choose $\mathcal{H}_i \in \mathbb{H}(M_i, T_i)$ such that $\text{net } x_\omega(\mathcal{H}_i) = \text{net } x_\omega(M_i, T_i)$. Again this is possible by Proposition 5.3 below. By transversality we may also assume each \mathcal{H}_i is disjoint from the marked points. As in the proof of [37, theorem 5.5], we may reverse orientations on the \mathcal{H}_i as necessary to ensure that their union with S is an oriented multiple orbifold Heegaard surface for (M, T) . The desired inequality follows.

THEOREM 4.10. *There exists an invariant system of summing spheres $S \subset W$ such that*

$$\text{net } x_\omega(W; G) \geq \text{net } x_\omega(W|_S; G) - 2|S|,$$

and $W|_S$ is irreducible. If each component of S is separating, then equality holds.

Proof. If W is not orbifold-reducible, then the theorem is vacuously true. Otherwise, it is orbifold-reducible. By the Equivariant Sphere Theorem, the quotient orbifold (M, T) is orbifold-reducible. We verify that (M, T) is nice. Note that as T is the singular set, no edge has infinite weight. Let $P \sqsubset \partial M$ be a 2-sphere. If $x_\omega(P) < 0$, then its pre-image in ∂W is the union of essential spheres, but there are none. Thus, $x_\omega(P) \geq 0$. Since (M, T) is covered by a manifold, there are no bad 2-suborbifolds. Consequently, (M, T) is nice.

By Lemma 4.4, $\text{net } x_\omega(W; G) = |G| \text{net } x_\omega(M, T)$. By Theorem 4.9, there is an efficient system of summing spheres \bar{S} for (M, T) such that $\text{net } x_\omega(M, T) \geq \text{net } x_\omega((M, T)|_{\bar{S}})$ and equality holds if every component of \bar{S} is separating. Let S be the lift of \bar{S} to W . If a component of \bar{S} is non-separating, each component of its preimage in W would be non-separating.

We have:

$$\text{net } x_\omega(W; G) = |G| \text{net } x_\omega(M, T) \geq |G| \text{net } x_\omega((M, T)|_{\bar{S}}) - |G|x_\omega(\bar{S}) = \text{net } x_\omega(W|_S; G) - x_\omega(S).$$

If each component of S is separating, then so is each component of \bar{S} and equality holds. Let W' be a component of $W|_S$. Its image in (M, T) is a component of $(M, T)|_S$. If W' were orbifold-reducible, then its image would be also, by the Equivariant Sphere Theorem. But this contradicts the fact that \bar{S} is an efficient system of summing spheres.

Suppose that some $S_0 \sqsubset S$ is inessential. Then it bounds a 3-ball $B \subset W$. Without loss of generality, we may assume that S_0 is innermost; i.e. the interior of B is disjoint from S . The image of B in (M, T) is then the quotient of B by its stabiliser. By [45], it is a trivial ball compressionbody and its boundary is inessential. This contradicts the fact that \bar{S} is efficient. Thus, each component of S is essential.

Remark 4.11. The proof of Theorem 4.9 demonstrates the advantage that invariant multiple vp-bridge surfaces have over invariant Heegaard surfaces. Although there is no guarantee that if W is reducible there is an equivariant sphere intersecting a minimal equivariant Heegaard splitting in a single closed loop, we can guarantee that there is an equivariant sphere showing up as a thin surface in an equivariant generalised Heegaard splitting of W .

5. Lower Bounds

The main purpose of this section is to find lower bounds on $\text{net } x_\omega(M, T)$ for an orbifold (M, T) and use that to prove Theorems 5.5 and 5.10. Along the way, we prove Proposition 5.3 which guarantees that $\text{net } x_\omega(M, T)$ is well-defined and that there exists $\mathcal{H} \in \mathbb{H}(M, T)$ with $\text{net } x_\omega(\mathcal{H}) = \text{net } x_\omega(M, T)$.

5.1. Analysing Orbifold Compressionbodies.

Definition 5.1. A **lens space** is a closed 3-manifold of Heegaard genus 1, other than S^3 or $S^1 \times S^2$. A **core loop** in a solid torus $D^2 \times S^1$ is a curve isotopic to $\{\text{point}\} \times S^1$. A **core loop** in a lens space or $S^1 \times S^2$ is a knot isotopic to the core loop of one half of a genus 1 Heegaard splitting. A **Hopf link** in S^3 , $S^1 \times S^2$, or a lens space is a 2-component link such that there is a Heegaard torus separating the components and so that each component is a core loop for the solid tori on opposite sides of a Heegaard torus. A **pillow** (C, T_C) is a vp-compressionbody with boundary a 4-punctured sphere that is the result either of joining two (3-ball, arc) trivial ball compressionbodies by an unweighted 1-handle or joining two (3-ball, trivalent graph) compressionbodies by a weighted 1-handle. (See Figure 2.) An orbifold that is a pillow or trivial ball compressionbody is **Euclidean** if the boundary surface is. A **Euclidean double**

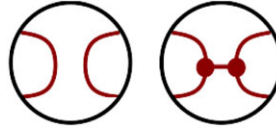


Fig. 2. The two types of pillow.

pillow is a pair (S^3, T) with an orbifold bridge surface H such that each of $(S^3, T) \setminus H$ is a Euclidean pillow. (The terminology stems from the fact that Euclidean orbifolds admit a complete metric locally modeled on Euclidean 2 or 3-space.)

If (C, T_C) is the disjoint union of orbifold compressionbodies, let $N(C, T_C) = x_\omega(\partial_+ C) - x_\omega(\partial_- C)$. Our key identity for an multiple orbifold Heegaard surface $\mathcal{H} \in \mathbb{H}(M, T)$ is:

$$2 \text{net } x_\omega(M, T) - x_\omega(\partial M) = \sum_{(C, T_C)} N(C, T_C), \tag{1}$$

where we sum over the vp-compressionbodies $(C, T_C) \sqsubset (M, T) \setminus \mathcal{H}$. This follows immediately from the fact that each component of \mathcal{H}^+ and each component of \mathcal{H}^- appears exactly twice as a boundary component of $(M, T) \setminus \mathcal{H}$.

LEMMA 5.2. *Suppose that (C, T_C) is an orbifold compressionbody with no component of $\partial_- C$ a once-punctured sphere. If $N(C, T_C) < 0$, then (C, T_C) is a trivial ball compressionbody. Also, if $N(C, T_C) = 0$ and $\partial_- C = \emptyset$, then (C, T_C) is one of:*

- (i) *Euclidean trivial ball compressionbody;*
- (ii) *Euclidean pillow;*
- (iii) *(solid torus, \emptyset);*
- (iv) *(solid torus, core loop).*

Proof. Let Δ be a complete collection of sc-discs for (C, T_C) such that ∂ -reducing (C, T_C) along Δ results in trivial vp-compressionbodies (C', T'_C) . Each disc of Δ , leaves 2 “scars” on $\partial_+ C'$. If E is a scar, let $\omega(E) = 1$ if it is unpunctured and otherwise let $\omega(E)$ be the weight of the puncture. Let $(C_0, T_0) \sqsubset (C', T'_C)$. Let $N'(C_0, T_0)$ be equal to the sum of $N(C_0, T_0)$ with $1/\omega(E)$ for all scars E on $\partial_+ C_0$. Observe that

$$N(C, T_C) = \sum_{(C_0, T_0)} N'(C_0, T'_0),$$

where the sum is taken over all $(C_0, T_0) \sqsubset (C', T')$.

If (C_0, T_0) is a product compressionbody then $N'(C_0, T_0) \geq N(C_0, T_0) = 0$ with equality if and only if every scar on $\partial_+ C_0$ has weight ∞ . Suppose that (C_0, T_0) is a trivial ball compressionbody.

Case 1: $T_0 = \emptyset$.

If $\Delta = \emptyset$, then $(C, T_C) = (C_0, T_0)$ and $N(C, T_C) = -2$. Otherwise, by the choice of Δ , $\partial_+ C_0$ contains at least 2 scars, each of weight 1. If it contains exactly 2, then (C, T_C) is (solid torus, \emptyset). If it has at least 3 scars, then $N'(C_0, T_0) \geq 1$.

Case 2: T_0 is an arc of weight k .

If $\Delta = \emptyset$, then $(C, T_C) = (C_0, T_0)$ and $N(C, T_C) = -2/k \geq -1$. If $N(C, T_C) = 0$, then $k = \infty$. If $\Delta \neq \emptyset$, then $\partial_+ C_0$ contains at least one scar. By our choice of Δ , either (C, T_C) is (solid torus, core loop) or $\partial_+ C_0$ contains at least 1 scar of weight 1. In which case, $N'(C_0, T_0) \geq 0$. Equality holds if and only if $k = 2$, there is exactly one scar and it has weight 1.

Case 3: T_0 contains an interior vertex.

Note that $N(C_0, T_0) = x_\omega(\partial_+ C) = x_\omega(v) < 0$ where v is the internal vertex of T_0 . If $\Delta = \emptyset$, then we have our result. If $\Delta \neq \emptyset$, then $\partial_+ C_0$ contains at least one scar and it either has weight 1 or has weight equal to the weight of one of the punctures on $\partial_+ C$. In which case, $N'(C_0, T_0) \geq 0$. Equality holds only when there is exactly one scar, it contains a puncture, and the two punctures not contained in the scar both have weight 2.

This concludes our analysis of the individual cases and, in particular, we may assume that $\Delta \neq \emptyset$ and that (C, T_C) is neither (solid torus, \emptyset) or (solid torus, core loop). By our analysis, each component $(C_0, T_0) \sqsubset (C', T'_C)$ has $N'(C_0, T_0) \geq 0$. Thus, $N(C, T_C) \geq 0$. Suppose that $N(C, T_C) = 0$. Then $N'(C_0, T_0) = 0$ for each component of (C', T'_C) . Consequently, each component is one of:

- (i) a product compressionbody such that every scar has infinite weight;
- (ii) a trivial ball compressionbody containing an arc and with a single scar of weight 1;
- (iii) a trivial ball compressionbody containing a trivalent vertex and with edges of weight $(2, 2, k)$ with $k \geq 2$. It has a single scar of weight k .

The compressionbody (C, T_C) can be reconstructed by attaching possibly weighted 1-handles to the scars on (C', T') . Thus our result holds if $N(C, T_C) \leq 0$.

We next have two propositions whose proofs are closely related.

PROPOSITION 5.3. *Suppose that (M, T) is a nice orbifold and that $\mathcal{H} \in \mathbb{H}(M, T)$. Then either $\text{net } x_\omega(\mathcal{H}) \geq (1/2)x_\omega(\partial M)$ or (M, T) is one of $\mathbb{S}(0)$, $\mathbb{S}(2)$ or $\mathbb{S}(3)$. Furthermore, there exists a locally thin $\mathcal{H} \in \mathbb{H}(M, T)$ with $\text{net } x_\omega(\mathcal{H}) = \text{net } x_\omega(M, T)$. If (M, T) is an $\mathbb{S}(i)$, for $i \in \{0, 2, 3\}$, then any locally thin \mathcal{H} is an i -punctured sphere.*

PROPOSITION 5.4. *Suppose that (M, T) is a nice, closed orbifold that is orbifold-irreducible and that $\mathcal{H} \in \mathbb{H}(M, T)$ is locally thin. Then either $\text{net } x_\omega(\mathcal{H}) \geq 1/6$ or one of the following exceptional cases holds:*

- (i) $\mathbb{S}(0)$, $\mathbb{S}(2)$, or $\mathbb{S}(3)$;
- (ii) $M = S^3$ or a lens space and T is a core loop, Hopf link and \mathcal{H} is an unpunctured torus;
or
- (iii) (M, T) is a Euclidean double pillow and \mathcal{H} is a four-punctured sphere.

The remainder of the section is devoted to the proofs of these propositions. A key book-keeping device for a vp-compressionbody (D, T_D) is its **ghost arc graph**. This is the graph Γ whose vertices are the components of $\partial_- D$ and the vertices of T_D . The ghost arcs of T_D

are the edges. For example, if (D, T_D) has a single ghost arc and it joins distinct components of $\partial_- D$, then Γ is a single edge. The key observation is that if $\partial_+ D$ is a sphere, then Γ is acyclic and if $\partial_+ D$ is a torus, then Γ contains at most one cycle. If it contains a cycle, then $\partial_- D$ is the union of spheres (that is, it does not contain a torus).

Begin by assuming only that (M, T) is a nice orbifold. Let $\mathcal{H} \in \mathbb{H}(M, T)$ be locally thin. Recall from Corollary 4.8 that \mathcal{H}^- contains an efficient system of summing spheres S . Assume, for the time being, that $S = \emptyset$; equivalently, that (M, T) is orbifold-irreducible. Since each component of \mathcal{H}^- is c-essential in (M, T) , this also implies that no $S_0 \sqsubset \mathcal{H}^-$ is a sphere with $|S_0 \cap T| \leq 3$ and $x_\omega(S_0) < 0$.

Case 1: Some $(C, T_C) \sqsubset (M, T) \setminus \mathcal{H}$ has $N(C, T_C) < 0$.

By Lemma 5.2, (C, T_C) is a trivial ball compressionbody. Observe that $\partial_+ C$ is a sphere with 0, 2 or 3 punctures. Let $(D, T_D) \sqsubset (M, T) \setminus \mathcal{H}$ be the other vp-compressionbody having $\partial_+ D = \partial_+ C$. If $\partial_- D = \emptyset$, then $M = C \cup D$ and (D, T_D) is also a trivial ball compressionbody. In this case, (M, T) is either $\mathbb{S}(0)$, $\mathbb{S}(2)$ or $\mathbb{S}(3)$. Assume there exists $F \sqsubset \partial_- D$.

Let Γ be the ghost arc graph for (D, T_D) . As $\partial_+ D$ is a sphere, Γ is acyclic and the components of $\partial_- D$ are all spheres. Since (M, T) is nice, none of them are once-punctured. Since $S = \emptyset$, F is at least thrice-punctured and has $x_\omega(F) \geq 0$. If Γ contains an isolated vertex, (D, T_D) is a product. Since \mathcal{H} is locally thin, $F = \partial_- D \subset \partial M$. If T_C contains an interior vertex v , we must have $0 > x_\omega(v) = x_\omega(F) \geq 0$, a contradiction. If T_C does not contain an interior vertex, then F is twice-punctured, contradicting our definition of nice 3-orbifold. Thus, we may assume that Γ does not have an isolated vertex. Since no component of $\partial_- D$ is a twice-punctured sphere, each leaf of Γ is incident to at least two vertical arcs, so there is at most one leaf. Since Γ is acyclic, this is a contradiction. Consequently, (M, T) is one of the exceptional cases in the statement of Proposition 5.3.

Case 2: Every $(C, T_C) \sqsubset (M, T) \setminus \mathcal{H}$ has $N(C, T_C) \geq 0$.

By (i), we see that $\text{net } x_\omega(\mathcal{H}) \geq x_\omega(\partial M)/2$. Let L be the product of all the finite weights on T . Note that for any $\mathcal{J} \in \mathbb{H}(M, T)$, the quantity $2L \text{net } x_\omega(\mathcal{J})$ is an integer, as is $2Lx_\omega(\partial M)/2$. By Theorem 4.7, for any $\mathcal{J} \in \mathbb{H}(M, T)$, there exists a locally thin $\mathcal{H} \in \mathbb{H}(M, T)$ with $\mathcal{J} \rightarrow \mathcal{H}$. By Lemma 4.6, $\text{net } x_\omega(\mathcal{J}) \geq \text{net } x_\omega(\mathcal{H})$. If (M, T) is one of the exceptional cases from Proposition 5.3, then by the analysis in Case 1, \mathcal{H} is connected and so $L \text{net } x_\omega(\mathcal{J})$ is bounded below by a constant depending only on (M, T) . If (M, T) is not one of the exceptional cases from Proposition 5.3, then we see that $2L \text{net } x_\omega(\mathcal{J}) \geq 2Lx_\omega(\partial M)/2 \geq 0$. Thus, in either case, since the invariant $2L \text{net } x_\omega$ defined on $\mathbb{H}(M, T)$ is integer-valued and bounded below by a number depending only on it achieves its minimum on a locally thin element of $\mathbb{H}(M, T)$. That element also minimizes $\text{net } x_\omega$. This concludes the analysis when $S = \emptyset$ for the proof of Proposition 5.3.

Now suppose that $S \neq \emptyset$. Expand S to include all summing spheres in \mathcal{H}^- ; continue to call it S . As we have observed previously,

$$\text{net } x_\omega(\mathcal{H}) = \text{net } x_\omega(\mathcal{H} \setminus S) - x_\omega(S).$$

Let $(M_0, T_0) \sqsubset (M, T)|_S$ and let $\mathcal{H}_0 = (\mathcal{H} \setminus S) \cap M_0$. If $\text{net } x_\omega(\mathcal{H}_0) < 0$, then (M_0, T_0) is one of the exceptional cases from Proposition 5.3. Since \mathcal{H} is locally thin, each component

of S is essential, so $(M_0, T_0) \neq \mathbb{S}(0)$. If $(M_0, T_0) = \mathbb{S}(2)$, then at least one of the components S' of S used to sum with (M_0, T_0) must be unpunctured. Thus, if T_0 has weight k , we have:

$$\text{net } x_\omega(\mathcal{H}_0) - \frac{1}{2}x_\omega(S') = -\frac{2}{k} + 1 \geq 0.$$

Similarly, if (M_0, T_0) is an $\mathbb{S}(3)$, then at least one of the components $S' \sqsubset S$ used to sum with (M_0, T_0) must be either unpunctured or twice punctured and with the weight of the punctures equal to the weight c of one of the edges of T_0 . In that case, letting a, b be the weights of the other punctures,

$$\text{net } x_\omega(\mathcal{H}_0) - \frac{1}{2}x_\omega(S') \geq 1 - \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \frac{1}{2} \cdot \frac{2}{c} \geq 0.$$

Consequently, $\text{net } x_\omega(\mathcal{H}) \geq x_\omega(\partial M)/2$, even in this situation. As before, the quantity $L \text{net } x_\omega$ is an integer-valued invariant on $\mathbb{H}(M, T)$ bounded below by a constant depending only on (M, T) and so, as before, there is a locally thin $\mathcal{H} \in \mathbb{H}(M, T)$ with $\text{net } x_\omega(\mathcal{H}) = \text{net } x_\omega(M, T)$. This concludes the proof of Proposition 5.3.

Henceforth, suppose that (M, T) is closed and orbifold-irreducible and not one of the exceptional cases from Proposition 5.3. By our previous remarks, this implies that $N(C, T_C) \geq 0$ for every $(C, T_C) \sqsubset (M, T) \setminus \mathcal{H}$. The dual digraph to \mathcal{H} is acyclic, so it has at least one source and one sink. The sources and sinks are exactly those $(C, T_C) \sqsubset (M, T) \setminus \mathcal{H}$ with $\partial_- C = \emptyset$. Suppose that (C, T_C) is one such. Note that $N(C, T_C) = x_\omega(\partial_+ C)$. Let $(D, T_D) \sqsubset (M, T) \setminus \mathcal{H}$ be the other orbifold compressionbody with $\partial_+ D = \partial_+ C = H$.

Observe that $x_\omega(H) \geq -\chi(H) + |H \cap T|/2$. Equality holds only if every puncture on H has weight 2. Consequently, if $1/6 > x_\omega(H)$, then H is a sphere with $|H \cap T| \leq 4$. If $|H \cap T| \leq 3$, then by our analysis above (M, T) is one of the exceptional cases from Proposition 5.3. Consider, therefore, the case that $|H \cap T| = 4$. If at least one puncture does not have weight 2, then $x_\omega(H) \geq 1/6$, so assume that each puncture has weight 2. If $\partial_- D = \emptyset$, then \mathcal{H} divides (M, T) into two Euclidean pillows. Suppose $\partial_- D \neq \emptyset$ and let Γ be the ghost arc graph for (D, T_D) as above. It is acyclic. Each component of $\partial_- D$ is a sphere with at least three punctures, since (M, T) is orbifold-irreducible and nice. Also $\partial_- D \subset \partial M$ since M is closed. An isolated vertex of Γ is a sphere incident to at least three vertical arcs and a leaf is a sphere incident to at least two vertical arcs. Since H has four punctures, if Γ has an isolated vertex, that vertex is the entirety of Γ and it is incident to four vertical arcs. This implies (D, T_D) is a product and contradicts local thinness of \mathcal{H} . Thus, Γ has two leaves, each incident to two vertical arcs. At least one of those leaves F is a component of $\partial_- D$ (the other may be a vertex of T). Since (M, T) is orbifold-irreducible, $x_\omega(F) \geq 0$. Consequently, at least two of the arcs incident to F have weight at least 3. At least one of those is a vertical arc, contradicting the fact that each puncture of H has weight 2. Consequently, $N(C, T_C) = x_\omega(H) \geq 1/6$.

Since the dual digraph to \mathcal{H} has at least one source and one sink either (M, T) is one of the exceptional cases of Proposition 5.3, or \mathcal{H} is a four punctured sphere dividing (M, T) into two Euclidean pillows, or $2 \text{net } x_\omega(\mathcal{H}) \geq 2 \cdot (1/6)$. This concludes the proof of Proposition 5.4.

Figure 3 shows that our bound of $1/6$ is asymptotically sharp.

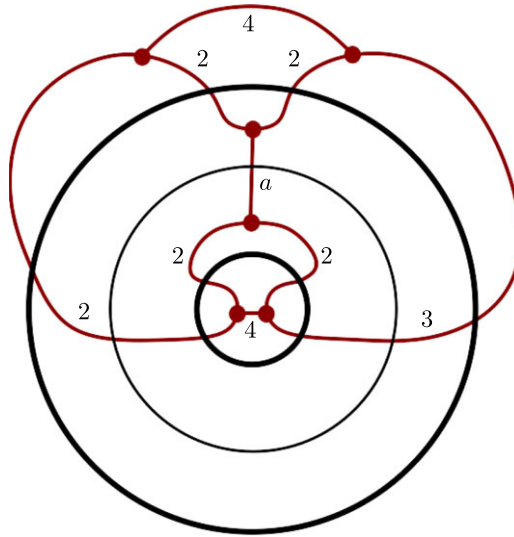


Fig. 3. An example of an orbifold with underlying 3-manifold S^3 . The thick circles represent thick spheres and the thin circle is a thin sphere of a multiple vp-bridge surface \mathcal{H} . Arbitrary gluing maps preserving the punctures pointwise can be used along the thick spheres. For $a \in \mathbb{N}_2^\infty$ we have $\text{net } x_\omega(\mathcal{H}) = 1/6 + 1/a$ and the orbifold characteristic of the thin sphere is $1/6 - 1/a$. Thus, for $a \geq 6$, \mathcal{H}^- does not contain a spherical orbifold. As $a \rightarrow \infty$, we approach $1/6$.

5.2. Equivariant Heegaard Genus and the Order of the Group.

THEOREM 5.5. *Suppose that W is closed and connected and that G does not act freely. If $S \subset W$ is an invariant system of summing spheres, then*

$$g(W;G) \geq |G|(v/12 - \mu/2) + |S| + 1,$$

where v is the number of orbits of the components of $(W|_S)$ that are not S^3 or lens spaces and μ is the number of orbits of components of $W|_S$ that are homeomorphic to S^3 .

Proof. Let (M,T) be the quotient orbifold and note that no edge of T has infinite weight. Let \bar{S} be the image of S and note that each component of $(M,T)|_{\bar{S}}$ is orbifold-irreducible by the definition of “system of summing spheres” and Theorem 2.5. By Theorem 2.4, \bar{S} contains an efficient subset. Let $S_0 \subset S$ be the preimage of a component of \bar{S} that is not in our chosen efficient subset. Since W is closed, each component of $W|_S$ that is not a component of $W|_{S_0}$ is a homeomorphic to S^3 . Thus, passing from S to $S \setminus S_0$ increases the right-hand side of our inequality by $|G|/2 - 1$. Since G does not act freely, it is not the trivial group. We conclude that it is enough to prove our result when \bar{S} is efficient. Furthermore, by Theorem 2.4, we may prove it when \bar{S} is any efficient system of summing spheres for (M,T) , not merely the given one.

Recall that $x_\omega(W;G) \geq |G| \text{net } x_\omega(M,T)$. By Theorem 4.7 and Proposition 5.3, there is a locally thin $\mathcal{H} \in \mathbb{H}(M,T)$ such that $\text{net } x_\omega(\mathcal{H}) = \text{net } x_\omega(M,T)$. Furthermore, \mathcal{H}^- contains an efficient set of summing spheres (Corollary 4.8). Call them \bar{S} . Let S be the preimage of \bar{S} in W . By Theorem 4.9, we have

$$\text{net } x_\omega(M,T) = \text{net } x_\omega((M,T)|_{\bar{S}}) - x_\omega(\bar{S}).$$

Suppose some $(M_i, T_i) \sqsubset (M, T)|_{\bar{S}}$ is an $\mathbb{S}(0)$. Since \bar{S} is efficient, the matching scar to every scar in (M_i, T_i) is also in (M_i, T_i) . It follows that $(M, T)|_{\bar{S}} = (M_i, T_i)$. In which case, $T = \emptyset$, and G acts freely, contrary to hypothesis. Henceforth, we may assume no $(M_i, T_i) \sqsubset (M, T)|_S$ is an $\mathbb{S}(0)$. Suppose that (M_i, T_i) is an $\mathbb{S}(2)$. Then $x_\omega(M_i, T_i) \geq -1$ and each component of the pre-image of (M_i, T_i) in $W|_S$, is a copy of S^3 . Similarly, if (M_i, T_i) is an $\mathbb{S}(3)$, then $x_\omega(M_i, T_i) \geq -1/2$ and each component of the pre-image of (M_i, T_i) in $W|_S$ is also a copy of S^3 . Conversely, by the classification of finite groups of diffeomorphisms of S^3 , if $W_i \sqsubset W|_S$ is a copy of S^3 , then its image in $(M, T)|_S$ is an $\mathbb{S}(k)$ for some $k \in \{0, 2, 3\}$. Consequently, μ both the number of orbits of S^3 components of $W|_S$ and the number of (M_i, T_i) that are $\mathbb{S}(2)$ or $\mathbb{S}(3)$.

If (M_i, T_i) is a $(S^3, \text{Hopf link})$, (lens space, core loop), (lens space, Hopf link) or a Euclidean double pillow, then $x_\omega(M_i, T_i) = 0$. If a component W_i of $W|_S$ covers (M_i, T_i) , then W_i admits an invariant Heegaard torus, but no Heegaard sphere. Indeed, any W_i that is a lens space has $x_\omega(W_i) \geq 0$. Amalgamating a generalised Heegaard splitting of a 3-manifold produces a Heegaard surface and does not change x_ω . Thus, $\text{net } x_\omega(W_i; G) \geq 0$, whenever W_i is a lens space.

If (M_i, T_i) is neither an $\mathbb{S}(k)$ for $k \in \{0, 2, 3\}$ nor a $(S^3, \text{Hopf link})$, (lens space, core loop), (lens space, Hopf link), or a Euclidean double pillow, then by Proposition 5.4, $\text{net } x_\omega(M_i, T_i) \geq 1/6$.

Consequently,

$$x_\omega(W; G) \geq |G| \text{net } x_\omega(M, T) \geq |G|(v/6 - \mu - x_\omega(\bar{S})) = |G|(v/6 - \mu) - x_\omega(S).$$

Converting to genus, we have

$$g(W; G) \geq |G|(v/12 - \mu/2) + |S| + 1$$

EXAMPLE 5.6. Consider a lens space M containing an unknot T such that there is a 2-sphere \bar{S} in M bounding a 3-ball in M containing T . Give T weight 2 and let W be the 3-manifold such that there is an orientation preserving involution of W whose quotient produces the orbifold (M, T) . Observe that W is homeomorphic to the connected sum of M with itself. This is depicted in Figure 4.

Let S be the preimage of \bar{S} in W . Since $\bar{S} \cap T = \emptyset$, the surface S is the union of two spheres and $W|_S$ is the disjoint union of two lens spaces (each homeomorphic to M) and a copy of S^3 . Thus, $|G|v/12 - |G|\mu/2 + |S| + 1 = 2$. Notice that (M, T) has an orbifold Heegaard surface \bar{H} with $x_\omega(\bar{H}) = 1$. The surface \bar{H} is a torus intersecting T twice. The preimage of \bar{H} in W is an invariant Heegaard surface $H \subset W$ of genus 2. Thus, in this case, our inequality is sharp.

EXAMPLE 5.7. Let $M_1 = S^3$ and let T_1 be the spatial graph constructed as follows, and depicted in Figure 5. Choose a 2-bridge knot K and attach both an upper and lower tunnel to obtain T_1 . The graph T_1 has four edges that intersect a bridge sphere H_1 and two edges (the upper and lower tunnels) that are disjoint from H_1 . Give one of the edges intersecting H_1 a weight of 3 and give all the other edges weight 2. The subgraph T_0 of T_1 with edges of weight 2 is a trivial θ -curve in S^3 . The double-branched cover of S^3 over a cycle in T_0 is again S^3 and the third edge of T_0 lifts to an unknot. Taking another double-branched cover over that unknot again produces S^3 . Thus, S^3 is a 4-fold orbifold cover over (M_1, T_0) . The edge of weight 3 in T lifts to a knot κ in the cover. The sphere H_1 lifts to a bridge sphere for (S^3, κ) such that $|H_1 \cap \kappa| = 4$. In particular, κ is either the unknot or a 2-bridge knot. Since K is knotted, κ is a 2-bridge knot.

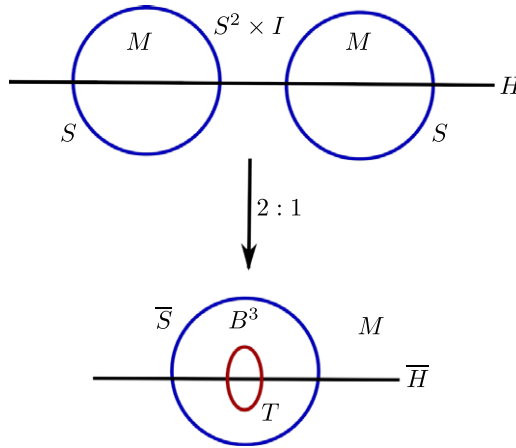


Fig. 4. The covering of the orbifold (M, T) by (W, \emptyset) in Example 5·6. The manifold M is a lens space and $W = M \times M$. The surface S is the union of two disjoint spheres; it is a double cover of the unpunctured sphere \bar{S} . The knot T is an unknot contained in 3-ball bounded by the sphere S . The horizontal lines represent bridge surfaces for (M, T) and (W, \emptyset) , with \bar{H} being a twice punctured torus and H being a genus 2 surface.

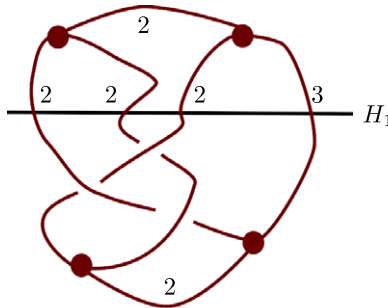


Fig. 5. The singular set T_1 and the bridge sphere H_1 for the orbifold (M_1, T_1) in Example 5·7

Let W_1 be the 3-fold branched cover of S^3 over κ . Thus, (M_1, T_1) is the quotient of W_1 by a group of diffeomorphisms of order 12 (isomorphic to the product of two groups of order 2 and one of order 3). As long as κ is not a torus knot or the figure-eight knot, W_1 is hyperbolic [2, chapter 1]. Let (M, T) be the distant sum of (M_1, T_1) with (M_2, \emptyset) , a lens space having empty singular set. Let \bar{S} be the summing sphere. The action of G on W_1 extends to an action of (a group isomorphic to) G on the connected sum W of W_1 with 12 copies of M_2 . The sphere \bar{S} lifts to 12 copies of S^2 . The right hand-side of the inequality in Theorem 5·5 is then 14. The orbifold (M, T) admits an orbifold Heegaard surface with orbifold characteristic $13/6$. (It is a torus having three punctures of weight 2 and one of weight 3.) This lifts to an invariant Heegaard surface of W having genus 14. So our lower bound is sharp in this case as well.

5·3. Comparatively Small Factors.

A 3-orbifold (M, T) is **comparatively small** if each c-essential surface $F \subset (M, T)$ has $x_\omega(F) > x_\omega(M, T)$.

LEMMA 5.8. *Suppose that (M, T) is a comparatively small nice orbifold. Then $\text{net } x_\omega(M, T) = x_\omega(M, T)$.*

Proof. By definition, $x_\omega(M, T) \geq \text{net } x_\omega(M, T)$. We now show $x_\omega(M, T) \leq \text{net } x_\omega(M, T)$. Let $\mathcal{H} \in \mathbb{H}(M, T)$ be locally thin, with $\text{net } x_\omega(\mathcal{H}) = \text{net } x_\omega(M, T)$. We claim $\mathcal{H}^- = \emptyset$.

Suppose, for a contradiction, that $\mathcal{H}^- \neq \emptyset$. The dual digraph to \mathcal{H} has at least one source and at least one sink. Each such source and sink corresponds to a $(C, T_C) \sqsubset (M, T) \setminus \mathcal{H}$ with $\partial_- C \subset \partial M$. Let (C, T_C) be one such and let $(D, T_D) \sqsubset (M, T) \setminus \mathcal{H}$ be distinct from (C, T_C) but have $\partial_+ D = \partial_+ C$. Then $\mathcal{H}^- \cap \partial_- D \neq \emptyset$; let F be a component. Since \mathcal{H} is locally thin, F is c-essential in (M, T) (Theorem 4.7). Since (M, T) is comparatively small, $x_\omega(F) > x_\omega(M, T)$. By Lemma 5.2,

$$N(C, T_C) + x_\omega(\partial_- C) = x_\omega(\partial_+ C) = x_\omega(\partial_+ D) \geq x_\omega(F) > x_\omega(M, T) \geq \text{net } x_\omega(M, T).$$

As there are at least two such (C, T_C) , by (1) and the niceness of (M, T) , we conclude $\text{net } x_\omega(M, T) > \text{net } x_\omega(M, T)$, a contradiction. Thus, $\mathcal{H}^- = \emptyset$. Consequently, \mathcal{H} is an orbifold Heegaard surface for (M, T) and so $\text{net } x_\omega(\mathcal{H}) \geq x_\omega(M, T)$.

THEOREM 5.9. *Suppose that (M, T) is a nice 3-orbifold. Assume also that for some (hence, any) efficient system of summing spheres S each component of $(M, T)|_S$ is comparatively small. Then*

$$x_\omega(M, T) \geq \text{net } x_\omega(M, T) \geq x_\omega((M, T)|_S) - x_\omega(S).$$

Proof. If (M, T) is orbifold-irreducible, the result is vacuously true. Suppose that (M, T) is orbifold-reducible. By Theorem 2.4, each component of $(M, T)|_S$ is comparatively small for any efficient system of summing spheres S . By definition, $x_\omega(M, T) \geq \text{net } x_\omega(M, T)$. By Theorem 4.9, $\text{net } x_\omega(M, T) \geq \text{net } x_\omega((M, T)|_S) - x_\omega(S)$ for an efficient system of summing spheres S . By Lemma 5.8, $\text{net } x_\omega((M, T)|_S) = x_\omega((M, T)|_S)$, and the result follows.

THEOREM 5.10. *Suppose that when $S \subset W$ is an invariant system of summing spheres, then every component of $W|_S$ is equivariantly comparatively small. Then:*

$$g(W; G) \geq g(W|_S; G).$$

Proof. Without loss of generality, we may assume that W is reducible. Let (M, T) be the quotient orbifold. Observe that it is nice and orbifold-reducible. Let \bar{S} be an efficient system of summing spheres for (M, T) . We claim that each component of $(M, T)|_{\bar{S}}$ is comparatively small. To see this, suppose that $(M_i, T_i) \sqsubset (M, T)|_{\bar{S}}$ and that $\bar{F} \subset (M_i, T_i)$ is a c-essential surface. Let W_i be a component of the preimage of (M_i, T_i) . Since (M_i, T_i) is orbifold-irreducible, \bar{F} is not a sphere. Thus, no component of the preimage of \bar{F} to W is a sphere. Let F be the preimage of \bar{F} in $W|_{\bar{S}}$, with $F_i = F \cap W_i$. Let $G_i \subset G$ be the stabiliser of W_i . We have $x_\omega(\bar{F}) = x_\omega(F_i)/|G_i|$.

By the equivariant loop theorem [17], F is incompressible. Suppose that some $F' \sqsubset F$ is ∂ -parallel in W . Let $W' \sqsubset W \setminus F'$ be homeomorphic to $F' \times I$. We may assume that F' was chosen so that the interior of W' is disjoint from F . The image of W' in (M, T) is the quotient of a trivial product compressionbody (namely W') by a finite group of diffeomorphisms (the stabiliser of F'). By Lemma 2.7, it a product trivial compressionbody. Thus, \bar{F} bounds

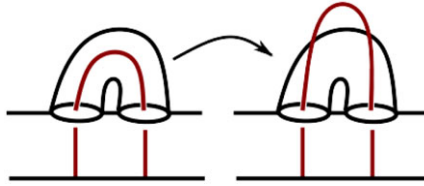


Fig. 6. Creating a removable arc.

a trivial product compressionbody with a component of ∂W and so \bar{F} is not c -essential, a contradiction. Thus, F is essential in $W|_S$. Since W_i is equivariantly comparatively small, $x_\omega(F_i) > x_\omega(W_i)$. Thus,

$$x_\omega(\bar{F}) = x_\omega(F_i)/|G_i| > x_\omega(W_i)/|G_i| = x_\omega(M_i, T_i).$$

Thus, each component of $(M, T)|_{\bar{S}}$ is comparatively small. Our result follows from Theorem 5.9 after multiplying by $|G|$.

6. Upper Bounds

In this section we use multiple orbifold Heegaard surfaces to construct upper bounds on the equivariant Heegaard genus of W and explain why the bound is sharp. Throughout this section, we use the inverse operation to “undoing a removable arc” mentioned earlier.

Definition 6.1. Suppose that (M, T) is a nice orbifold and that $\mathcal{H} \in \mathbb{H}(M, T)$. Suppose also that an edge α of $T \setminus \mathcal{H}$ is a ghost arc contained in $(C, T_C) \sqsubset (M, T) \setminus \mathcal{H}$. Choose a cut disc or semi-cut disc $D \subset (C, T_C)$ intersecting α exactly once. Choose an arc κ in D from ∂D to $D \cap \alpha$. In a neighborhood of D in M , isotope α by an isotopy following κ , so that the interior of α is pushed across $\partial_+ C$, as in Figure 6. Dually, we may isotope \mathcal{H} . This converts \mathcal{H} into a new $\mathcal{J} \in \mathbb{H}(M, T)$ such that

$$\text{net } x_\omega(\mathcal{J}) = \text{net } x_\omega(\mathcal{H}) + 2(1 - 1/\omega(\alpha)).$$

The ghost arc α is converted into the union of two vertical arcs and a bridge arc on the opposite side of $\partial_+ C$ from the vertical arcs. We call this move **creating a removable arc from α** .

In 3-manifold theory, generalised Heegaard splittings may be amalgamated to create Heegaard splittings [35]. In our situation, it is not always possible to amalgamate multiple orbifold Heegaard surfaces to create Heegaard surfaces. Ghost arcs are the obstruction and creating removable arcs allows us to amalgamate.

Definition 6.2. Suppose that (M, T) is an orbifold and that $\mathcal{H} \in \mathbb{H}(M, T)$. Suppose that $(M_i, T_i) \sqsubset (M, T) \setminus \mathcal{H}^-$ for $i = 1, 2$ and that $F = \partial M_1 \cap \partial M_2 \neq \emptyset$. Let $H_i = \mathcal{H}^+ \cap M_i$. We say that H_1 and H_2 are **amalgable** if whenever $e_i \subset T_i \setminus \mathcal{H}$ for $i = 1, 2$ are edges sharing an endpoint, then at least one is not a ghost arc.

PROPOSITION 6.3 (Amalgamation). *If H_1 and H_2 are amalgable, then there exists $\mathcal{J} \in \mathbb{H}(M, T)$ and $H \sqsubset \mathcal{J}^+$ such that $\mathcal{J} \setminus H = \mathcal{H} \setminus (H_1 \cup H_2 \cup F)$ and $\text{net } x_\omega(\mathcal{J}) = \text{net } x_\omega(\mathcal{H})$. Furthermore, each ghost arc of $T \setminus \mathcal{J}$ contains at least one ghost arc of $T \setminus \mathcal{H}$.*

Proof. Our proof is similar to that of [26, theorem 4.1]. Let $(C, T_C), (D, T_D) \sqsubset (M, T) \setminus \mathcal{H}$ be the components with $H_1 = \partial_+ C$ and $H_2 = \partial_+ D$. Choose complete collections of sc-discs $\Delta_C \subset (C, T_C)$ and $\Delta_D \subset (D, T_D)$. We choose these discs somewhat carefully. They must have the properties that the only edges of T_C and T_D intersecting Δ_C and Δ_D respectively are ghost arcs or core loops and that each ghost arc and core loop intersects exactly one such disc. It is not difficult to see that Δ_C and Δ_D can be chosen to satisfy these conditions. For simplicity in the discussion, assume that F is connected. If it is not, the proof goes through with only minor changes in wording.

Boundary reduce (C, T_C) and (D, T_D) using Δ_C and Δ_D to obtain (C', T'_C) and (D', T'_D) respectively. One component of each is a trivial product compressionbody having F as a boundary component. Observe that these components are disjoint from any core loops in T_C or T_D . Let X be their union with $T_X = T \cap X$. Notice that X is homeomorphic to $F \times I$. Let $\partial_C X$ be the component of ∂X intersecting $\partial_+ C$ and $\partial_D X$ the component intersecting $\partial_+ D$. Each component of T_X is a vertical arc. Consider the scars δ_C on $\partial_C X$ resulting from the ∂ -reduction along Δ_C . Some of them are unpunctured discs and others are punctured discs. Each punctured disc is incident to an arc of T_X whose other endpoint may lie on F , but does not lie in the scars $\delta_D \subset \partial_D X$ resulting from the ∂ -reduction along Δ_D . Extend $B = \delta_C \cap \partial_C X$ vertically through X via a solid tubes $B \times I$. The **frontier** of the tubes is $\partial B \times I$. Since no component of T_X is incident to both δ_C and δ_D , we may shrink the tubes of B containing an arc of T_X and isotope the ends on $\partial_D X$ of the other tubes so that the tubes are disjoint from δ_D .

Delete the discs that are the ends of the tubes from H_2 and attach the frontiers of the tubes. When we compressed H_1 along Δ_C , we used a regular neighbourhood that can be parameterised as $\Delta_C \times I$. Reattach $\partial \Delta_C \times I$ to the ends on $\partial_C X$ of the frontiers of the tubes. Call the surface we created H .

We claim that H is a vp-bridge surface for $(M', T') = (M_1, T_1) \cup_F (M_2, T_2)$. Let (D', T'_D) be the other vp-compressionbody of $(M, T) \setminus \mathcal{H}$ with $\partial_+ D' = H_2$ and (C', T'_C) the other vp-compressionbody with $\partial_+ C' = H_1$. Let (U, T_U) be the component of $(M', T') \setminus H$ containing (D', T'_D) . Let (V, T_V) be the other component. Observe that the union of Δ_C with a complete set of sc-discs for (D', T'_D) is a complete set of sc-discs for (U, T_U) . Thus, (U, T_U) is a vp-compressionbody. Let Δ'_C be a complete set of sc-discs for (C', T'_C) . We can extend $\partial \Delta'_C \cap \partial_C X$ through X to lie on $\partial_D X$. As before, we can ensure they miss the scars of Δ_D . The union of these discs with Δ_D is then a complete set of sc-discs for (V, T_V) , showing that it is also a vp-compressionbody. Thus, H is a vp-bridge surface for (M', T') .

Let \mathcal{J} be as in the statement of the proposition. The dual digraph to \mathcal{H} contains a connected subgraph α with edges that corresponding to the surfaces H_1, F , and H_2 . It has a single source and a single sink. Give H the transverse orientation inherited from H_1 and H_2 . The dual digraph to \mathcal{J} is obtained from that of \mathcal{H} by replacing α with a single edge. Since the dual digraph to \mathcal{H} was acyclic, so is the dual digraph to \mathcal{J} . Thus, $\mathcal{J} \in \mathbb{H}(M, T)$. The computation $\text{net } x_\omega(\mathcal{J}) = \text{net } x_\omega(\mathcal{H})$ is easily verified by compressing H along the discs Δ_C to recover H_2 .

Finally, suppose that α is a ghost arc of $T \setminus \mathcal{J}$. Traversing α we start at a thin surface in \mathcal{J}^- or vertex of T and end on a thin surface of \mathcal{J}^- or vertex of T . If during the traversal, we never traverse an arc of T_X , then α is a ghost arc of $T \setminus \mathcal{H}$. If we do traverse an arc of T_X , then α must have one endpoint in $\delta_C \cup \delta_D$, as otherwise it wouldn't be a ghost arc of $T \setminus \mathcal{J}$. But this implies that it contains a ghost arc of $T \setminus \mathcal{H}$.

6.1. Examples of super additivity

THEOREM 6.4. *For $k \geq 2$, there exist reducible W having a finite cyclic group of diffeomorphisms G of order k , and an invariant essential sphere S dividing W into two irreducible manifolds, such that*

$$g(W;G) = g(W|_S;G) + k - 1.$$

Furthermore, $g(W;G)$ can be arbitrarily high when G is a cyclic group of fixed order.

Associated to each Heegaard surface H of genus at least 2 of a compact 3-manifold X , is a nonnegative integer invariant $d(H)$, called **Hempel distance** [12]. When $d(H) = 0$, the Heegaard surface H is reducible. When $d(H) = 1$, the Heegaard surface H can be untelescoped to a generalised Heegaard surface with multiple components. Essential surfaces or strongly irreducible Heegaard surfaces can often be used to provide upper bounds on Hempel distance. Furthermore, if every thick component of a multiple Heegaard surface has high enough Hempel distance, the generalised Heegaard surface is locally thin. We apply this philosophy in our context.

By [18], for each $t \in \mathbb{N}$ and $N \in \mathbb{N}$, there exists a knot $K \subset S^3$ such that $S^3 \setminus K$ admits a Heegaard surface of genus $t + 1$ and Hempel distance $d(H) \geq N$. Fix $t, N, k \in \mathbb{N}$. For $i = 1, 2$, choose knots K_i in S^3 such that each has a Heegaard surface H_i of its exterior of genus $t + 1$ (and $x_\omega(H_i) = 2t$) and Hempel distance at least $\zeta = 2(4t + N + 2(1 - 1/k) + 2)/(1 - 1/k) + 3$. We use the following, drawn from work of Scharlemann, Scharlemann–Tomova and Tomova. The statement for F is due to Scharlemann [30, theorem 3.1] and is a generalisation of [8]. The case when J is disjoint from K_i is the main result of [32] and when J intersects K_i , it is the main result of [41]. We rephrase their results, using our terminology.

THEOREM 6.5 (Scharlemann, Scharlemann–Tomova, Tomova). *If $F \subset (S^3, K_i)$ is an essential connected surface, then $-\chi(F) + |F \cap K_i| > \zeta - 3$. If $J \in \mathbb{H}(S^3, K_i)$ is connected, then either $-\chi(J) + |J \cap T| > \zeta - 3$ or $J \rightarrow H_i$.*

Let $K = K_1 \# K_2$. Consider (S^3, K) as an orbifold where K is given weight $k \in \mathbb{N}_2^\infty$. Note that (S^3, K) has a multiple orbifold Heegaard surface \mathcal{H} with $\mathcal{H}^+ = H_1 \cup H_2$ and $S = \mathcal{H}^-$ the twice-punctured summing sphere realising K as a connected sum of K_1 and K_2 . (We also need to give \mathcal{H} one of the two orientations making it an oriented multiple vp-bridge surface.) For the record, we have $\text{net } x_\omega(\mathcal{H}) = 4t + 2/k$. By the definitions of Hempel distance and the partial order \rightarrow (both of which we have omitted), \mathcal{H} is locally thin.

LEMMA 6.6. *We have $\text{net } x_\omega(S^3, K) = \text{net } x_\omega(\mathcal{H})$. Furthermore, if $\mathcal{J} \in \mathbb{H}(M, T)$ is locally thin and has $\text{net } x_\omega(\mathcal{J}) \leq \text{net } x_\omega(S^3, K) + N + 2(1 - 1/k)$, then up to isotopy and orientation reversal, \mathcal{H} can be obtained from \mathcal{J} by deleting pairs of twice-punctured spheres from \mathcal{J}^- and \mathcal{J}^+ . In particular, $\text{net } x_\omega(\mathcal{J}) = \text{net } x_\omega(\mathcal{H})$.*

Proof. Let $\mathcal{J} \in \mathbb{H}(S^3, K)$ be locally thin and have $\text{net } x_\omega(\mathcal{J}) \leq \text{net } x_\omega(S^3, K) + N + 2(1 - 1/k)$. Since (S^3, K) is orbifold-reducible, \mathcal{J}^- contains an efficient system of summing spheres for (S^3, K) . By Theorem 6.5 (or, indeed, by [12]), both K_1 and K_2 are prime knots. Consequently, there is a unique essential summing sphere for (S^3, K) , up to isotopy. Isotope \mathcal{J} so that $S \sqsubset \mathcal{J}^-$. Suppose that there exists $F \sqsubset \mathcal{J}^- \setminus S$. Choose F so that it bounds a 3-submanifold $X \subset S^3$ with interior disjoint from \mathcal{J}^- (i.e. F is outermost). Let $J = \mathcal{J}^+ \cap X$

and let $(C, T_C) \sqsubset (S^3, K) \setminus \mathcal{J}$ be the orbifold compressionbody with $J = \partial_+ C$ and $\partial_- C = \emptyset$. Without loss of generality, we may assume that F is on the same side of S as K_1 . If F is essential in $(M, T) \setminus S$, then by Theorem 6.5,

$$\begin{aligned} 2(4t + N + 2(1 - 1/k) + 2) &= (1 - 1/k)(\zeta - 3) \\ &< (1 - 1/k)(-\chi(F) + |F \cap K_1|) \\ &\leq -\chi(F) + (1 - 1/k)|F \cap K_1| \\ &= x_\omega(F). \end{aligned}$$

By Lemma 5.2, $N(C, T_C) = x_\omega(H) \geq x_\omega(F)$ and so, by (1),

$$\begin{aligned} \text{net } x_\omega(S^3, K) + N + 2(1 - 1/k) &\geq \text{net } x_\omega(\mathcal{J}) \\ &> 4t + N + 2(1 - 1/k) + 2 \\ &= \text{net } x_\omega(\mathcal{H}) + N + 2(1 - 1/k) - 2/k + 2 \\ &\geq \text{net } x_\omega(S^3, K) + N + 4(1 - 1/k). \end{aligned}$$

This is a contradiction. Thus, F must be inessential in $(S^3, K) \setminus S$. Since F is c-essential in (S^3, K) , it must be isotopic to S . Perform the isotopy to make F coincide with S . After the isotopy, X is the side of $S = F$ containing K_1 . By Theorem 6.5, either $-\chi(J) + |J \cap T| > \zeta - 3$ or $J \rightarrow H_i$. In the former case, we have:

$$\begin{aligned} N(C, T_C) = x_\omega(J) &= -\chi(J) + (1 - 1/k)|J \cap T| > (1 - 1/k)(\zeta - 3) \\ &= 2(4t + N + 2(1 - 1/k) + 2). \end{aligned}$$

The same arithmetic as before establishes a contradiction. Thus, $J \rightarrow H_i$. However, \mathcal{J} is locally thin, so in fact, J is isotopic to H_i (ignoring orientations).

We have shown that each outermost $F \sqsubset \mathcal{J}^-$ is isotopic to S . Furthermore, if \mathcal{J}^- is connected, then \mathcal{J} is isotopic to \mathcal{H} , ignoring orientations. Suppose that \mathcal{J}^- is disconnected. Since every surface in S^3 separates, there are at least two outermost $F_1, F_2 \sqsubset \mathcal{J}^-$. They cobound $(Y, T_Y) \sqsubset (S^3, K)$ that is a product compressionbody homeomorphic to $(S^2 \times I, \{p_1, p_2\} \times I)$, where $p_1, p_2 \in S^2$. Since each component of \mathcal{J}^- is c-essential, each component of \mathcal{J}^- must be parallel to each of F_1 and F_2 in (Y, T_Y) . Suppose that $F_i, F_{i+1} \sqsubset \mathcal{J}^-$ cobound a submanifold $(Y', T'_{Y'}) \sqsubset (Y, T_Y) \setminus \mathcal{J}^-$. Note that $(Y', T'_{Y'})$ is homeomorphic to (Y, T_Y) . Let $J' = \mathcal{J}^+ \cap Y'$. Since \mathcal{J} is locally thin, J' must be a twice-punctured sphere bounding a trivial ball compressionbody $B(J')$ to one side. We conclude that

$$\begin{aligned} \text{net } x_\omega(\mathcal{J}) &= x_\omega(H_1) + x_\omega(H_2) + \frac{-2}{k}(|\mathcal{J}^-| - 1) - \frac{-2}{k}|\mathcal{J}^-| = x_\omega(H_1) + x_\omega(H_2) + \frac{2}{k} \\ &= \text{net } x_\omega(\mathcal{H}). \end{aligned}$$

Thus, $\text{net } x_\omega(\mathcal{H}) = \text{net } x_\omega(S^3, K)$. Furthermore, after deleting all the components of $\mathcal{J} \cap Y$ except F_1 , we obtain a multiple vp-bridge surface isotopic to \mathcal{H} , ignoring orientations.

We will also need the following:

Definition 6.7. Suppose that $\mathcal{H} \in \mathbb{H}(M, T)$. The **net geometric intersection number** of \mathcal{H} is $\text{net } \iota(\mathcal{H}) = |\mathcal{H}^+ \cap T| - |\mathcal{H}^- \cap T|$.

We can now calculate the Heegaard characteristic of (S^3, K) .

LEMMA 6.8. $x_\omega(S^3, K) = 4t + 2$

Proof. There are four orbifold compressionbody components of $(S^3, K) \setminus \mathcal{H}$. The two not adjacent to $S = \mathcal{H}^-$ are disjoint from K and each of the other two contain a single ghost arc, both of whose endpoints are on S . Choose one of them and use it to create a removable arc. Call the new multiple orbifold Heegaard surface \mathcal{H}' with two thick surfaces and thin surface S . Note that the thick surfaces are amalgable. Amalgamate them, converting \mathcal{H}' into a connected $H' \in \mathbb{H}(M, T)$ with

$$\text{net } x_\omega(H') = \text{net } x_\omega(\mathcal{H}') = \text{net } x_\omega(M, T) + 2(1 - 1/k) = 4t + 2.$$

Let J be an orbifold Heegaard surface for (S^3, K) such that $x_\omega(J) = x_\omega(S^3, K)$. There is a locally thin $\mathcal{J} \in \mathbb{H}(S^3, K)$ such that $J \rightarrow \mathcal{J}$. By Lemma 6.6,

$$\text{net } x_\omega(M, T) + 2(1 - 1/k) = x_\omega(H') \geq x_\omega(J) \geq \text{net } x_\omega(\mathcal{J}) \geq \text{net } x_\omega(\mathcal{H}) = \text{net } x_\omega(M, T).$$

By Lemma 6.6, $\text{net } x_\omega(\mathcal{J}) = \text{net } x_\omega(\mathcal{H})$ and after deleting pairs of twice-punctured spheres from \mathcal{J}^- and \mathcal{J}^+ , \mathcal{J} is isotopic to \mathcal{H} (ignoring orientations). As in Lemma 4.5, the thinning moves that create \mathcal{J} from J potentially consist of four types of moves. Since S^3 is closed and K is a knot, of the moves listed in Type (I), we never need to perform a ∂ -destabilisation, meridional ∂ -destabilisation or meridional ghost ∂ -destabilisation. Performing a ghost ∂ -destabilisation, involves compression along a separating compressing disc and the discarding of a torus boundary component. (That torus is isotopic to the frontier of a regular neighbourhood of K .) Such a move decreases negative orbifold Euler characteristic of a thick surface by $2 \geq 2(1 - 1/k)$. Destabilisation also decreases the negative orbifold Euler characteristic of a thick surface by 2. Meridional destabilisation decreases it by $2/k$. The moves in Type (II) decrease it by $2(1 - 1/k)$. Consolidation and untelescoping leave $\text{net } x_\omega$ unchanged.

We see, therefore, that in the thinning sequence producing \mathcal{J} from J , there can be at most one move that is a ghost ∂ -destabilisation, destabilisation, unperturbation, or undoing a removable arc. All other moves are either meridional destabilisation, untelescoping, or consolidation. Ghost ∂ -stabilisation, destabilisation, meridional destabilisation, untelescoping, and consolidation do not decrease $\text{net } \iota$. Unperturbing and undoing a removable arc decrease $\text{net } \iota$ by 2.

Observe that $\text{net } \iota(\mathcal{J}) = \text{net } \iota(\mathcal{H}) = -2$. Since J is connected, $K \setminus J$ contains no ghost arcs. Thus, $\text{net } \iota(J) \geq 0$. Thus, at least one unperturbing or undoing a removable arc are required in the thinning sequence producing \mathcal{J} from J . We have already seen that there is at most one, so there must be exactly one and we cannot have any meridional destabilisations. We conclude that the thinning sequence producing \mathcal{J} from J consists of exactly one unperturbation or undoing a removable arc and some number of untelescoping and consolidations. We conclude that $x_\omega(J) = x_\omega(H')$. Thus, $x_\omega(S^3, K) = x_\omega(H') = 4t + 2$.

The first examples of pairs of knots producing super-additivity of tunnel number were given in [19, 20]. Setting $k = \infty$ produces other examples, using a similar method to [14, 43].

Proof of Theorem 6.4. Fix $t, k \in \mathbb{N}$ with $k \geq 2$. For each $N \in \mathbb{N}$, construct K_1, K_2 as above. By Lemma 6.8, $x_\omega(S^3, K) = 4t + 2$. Let W be the k -fold cyclic branched cover over K , with G the deck group. Then, by Lemma 2.7, $x_\omega(W; G) = 4tk + 2k$. The manifold W is the connected

sum of W_1 and W_2 which are the k -fold branched covers over K_1 and K_2 respectively. By Lemma 6.6, $x_\omega(W_i) = 2t$. Let S be the lift of a summing sphere for (S^3, K) . Observe that S is efficient as K_1 and K_2 are prime. Thus $x_\omega(W|_S; G) = 4tk$ and $x_\omega(W; G) = x_\omega(W|_S; G) + 2k$. Converting to genus, we have

$$g(W; G) = g(W|_S; G) + k - 1.$$

6.2. A General Upper Bound

In this section, we adapt our example to produce a general upper bound for equivariant Heegaard genus of composite manifolds. As usual, we start by considering orbifolds.

THEOREM 6.9. *Suppose that (M, T) is orbifold-reducible and that S is a system of summing spheres, with each component of S separating. Then*

$$x_\omega(M, T) \leq x_\omega((M, T)|_S) - x_\omega(S)(1 - c) + 2c|S|$$

where $c = 1$ if T has no vertices and $c = 2$ if it does.

Proof. Let (M_i, T_i) for $i = 1, \dots, n$ be the components of $(M, T)|_S$. Choose an orbifold Heegaard surface $H_i \subset (M_i, T_i)$ such that $x_\omega(M_i, T_i) = x_\omega(H_i)$. Assign transverse orientations chosen so that if we set $\mathcal{H}^+ = \bigcup H_i$ and $\mathcal{H}^- = S$, then $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^- \in \mathbb{H}(M, T)$. This is possible since the dual graph to S is a tree. Thus,

$$\text{net } x_\omega(\mathcal{H}) = \sum x_\omega(H_i) - x_\omega(S).$$

Consider a point $p \in T \cap S$ such that both edges of $T \setminus \mathcal{H}$ incident to p are ghost arcs. Let $S_0 \sqsubset S$ contain p . As the two ghost arcs lie in the same edge of T , they have the same weight $\omega(p)$. Let α be the one lying on the same side of S as its normal vector and suppose $\alpha \subset (M_i, T_i)$. Perform the isotopy of Definition 6.1 to create a removable arc from α . The isotopy converts α into the union of a bridge arc and two vertical arcs, with the bridge arc on the opposite side of H_i from the two vertical arcs. Dually, we may isotope H_i . After the isotopy, $x_\omega(H_i)$ has increased by $2(1 - 1/\omega(p))$. Do this for each such ghost arc in (M_i, T_i) incident at a puncture of S_0 to a ghost arc on the opposite side of S . The most we increase $x_\omega(H_i)$ by is $2 \sum (1 - 1/\omega(p))$, where the sum is over the punctures of S_0 . This is equal to $4 + 2x_\omega(S_0)$. Doing the same thing for each component of S increases $\text{net } x_\omega(\mathcal{H})$ by at most $4|S| + 2x_\omega(S)$. However, notice that if a ghost arc has both its endpoints on S (rather than on S and a vertex of T) then at worst we only need to perform our move once for each point of $|S \cap T|$, rather than twice. Such will be the case if T has no vertices, for example. In that case, we increase $x_\omega(\mathcal{H})$ by at most $\sum_p (1 - 1/p)$ where the sum is over all punctures p of S . In such a case, we increase $x_\omega(\mathcal{H})$ by at most $2|S| + x_\omega(S)$. We then have a new $\mathcal{J} \in \mathbb{H}(M, T)$ such that

$$\text{net } x_\omega(\mathcal{H}) \leq \text{net } x_\omega(\mathcal{J}) \leq \text{net } x_\omega(\mathcal{H}) + D,$$

where $D = 2|S| + x_\omega(S)$ if T is a link and $D = 4|S| + 2x_\omega(S)$ otherwise. Notice that no two ghost arcs of $T \setminus \mathcal{J}$ are incident to the same point of $T \cap S$.

Amalgamation does not create additional ghost arcs, so by Proposition 6.3 we may amalgamate the thick surfaces of \mathcal{J} two at a time to eventually obtain a connected $J \in \mathbb{H}(M, T)$

such that $\text{net } x_\omega(\mathcal{J}) = x_\omega(J)$. Thus,

$$x_\omega((M, T)|_S) - x_\omega(S) + D \geq \text{net } x_\omega(\mathcal{J}) = x_\omega(J) \geq x_\omega(M, T).$$

This can be rearranged into the claimed inequality.

THEOREM 6.10. *If every sphere in W separates, then for any equivariant system of summing spheres $S \subset W$,*

$$g(W;G) \leq g(W|_S;G) + (c(|G| + 1) - 2)(|W|_S - 1)$$

where $c = 1$ if every point of W has cyclic stabiliser and $c = 2$ otherwise.

Proof. If W is irreducible, then we can take $S = \emptyset$ and $n = 1$ and the result is vacuously true. Suppose that W is reducible. By Theorem 2.5, the quotient orbifold (M, T) is orbifold-reducible. Recall that T has no vertices if and only if every point of W has cyclic stabiliser. Let \bar{S} be an efficient system of summing spheres and S its lift to W . Note that $-2(n - 1) = -2|S| = |G|x_\omega(S)$. By Lemma 2.7, $x_\omega(W;G) = |G|x_\omega(M, T)$ and $x_\omega(W|_S;G) = |G|x_\omega((M, T)|_S)$. The result follows from Theorem 6.9 after converting the inequalities in the conclusion of that theorem to be in terms of genus.

Remark 6.11. The examples of Theorem 6.4 show that our upper bound is sharp when every point of W has cyclic stabiliser. It is likely possible to adapt those examples to show that the inequality is sharp even when some points of W do not have cyclic stabiliser.

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Appendix A. Efficient Factorisations

In this section we sketch the proof that, even when there are nonseparating spherical suborbifolds, efficient factorizations exist and the factors are unique up to orbifold homeomorphism.

COROLLARY 4.8. *Suppose that (M, T) is a nice orbifold and let $\mathcal{H} \in \mathbb{H}(M, T)$ be locally thin. Then \mathcal{H}^- contains a system of summing spheres for (M, T) .*

Proof. If (M, T) is not orbifold-reducible, there is nothing to prove, so suppose that it is. Let $S \subset (M, T)$ be an essential summing sphere. Isotope it to intersect \mathcal{H}^- in the minimal number of loops. Since S has three or fewer punctures, each component of $S \cap \mathcal{H}^-$ is inessential in S . Let ξ be an innermost such loop on S , with $D \subset S$ the unpunctured or once-punctured disc it bounds. Let $F \sqsubset \mathcal{H}^-$ contain ξ . Since F is c-essential, the curve ξ must be inessential on F . Let $E \subset F$ be the unpunctured or once-punctured disc it bounds. Since (M, T) is nice, $|E \cap T| = |D \cap T|$. If both E and D are once-punctured, then since (M, T) is nice, the weights of the punctures are the same. Observe that if S intersects the interior of E , the intersection curves are also inessential in F . Let $\zeta \subset E$ be an innermost such component; possibly $\zeta = \xi$. Let $E' \subset E$ be the unpunctured or once-punctured disc it bounds in F and $D' \subset S$ the unpunctured or once-punctured disc it bounds in S . Again, we have $|E' \cap T| = |D' \cap T|$ and if E' and D' are both once-punctured, then the punctures have the same weight.

Compress S using E' to arrive at S' . Note that S' consists of two components, neither with more punctures than S . Although the total number of punctures may have gone up, it does so if and only if E' and D' are once-punctured. In which case, one component of S' is a sphere with the same number of punctures and orbifold Euler characteristic as S and the other is a twice-punctured sphere with both punctures of the same weight. If E' was once-punctured, then neither component of S' is an unpunctured sphere or a twice-punctured sphere bounding a trivial ball compressionbody, as $|S \cap \mathcal{H}^-|$ was minimised, up to isotopy. Similarly, if E' was unpunctured, then neither component of S' is an unpunctured sphere bounding a ball disjoint from T . Thus, some component of S' is an essential summing sphere with no more punctures than S and with $x_\omega(S) \leq x_\omega(S')$. Repeating the argument, we arrive at an essential summing sphere S'' for (M, T) with $x_\omega(S'') \leq x_\omega(S)$, with S'' having no more punctures than S , and with $S'' \cap \mathcal{H}^- = \emptyset$. The argument of [36, theorem 7.2] shows that we can further isotope S'' to be disjoint from \mathcal{H} .

Let $(C, T_C) \sqsubset (M, T) \setminus \mathcal{H}$ be the component containing S'' . Let $\Delta \subset (C, T_C)$ be a complete collection of sc-discs for (C, T_C) . Isotope S'' in (C, T_C) to minimise $|\Delta \cap S''|$. The intersection consists of loops, since S'' is closed. An argument identical to that above shows that we can compress S'' to make it disjoint from Δ . We may as well assume that S'' was disjoint from Δ to begin with. The sphere S'' then lies in a trivial compressionbody. Since it is c-essential, it is parallel to a component of $\partial_- C$. All that is needed for the above argument to work is that each component of \mathcal{H}^- is c-incompressible and each component of \mathcal{H}^+ is sc-strongly irreducible.

Thus, if (M, T) contains an essential unpunctured sphere, so does \mathcal{H}^- . Let S be the union of all unpunctured spheres in \mathcal{H}^- . Notice that $\mathcal{H}_S = \mathcal{H} \setminus S$ restricts to a multiple vp-bridge surface for each component of $(M, T)|_S$. It is easy to see that each component of \mathcal{H}_S^- remains c-incompressible and each component of \mathcal{H}_S^+ is sc-strongly irreducible. Thus, $(M, T)|_S$ does not contain an essential unpunctured sphere. The previous argument then shows that if $(M, T)|_S$ contains an essential twice-punctured summing sphere, then \mathcal{H}_S^- does as well. Let $Q \subset \mathcal{H}_S^-$ be the union of all the twice-punctured summing spheres. The same argument as in the unpunctured case shows that $(M, T)|_{S \cup Q}$ contains no essential unpunctured or twice-punctured summing spheres. Finally, repeat the analysis for thrice-punctured spheres to see that \mathcal{H}^- contains a collection of summing spheres P such that $(M, T)|_P$ contains no essential summing spheres.

THEOREM 2.4 (after Petronio, Hog-Angeloni–Matveev). *Suppose that (M, T) is a nice orbifold that is orbifold-reducible. Then there exists an efficient system of summing spheres $S \subset (M, T)$; indeed, any system of summing spheres contains an efficient subset. Furthermore, any two such systems S, S' are orbifold-homeomorphic, as are $(M, T)|_S$ and $(M, T)|_{S'}$.*

Proof. If every spherical 2-suborbifold of (M, T) is separating, and if no edge of T has infinite weight and if M is closed, then this is exactly Petronio’s theorem. If T has some edges of infinite weight or if M is not closed, then this can be deduced as in [13, theorems 8 and 9]. The case when (M, T) contains nonseparating spherical 2-suborbifolds needs to be handled somewhat differently; although the essence of the argument is the same. In Corollary 4.8 above we proved that if (M, T) is a nice orbifold, then there exists a system of summing spheres $\widehat{S} \subset (M, T)$. A subset of the components of \widehat{S} will be efficient, as we now explain. Begin by setting $S = \widehat{S}$. Consider the components of $(M, T)|_S$ one at a time, in some order. Suppose a component is an $\mathbb{S}(0)$ and contains a (necessarily unpunctured) scar but not the matching scar. Let $P \sqsubset S$ be the sphere producing the scar. Remove P from S . The effect on $(M, T)|_S$ is, up to homeomorphism, to remove an $\mathbb{S}(0)$ component. We then restart the process. We handle the $\mathbb{S}(2)$ and $\mathbb{S}(3)$ components similarly. The end result is that after removing some components from \widehat{S} we arrive at a subset S that is efficient.

We now consider uniqueness, adapting the arguments of [13, 24]. We start by establishing some basic moves on systems of spheres. Suppose that $S_0 \subset (M, T)$ is a sphere with $n_0 \leq 2$ punctures and that $S_1 \subset (M, T)$ is a disjoint sphere with $n_0 \leq n_1 \leq 3$ punctures. Let α be an embedded arc joining S_1 to S_0 . If $n_0 = 0$, assume α is disjoint from T . If $n_0 = 2$, assume that α is an arc lying in the interior of an edge of T . Let S_2 be the result of tubing S_1 to S_0 along α . Notice that S_2 has the same number of punctures as S_1 . We say that S_2 is obtained by **sliding** S_1 over S_0 using the **sliding arc** α . Observe that $(S_2 \cup S_0)$ is orbifold-homeomorphic to $(S_1 \cup$

S_0) and that $(M, T)|_{(S_2 \cup S_0)}$ is orbifold-homeomorphic to $(M, T)|_{(S_1 \cup S_0)}$. If S_1 is essential, then in most cases S_2 will also be essential. The only time it is not, is if either:

- (i) S_1 and S_2 are parallel unpunctured or twice-punctured spheres in (M, T) and the region of parallelism contains α , or
- (ii) if S_1 is thrice-punctured, and the component of $(M, T)|_{S_0 \cup S_1}$ is an $\mathbb{S}(3)$.

Suppose that $S \subset (M, T)$ is an efficient system of summing spheres and that $S_0 \subset (M, T)$ is an essential sphere having $n_0 \leq 2$ punctures which is either a component of S or is disjoint from S and not parallel to a component of S . Let $S_1 \sqsubset S$ be distinct from S_0 . Suppose that α is a sliding arc such that we may slide S_1 over S_0 using α and that the interior of α is disjoint from S . Let S_2 be the sphere that results from the sliding. By our previous remarks and the definition of efficient, S_2 is essential. Furthermore, in $(M, T)|_{S_0}$, the sliding can be achieved by an isotopy of S_1 . It is then easy to see that $S' = (S \setminus S_1) \cup S_2$ is efficient.

Suppose that $S, P \subset (M, T)$ are both efficient systems of summing spheres but that either S and P are not orbifold-homeomorphic or $(M, T)|_S$ and $(M, T)|_P$ are not orbifold-homeomorphic. Suppose that there are such choices for S and P so that $S \cap P$ is the union of components of S (equivalently, of P). Out of all such choices, choose S and P so that $|S \cap P|$ is maximal. Let $P_0 \sqsubset P \setminus S$, if such exists. Since P_0 is inessential in $(M, T)|_S$ either it bounds a 3-ball in $M|_S$ disjoint from the graph or it is ∂ -parallel. Let B be either the 3-ball or the region of parallelism. We may assume that P_0 was chosen so that the interior of B is disjoint from $P \setminus S$.

Consider, first, the possibility that B is a 3-ball; that is P_0 is unpunctured. If all the spheres of S with scars in B , lie in $S \cap P$, then there is an $\mathbb{S}(0)$ component of $(M, T)|_P$ containing one scar from P_0 , but not the other. This contradicts the fact that P is efficient. Thus, there is a sphere $S_0 \sqsubset S \setminus P$ with at least one scar in B . Suppose S_0 has both scars in B . We may join the scars by an arc in B that is disjoint from all other other scars from S in B . This arc can be closed up to a loop in $(M, T)|_P$ intersecting S_0 exactly once, contradicting the fact that $(M, T)|_P$ is irreducible. Thus, S_0 has exactly one scar in B . Using arcs in B , slide P_0 over each sphere of S , except for S_0 , having a scar in B . We arrive at the situation where B contains only one scar, and that scar is a scar from S_0 . Consequently, P_0 is then parallel to S_0 , contradicting our choice of S and P to maximize $|S \cap P|$. Thus, B is not a 3-ball.

Suppose therefore that B is a product region. The boundary of B consists of $P_0 \setminus T$ together with a subsurface F of $\partial(M_0 \setminus T_0)$. If B contains a pair of matching unpunctured scars, we may slide P_0 as before, to remove them. Using arcs in B , we may also slide P_0 over each unpunctured sphere in S having a single scar in B , to remove that scar. Thus, we may assume that B contains no unpunctured scars. If every scar in B (and therefore in F) corresponds to a sphere of $S \cap P$, then as before we contradict the assumption that P is efficient. Similarly, if a scar from a sphere of $S \setminus P$ lying in B has its matching scar in B , then $(M, T)|_P$ is not irreducible. Suppose that P_0 is twice-punctured. This implies that F is an annulus joining the components of $\partial(P_0 \setminus T)$. Working from either end of F , we slide P_0 over components of $S \setminus S_0$, until the only scar left in B is that from S_0 . At this point, P_0 is parallel to S_0 , a contradiction. Thus, P_0 is not a twice-punctured sphere.

Suppose that P_0 is a thrice-punctured sphere. Using arcs in B , we can slide P_0 over the twice-punctured spheres with scars in B , to arrive at the situation where either there are no scars in B or there is a single thrice-punctured scar in B and no other scars. If there is no scar in B , then (after the slides) P_0 is inessential in (M, T) . By our earlier remarks, this contradicts

the fact that P is efficient. Thus, there is a thrice-punctured scar in B . Let $S_0 \sqsubset S$ correspond to it. If $S_0 \sqsubset P$, then $(M, T)|_P$ contains an $\mathbb{S}(3)$ component having exactly one scar from P_0 . This contradicts the assumption that P is efficient. Thus, $S_0 \sqsubset S \setminus P$. Since P is then parallel in (M, T) to S_0 , we contradict our choice of S and P . We conclude that $P \subset S$. If $P \neq S$, then it is easy to see that S cannot be efficient, a contradiction. Thus, $S = P$. This contradicts the hypothesis that either S and P are not orbifold-homeomorphic or that $(M, T)|_S$ and $(M, T)|_P$ are not orbifold-homeomorphic.

As a result of the preceding argument, the result holds when $S \cap P = \emptyset$. Suppose that S and P are chosen so that they are transverse and $|S \cap P|$ is minimal. Since both S and P are transverse to T , $S \cap P$ consists of circles. Choose one ζ that is innermost in S . It bounds an unpunctured or once-punctured disc $D \subset S$ with interior disjoint from P . Use D to compress P . This separates some component P_0 of P into two components P_1 and P_2 . Both of which can be isotoped in (M, T) to be disjoint from P . Thus, both are inessential in $(M, T)|_P$. A sequence of slides and isotopies, then allows us to replace P with an efficient system of summing spheres $\widehat{P} \subset (M, T)$ such that P and \widehat{P} are pairwise orbifold-homeomorphic, as are $(M, T)|_P$ and $(M, T)|_{\widehat{P}}$ and for which $|S \cap \widehat{P}| < |S \cap P|$, a contradiction. Thus, it must be the case that an two efficient systems of summing spheres S and P are orbifold-homeomorphic and we also have $(M, T)|_S$ and $(M, T)|_P$ orbifold-homeomorphic.