QUADRATIC ISOPARAMETRIC SYSTEMS IN $\mathbb{R}_p^{n+m}$

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1. Introduction. The notions of isoparametric maps and submanifolds in semi-Riemannian spaces are the generalizations of such notions in Riemannian spaces. The generalizations are different according to the purposes. We take the definitions as in the Riemannian case. Quadratic isoparametric maps and submanifolds are interesting examples which can be studied in detail. In this paper we study what we call quadratic isoparametric systems. In fact we give a classification of such systems of codimension 2. We use three different methods to show that quadratic isoparametric submanifolds of codimension 2 are homogeneous. The classification of quadratic isoparametric systems is done algebraically. By this we have changed the geometric problem of classifying quadratic submanifolds of codimension 2 into the algebraic problem of classifying quadratic isoparametric systems of codimension 2. The classification of such systems with arbitrary codimension is still open.

2. Preliminaries

DEFINITION 1. A smooth function $f=(f_1,\ldots,f_m):\mathbb{R}_p^{n+m}\to\mathbb{R}^m$ is called *isoparametric* if

(i) $\langle \operatorname{grad} f_\alpha, \operatorname{grad} f_\beta\rangle$ and $\Delta f_\alpha = \operatorname{div}(\operatorname{grad} f_\alpha)$ are smooth functions of f for all α, β, $1\le\alpha$, $\beta\le m$;

(ii) $[\operatorname{grad} f_\alpha, \operatorname{grad} f_\beta]$ is a linear combination of $\operatorname{grad} f_1,\ldots,\operatorname{grad} f_m$ with coefficients being smooth functions of f for all α,β, $1\le\alpha$, $\beta\le m$.

DEFINITION 2. If $f:\mathbb{R}_p^{n+m}\to\mathbb{R}^m$ is isoparametric, c a regular value of f such that $f^{-1}(c)\neq\varnothing$, and $\langle,\rangle|_{Tf^{-1}(c)}$ is nondegenerate, then each connected component of $f^{-1}(c)$ is called an *isoparametric submanifold.*

Let $\operatorname{Sym}(\mathbb{R}_p^{n+m})$ be the space of self-adjoint linear endomorphisms of $\mathbb{R}_p^{n+m}$. $\mathbb{R}_p^{n+m}$ is the real vector space $\mathbb{R}^{n+m}$ with the inner product which has signature $p\ge 0$.

By a *quadratic map* we mean a map $f:\mathbb{R}_p^{n+m}\to\mathbb{R}^m$ defined by

$$f(x)=(\langle A_1x,x\rangle+2\langle a^1,x\rangle,\ldots,\langle A_mx,x\rangle+2\langle a^m,x\rangle)$$

where $A_\alpha\in\operatorname{Sym}(\mathbb{R}_p^{n+m})$, $a^\alpha\in\mathbb{R}_p^{n+m}$, $\forall\alpha$, $1\le\alpha\le m$ and the set $\{A_\alpha \mid 1\le\alpha\le m\}$ is a linearly independent set. We have the following interesting theorem.

THEOREM 3. *Let $f:\mathbb{R}_p^{n+m}\to\mathbb{R}^m$ be a quadratic map. Then f is isoparametric iff there exist constants $\lambda_\gamma^{\alpha\beta}=\lambda_\gamma^{\beta\alpha}$ such that*

$$A_\alpha A_\beta=\sum_\gamma \lambda_\gamma^{\alpha\beta}A_\gamma,\ A_\alpha a^\beta=\sum_\gamma\lambda_\gamma^{\alpha\beta}a^\gamma,\qquad 1\le\alpha,\beta,\gamma\le m.$$

Proof. We calculate that $(\operatorname{grad} f_\alpha)(x)=2(A_\alpha x+a^\alpha)$ $\forall\alpha$, $1\le\alpha\le m$, and $\forall x\in\mathbb{R}_p^{n+m}$. So

$$\langle \operatorname{grad} f_\alpha, \operatorname{grad} f_\beta\rangle(x)=2\langle (A_\alpha A_\beta+A_\beta A_\alpha)(x),x\rangle+4(\langle A_\alpha a^\beta+A_\beta a^\alpha,x\rangle+\langle a^\alpha,a^\beta\rangle),$$

$$[\operatorname{grad} f_\alpha,\operatorname{grad} f_\beta](x)=4[(A_\beta A_\alpha-A_\alpha A_\beta)(x)+A_\beta a^\alpha-A_\alpha a^\beta]$$

and $\Delta f_\alpha=2\operatorname{trace} A_\alpha=\text{constant}$.

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Now suppose that f is isoparametric; then we have

$$\langle \operatorname{grad} f_\alpha, \operatorname{grad} f_\beta \rangle = F_{\alpha\beta}(f) \quad \text{and} \quad [\operatorname{grad} f_\alpha, \operatorname{grad} f_\beta] = \sum_\gamma \mu_\gamma^{\alpha\beta}(f) \operatorname{grad} f_\gamma.$$

So we get that

$$\begin{cases} A_\alpha A_\beta + A_\beta A_\alpha = 2\sum_\gamma \lambda_\gamma^{\alpha\beta} A_\gamma \\ A_\alpha a^\beta + A_\beta a^\alpha = 2\sum_\gamma \lambda_\gamma^{\alpha\beta} a^\gamma \end{cases} \quad \text{and} \quad \begin{cases} A_\beta A_\alpha - A_\alpha A_\beta = \sum_\gamma \mu_\gamma^{\alpha\beta} A_\gamma \\ A_\beta a^\alpha - A_\alpha a^\beta = \sum_\gamma \mu_\gamma^{\alpha\beta} a_\gamma \end{cases}$$

where $\lambda_\gamma^{\alpha\beta}$, $\mu_\gamma^{\alpha\beta}$ are constant for each α, β, γ.

By solving the two systems of linear equations, using the fact that $A_\alpha \in \operatorname{Sym}(\mathbb{R}_p^{n+m})$ for each α, and that the set $\{A_\alpha \mid 1 \le \alpha \le m\}$ is linearly independent we deduce that

$$\lambda_\gamma^{\alpha\beta} = \lambda_\gamma^{\beta\alpha}, \ \mu_\gamma^{\alpha\beta} = 0,$$

$$A_\alpha A_\beta = A_\beta A_\alpha = \sum_\gamma \lambda_\gamma^{\alpha\beta} A_\gamma \quad \forall \alpha,\beta,\gamma, \quad 1 \le \alpha,\beta,\gamma \le m,$$

$$A_\alpha a^\beta = A_\beta a^\alpha = \sum_\gamma \lambda_\gamma^{\alpha\beta} a^\gamma.$$

Thus the conditions are necessary. By a simple calculation it is seen that the conditions are sufficient.

REMARK 4. Note that during the above proof we showed that $\mu_\gamma^{\alpha\beta} = 0 \ \forall \alpha,\beta,\gamma$, whence $[\operatorname{grad} f_\alpha, \operatorname{grad} f_\beta] = 0$. Thus we can give a sharper definition for quadratic isoparametric maps as follows.

DEFINITION 5. A quadratic map $f : \mathbb{R}_p^{n+m} \to \mathbb{R}^m$ is called *isoparametric* if

(i) $\langle \operatorname{grad} f_\alpha, \operatorname{grad} f_\beta \rangle$ and $\Delta f_\alpha = \operatorname{div}(\operatorname{grad} f_\alpha)$ are smooth functions of $f \ \forall \alpha,\beta$, $1 \le \alpha,\beta \le m$,

(ii) $[\operatorname{grad} f_\alpha, \operatorname{grad} f_\beta] = 0 \ \forall \alpha,\beta$, $1 \le \alpha,\beta \le m$.

REMARK 6. If we consider the affine maps (A_α, a^α), $1 \le \alpha \le m$, of the above theorem we see that they generate an m-dimensional commutative algebra $\mathscr{A}$ as follows.

Definitions of addition (+) of two such maps and multiplication (.) by scalars are as usual.

With these two operations $\mathscr{A}$ is a vector space. We define a multiplication $*$ in $\mathscr{A}$ as: $(A_\alpha, a^\alpha) * (A_\beta, a^\beta) = (A_\alpha A_\beta, \tfrac{1}{2}(A_\alpha a^\beta + A_\beta a^\beta))$ which by the above theorem is equivalent to $(A_\alpha, a^\alpha) * (A_\beta, a^\beta) = (A_\alpha A_\beta, A_\alpha \alpha^\beta)$. It is easily seen that $(\mathscr{A}, +, ., *)$ is an m-dimensional commutative algebra. Thus each quadratic isoparametric map gives rise to a commutative algebra.

Conversely, if we have any vector space $\mathscr{A}$ spanned by the set of affine maps $\{(A_\alpha, a^\alpha) : A_\alpha \in \operatorname{Sym}(\mathbb{R}_p^{n+m}), a^\alpha \in \mathbb{R}_p^{n+m}, 1 \le \alpha \le m, \{A_\alpha\}$ is a linearly independent set$\}$ and if $(\mathscr{A}, +, ., *)$ is an m-dimensional (commutative) algebra, then each basis $\{(B_\alpha, b^\alpha) : 1 \le \alpha \le m\}$ of $\mathscr{A}$ gives us a quadratic isoparametric map $g : \mathbb{R}_p^{n+m} \to \mathbb{R}^m$ defined by $g(x) = (\langle B_1 x, x \rangle + 2\langle b^1, x \rangle, \ldots, \langle B_m x, x \rangle + 2\langle b^m, x \rangle)$.

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REMARK 7. There exists an equivalence relation on the set of all quadratic isoparametric maps $f:\mathbb{R}_p^{n+m}\to\mathbb{R}^m$ defined by: $g\sim f$ if $g=T\circ f\circ h$ where $T\in GL(\mathbb{R}^m)$ and $h\in O_p(n+m)$, the linear isometry group of $\mathbb{R}_p^{n+m}$. We call each equivalence class of an isoparametric map, an *isoparametric system*. Certainly if we restrict the relation to the set of all quadratic isoparametric maps obtained from an algebra $\mathscr{A}$, we get just one equivalence class.

The following relation holds between quadratic isoparametric submanifolds and systems.

PROPOSITION 8. *If $M^n\subset\mathbb{R}_p^{n+m}$ is a full (i.e. M does not lie in any nondegenerate hyperplane of $\mathbb{R}_p^{n+m}$) quadratic isoparametric submanifold, then it determines a unique isoparametric system $\mathscr{A}$.*

Proof. Let M be a component of some nondegenerate regular level $f^{-1}(c)$ where $f:\mathbb{R}_p^{n+m}\to\mathbb{R}^m$ is a quadratic isoparametric map defined by $f(x)=(\langle A_1x,x\rangle+2\langle a^1,x\rangle,\dots,\langle A_mx,x\rangle+2\langle a^m,x\rangle)$. If g is another such map defined by $g(x)=(\langle B_1x,x\rangle+2\langle b^1,x\rangle,\dots,\langle B_mx,x\rangle+2\langle b^m,x\rangle)$ and M is a component of some nondegenerate regular level $g^{-1}(c)$, then we have $(\operatorname{grad} g_\alpha)(x)=\sum_\beta a_{\alpha\beta}(x)(\operatorname{grad} f_\beta)(x)\ \forall x\in M$. Since both $\operatorname{grad} f_\alpha$ and $\operatorname{grad} g_\beta$ are parallel normal fields, the coefficients $a_{\alpha\beta}(x)$ are constant on M. It is easily seen that the shape operator of M along $\operatorname{grad} g_\alpha$ is $-B_\alpha|_{TM}$ and along $\sum_\beta a_{\alpha\beta}\operatorname{grad} f_\beta$ is $-\sum_\beta a_{\alpha\beta}A_\beta|_{TM}$, so we have $B_\alpha|_{T_xM}=(\sum a_{\alpha\beta}A_\beta)|_{T_xM}\ \forall x\in M$. By using the relation $(\operatorname{grad} g_\alpha)(x)=\sum_\beta a_{\alpha\beta}(x)(\operatorname{grad} f_\beta)(x)$ we get that $B_\alpha x+b^\alpha=\sum_\beta a_{\alpha\beta}(A_\beta x+a^\beta)\ \forall x\in M$ so $b^\alpha=\sum_\beta a_{\alpha\beta}a^\beta\ \forall\alpha,\ 1\le\alpha\le m$.

Now we show that $B_\alpha=\sum_\beta a_{\alpha\beta}A_\beta\ \forall\alpha,\ 1\le\alpha\le m$.

Let W be the subspace of $\mathbb{R}_p^{n+m}$ spanned by the set $\{T_xM:x\in M\}$, i.e. each vector $v\in W$ is a finite direct sum of elements belonging to T_xM for various $x\in M$. If $W\neq\mathbb{R}_p^{n+m}$, then $W^\perp\neq\{0\}$, so there is a vector $0\neq v\in W^\perp$, i.e. $v\perp T_xM\ \forall x\in M$. This means that $\langle X,v\rangle=0$ for all tangent vector fields X on M, thus $\langle x,v\rangle=$ constant $\forall x\in M$, hence M lies in a hyperplane, and this contradicts the fact that M is full. Thus $B_\alpha=\sum_\beta a_{\alpha\beta}A_\beta$ on $\mathbb{R}_p^{n+m}$. So $(B_\alpha,b^\alpha)=\sum_\beta a_{\alpha\beta}(A_\beta,a^\beta)$.

3. The classification. To determine each quadratic isoparametric system of codimension 2, it is enough to find a basis $\{(A_1,a^1)(A_2,a^2)\}$ for the system; and in order to do that we try to find the simplest forms of elements of $\operatorname{Sym}(\mathbb{R}_p^n)$.

If $A\in\operatorname{Sym}(\mathbb{R}_p^n)$, there is some basis for $\mathbb{R}_p^n$ such that, with respect to it, A can be put in the form

$$A=\begin{bmatrix} B_1 & & & & & 0\\ & \ddots & & & & \\ & & B_k & & & \\ & & & C_1 & & \\ & & & & \ddots & \\ 0 & & & & & C_l\end{bmatrix}$$

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where B_i is an $s_i \times s_i$ matrix and C_j is a $2t_j \times 2t_j$ matrix, $\sum_{i=1}^{k} s_i + 2\sum_{j=1}^{l} t_j = n$, and we have

$$B_i = \begin{bmatrix} l_i\lambda_i & l_i & & \\ & \ddots & \ddots & \\ & & \ddots & l_i \\ & & & l_i\lambda_i \end{bmatrix}, \ l_i = \pm 1,$$

and

$$C_j = \begin{bmatrix} a_j & b_j & 1 & 0 & & & & \\ -b_j & a_j & 0 & 1 & & & & \\ & & a_j & b_j & 1 & 0 & & \\ & & -b_j & a_j & 0 & 1 & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & a_j & b_j \\ & & & & & & -b_j & a_j \end{bmatrix}, \ b_j \neq 0.$$

The matrix of the inner product is of the form

$$J = \begin{bmatrix} J_1 & & & & \\ & \ddots & & & \\ & & J_K & & \\ & & & J'_1 & \\ & & & & \ddots \\ & & & & & J'_l \end{bmatrix} \text{ with } J_i = \begin{bmatrix} & & l_i \\ & \iddots & \\ l_i & & \end{bmatrix} \text{ and } J'_m = \begin{bmatrix} & & & 1 & 0 \\ & & & 0 & -1 \\ & & \iddots & & \\ 1 & 0 & & & \\ 0 & -1 & & & \end{bmatrix}$$

Since the algebra $\mathscr{A}$ is two dimensional, the minimal polynomial $P_A(x)$ of each $A \in \mathscr{A}$ must satisfy $P_A(x) \mid (ax + bx^2 + cx^3)$, $a, b, c \in \mathbb{R}$. So each element of $\mathscr{A}$ can have (at most) three different simple real eigenvalues (if it has three different eigenvalues, one of them must be zero), or a simple zero eigenvalue and a nonsimple nonzero real eigenvalue with jordan block 2×2 or just an eigenvalue which appears as nonsimple eigenvalue with jordan block 2×2 (and also possibly as simple one), or a nonzero simple real eigenvalue and a zero nonsimple eigenvalue with jordan block 2×2, or a simple zero eigenvalue and a simple complex eigenvalue, (possibly with its conjugate), or just a simple complex eigenvalue (possibly with its conjugate). Using these facts one first finds the simplest forms of elements of $\mathrm{Sym}(\mathbb{R}^n_p)$ which satisfy the above condition. Choosing the simplest form of an $A_1 \in \mathrm{Sym}(\mathbb{R}^n_p)$ (satisfying the conditions) one looks for an $A_2 \in \mathrm{Sym}(\mathbb{R}^n_p)$ which also satisfies the required conditions and span $\{A_1, A_2\}$ is a 2-dimensional commutative algebra. At the end one chooses vectors $a^1, a^2 \in \mathbb{R}^n_p$ such that $\{(A_1, a^1), (A_2, a^2)\}$ is a basis for a 2-dimensional commutative algebra. By doing this process one gets (up to linear isomorphisms of $\mathbb{R}^n_p$) just one of the following systems.

(a) $\|x\|^2 = -x_1^2 - \ldots - x_p^2 + x_{p+1}^2 + \ldots + x_n^2$;
$f_1(x) = -x_1^2 - \ldots - x_p^2 + x_{p+1}^2 + \ldots + x_n^2 + 2(-a_1x_1 - \ldots - a_px_p + \ldots + a_nx_n)$;
$f_2(x) = \lambda(-x_1^2 - \ldots - x_p^2 + \ldots + x_i^2 - 2_1a_1x_1 - \ldots - 2a_px_p + \ldots + 2a_ix_i) +$
$\mu(x_{i+1}^2 + \ldots + x_n^2 + 2a_{i+1}x_{i+1} + \ldots + 2a_nx_n), \quad \lambda - \mu \neq 0, \quad 1 \leq i \leq n-1.$
Of course p can be greater than or equal to i.

(b) $\|x\|^2 = -x_1^2 - \ldots - x_p^2 + \ldots + x_n^2$;
$f_1(x) = \lambda(-x_{i+1}^2 - \ldots - x_p^2 + \ldots + x_{i+j}^2) + \mu(x_{i+j+1}^2 + \ldots + x_n^2) + 2(-a_{i+1}x_{i+1} - \ldots - a_px_p + \ldots + a_nx_n)$;
$f_2(x) = \lambda^2(-x_{i-1}^2 - \ldots - x_p^2 + \ldots + x_{i+j}^2) + \mu^2(x_{i+j+1}^2 + \ldots + x_n^2) + 2\lambda(-a_{i+1}x_{i+1} - \ldots - a_px_p + \ldots + a_nx_n) + 2\mu(a_{i+j+1}x_{i+j+1} + \ldots + a_nx_n)$; $\lambda, \ \mu, \ \lambda - \mu \neq 0$, $1 \leq i, j, n-i-j \leq n-2$.

There are other possibilities for $p : p \leq i$ or $p \geq i + j$.

(c) $\|x\|^2 = 2x_1x_2 - x_3^2 - \ldots - x_{p+1}^2 + \ldots + x_n^2$;
$f_1(x) = x_2^2 + 2(a_1x_2 + a_2x_1 - a_3x_3 - \ldots - a_{p+1}x_{p+1} + \ldots + a_nx_n)$;
$f_2(x) = 2x_2(\lambda x_1 + d_{p+1}x_{p+3} + \ldots + d_{n-2}x_n) + \langle A_2'x', x'\rangle' + 2(b_1x_2 + b_2x_1 - b_3x_3 - \ldots - b_{p+1}x_{p+1} + \ldots + b_nx_n)$
where $x' = (x_3, \ldots, x_n) \in \mathbb{R}^{n-2}_{p-1}$, $A_2' \in \mathrm{Sym}(\mathbb{R}^{n-2}_{p-1})$, $\|x'\|^2 = -x_3^2 - \ldots - x_{p+1}^2 + \ldots + x_n^2$ and $a^1 = (a_1, \ldots, a_n)$, $a^2 = (b_1, \ldots, b_n)$ satisfy the conditions in Theorem 3.

(d) $\|x\|^2 = 2(l_1x_1x_2 + \ldots + l_ix_{2i-1}x_{2i}) - x_{2i+1}^2 - \ldots - x_{p+i}^2 + \ldots + x_n^2$;
$f_1(x) = x_2^2 + x_4^2 + \ldots + x_{2i}^2 + 2\lambda(l_1x_1x_2 + \ldots + l_ix_{2i-1}x_{2i}) + \lambda(-x_{2i+1}^2 - \ldots - x_{p+1}^2 + \ldots + x_{2i+j}^2) + 2(l_1a_1x_2 + l_1a_2x_1 + \ldots + l_ia_{2i-1}x_{2i} + l_ia_{2i}x_{2i-1} - a_{2i+1}x_{2i+1} - \ldots - a_{p+i}x_{p+i} + \ldots + a_{2i+j}x_{2i+j})$;
$f_2(x) = 2\lambda(x_2^2 + x_4^2 + \ldots + x_{2i}^2) + 2\lambda^2(l_1x_1x_2 + \ldots + l_ix_{2i-1}x_{2i}) + \lambda^2(-x_{2i+1}^2 - \ldots - x_{p+i}^2 + \ldots + x_{2i+j}^2) + 2(a_2x_2 + \ldots + a_{2i}x_{2i}) + 2\lambda(l_1a_1x_2 + l_1a_2x_1 + \ldots + l_ia_{2i-1}x_{2i} + l_ia_{2i}x_{2i-1}) + 2\lambda(-a_{2i+1}x_{2i+1} - \ldots - a_{p+i}x_{p+i} + \ldots + a_{2i+j}x_{2i+j})$,
where $\lambda \neq 0$, each l_k is $+1$ or -1, $2 \leq 2i \leq \min\{2p, n\}$, $2i + j \leq n - 2$.

(e) $\|x\|^2$ as in the case (d);
$f_1(x) = (x_2^2 + x_4^2 + \ldots + x_{2i}^2) + \lambda(-x_{2i+j+1}^2 - \ldots - x_{p+i}^2 + \ldots + x_n^2) + 2(l_1a_1x_2 + l_2a_3x_4 + \ldots + l_ia_{2i-1}x_{2i} - a_{2i+1}x_{2i+1} - \ldots - a_{p+i}x_{p+i} + \ldots + a_nx_n)$;
$f_2(x) = \lambda^2(-x_{2i+j+1}^2 - \ldots - x_{p+i}^2 + \ldots + x_n^2) + 2\lambda(-a_{2i+j+1}x_{2i+j+1} - \ldots - a_{p+i}x_{p+i} + \ldots + a_nx_n)$;
$\lambda \neq 0$, each l_k is $+1$ or -1, $2 \leq 2i \leq \min\{2p, n-1\}$, $1 \leq n - i - j \leq n - 2$, $0 \leq j \leq n - 3$,

(f) $\|x\|^2$ as in the case (d);
$f_1(x)$ is as in the case (e) with $i + j = n$. If $i = 1$, then, this is the case (c).
If $i > 1$, we can not find the simplest form of $f_2(x)$ so we consider just the special case $n = 4$. We have then two subcases as follows.

(f.1) $f_1(x) = x_2^2 + x_4^2 + 2(l_1ax_2 + l_2bx_4)$;
$f_2(x) = l_1a_2(x_2^2 + x_4^2) + 2a_1(l_1x_1x_2 + l_2x_3x_4) + 2(l_1cx_2 + a_1ax_1 + l_2dx_4 + a_1bx_3)$.

(f.2) $f_1(x) = x_2^2 + x_4^2 + 2(l_1ax_2 + l_2bx_4)$;
$f_2(x) = l_1a_1x_2^2 + l_2^2 + l_2a_3x_4^2 + 2l_1a_2x_2x_4 + 2(cx_2 + dx_4)$.

In (f.1), (f.2) each l_i is $+1$ or -1.

(g) $\|x\|^2 = x_1^2 - x_2^2 + x_3^2 - x_4^2 + \ldots + x_{2i-1}^2 - x_{2i}^2 - (x_{2i+1}^2 + \ldots + x_{p+i}^2) + x_{p+i+1}^2 + \ldots + x_n^2$;
$f_1(x) = \alpha(x_1^2 - x_2^2 + \ldots + x_{2i-1}^2 - x_{2i}^2) + 2\beta(\epsilon_1x_1x_2 + \ldots + \epsilon_ix_{2i-1}x_{2i}) + 2(a_1x_1 - a_2x_2 + \ldots + a_{2i-1}x_{2i-1} - a_{2i}x_{2i})$;
$f_2(x) = (\alpha^2 - \beta^2)(x_1^2 - x_2^2 + \ldots + x_{2i-1}^2 - x_{2i}^2) + 2\alpha\beta(\epsilon_1x_1x_2 + \ldots + \epsilon_ix_{2i-1}x_{2i}) + 2[(\alpha a_1 + \epsilon_1\beta a_2)x_1 - (\alpha a_2 - \epsilon_1\beta a_1)x_2 + \ldots + (\alpha a_{2i-1} + \epsilon_i\beta a_{2i})x_{2i-1} - (\alpha a_{2i} - \epsilon_i\beta a_{2i-1})x_{2i}]$,

where $\epsilon_j = +1$ or -1, $1 \leq j \leq i$, $\beta \neq 0$, $2 \leq 2i \leq \min\{2p, n\}$. The systems obtained in (a) – (g) are all geometrically different, i.e. there is no invertible linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ and no $g \in O_p(n)$ such that $f^\beta = T \circ f \circ g$. f^α is the isoparametric map associated to the system α, (i.e. $f^\alpha(x) = (\langle A_1^\alpha x, x\rangle + 2\langle a^{1\alpha}, x\rangle, \langle A_2^\alpha x, x\rangle + 2\langle a^{2\alpha}, x\rangle)$, $\alpha, \beta \in \{a, \ldots, g\}$.

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4. Homogeneity of quadratic isoparametric submanifolds of codimension 2. We prove that the submanifold is homogeneous by using one of the three methods described as follows.

(i) The submanifold is a product of homogeneous spaces, so it is homogeneous.

(ii) We prove that the submanifold is "extrinsically symmetric" in $\mathbb{R}^n_p$, thus it is homogeneous.

(iii) By finding enough killing vector fields on the submanifold we show that it is homogeneous.

Before we begin the proof we give the definitions of ambient homogeneous and extrinsically symmetric submanifold.

Definition 9. The semi-Riemannian submanifold $M \subset \mathbb{R}^n_p$ is *ambient homogeneous* if the subgroup $G_M \subset I(\mathbb{R}^n_p)$ which leaves M invariant acts transitively on M.

Definition 10. The semi-Riemannian submanifold $M \subset \mathbb{R}^n_p$ is *extrinsically symmetric* in $\mathbb{R}^n_p$ if M is connected and $\forall x \in M$ there is a $\sigma_x \in I(\mathbb{R}^n_p)$ such that $\sigma_x(M) = M$ and $d\sigma_x|_{T_xM} = -\mathrm{id}_{T_xM}$.

In what follows $f:\mathbb{R}^n_p \to \mathbb{R}^2$ is a quadratic isoparametric map defined by $f(x) = (\langle A_1x, x\rangle + 2\langle a^1, x\rangle, \langle A_2x, x\rangle + 2\langle a^2, x\rangle)$ where $\langle , \rangle$ is specified in each case by giving the matrix of the inner product J. M is a component of some nondegenerate regular level $f^{-1}((c_1, c_2))$.

Note that as it is mentioned in Proposition 8 the shape operator of M (considered in Propositions 11–15) along $(\operatorname{grad} f_i)(x)$ is $-A_i|_{T_xM}$.

Proposition 11. *The submanifold M obtained from the system* (a) *or* (b) *is a product of some known homogeneous spaces, so it is homogeneous.*

Proof. If M is obtained from the system (a), then it is isometric to (a component of) $S^{i-1}_i(r_1) \times S^{n-i-1}_{p-i}(r_2)$ or $S^{i-1}_i(r_1) \times H^{n-i-1}_{p-i-1}(r_2)$ if $p \geq i$ and it is isometric to (a component of) $S^{i-1}_p(r_1) \times S^{n-i-1}(r_2)$ or $H^{i-1}_{p-1}(r_1) \times S^{n-i-1}(r_2)$ if $p < i$. Note that S^{i-1} is the $(i-1)$-dimensional sphere in $\mathbb{R}^i_i$. Thus in any case M is homogeneous.

If M is obtained from the system (b), it is isometric to $\mathbb{R}^i_p \times S^{j-1}(r_1) \times S^{k-1}(r_2)$ when $p \leq i$, and is isometric to (a component of) $\mathbb{R}^i_i \times S^{j-1}_{p-1}(r_1) \times S^{k-1}(r_2)$ or $\mathbb{R}^i_i \times H^{j-1}_{p-i-1}(r_1) \times S^{k-1}(r_2)$ when $i < p \leq i+j$ and is isometric to (a component of) $\mathbb{R}^i_i \times S^{j-1}_j(r_1) \times S^{k-1}_{p-i-j}(r_2)$ or (a component of) $\mathbb{R}^i_i \times S^{j-1}_j(r_1) \times H^{k-1}_{p-i-j-1}(r_2)$ when $p > i+j$.

Now we come to the cases for which we prove the submanifold is extrinsically symmetric in $\mathbb{R}^n_p$.

Proposition 12. *If M is obtained from the system* (c) (*with* $a^1 = a^2 = 0$), *it is extrinsically symmetric in* $\mathbb{R}^n_p$.

Proof. The isometries σ_x as in Definition 10 are defined by $\sigma_x(x) = x$, $d\sigma_x(x) = -x$, $\forall x \in T_xM$ and $d\sigma_x(v) = v$ $\forall v \in N_xM$. Then we obtain that $\sigma_x(y) = 2\dfrac{\langle y, A_2x\rangle}{\lambda c_1}A_1x - y + 2x$ $\forall y \in M$. If we use the relations $A_1^2 = 0$, $A_1A_2 = \lambda A_1$ and $A_2^2 = \left(\sum_{i=1}^{n-2} d_1^2\right)A_1 + \lambda A_2$, by an easy calculation we get that $f_i(\sigma_x(y)) = \langle A_i(\sigma_x(y)), \sigma_x(y)\rangle = \langle A_ix, x\rangle = f_i(x)$ $\forall x,y \in M$, $i = 1,2$. Thus M is extrinsically symmetric in $\mathbb{R}^n_p$, so it is homogeneous.

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PROPOSITION 13. *The semi-Riemannian submanifold M obtained from the system* (e) *is isometric to (a component of) $M_1 \times S$ where M_1 is some $(2i+j-1)$-dimensional semi-Riemannian submanifold of $\mathbb{R}_p^n$ which is extrinsically symmetric in some $(2i+j)$-dimensional subspace V and S is some $(n-2i-j-1)$-dimensional sphere or pseudo sphere or pseudo hyperbolic space.*

Proof. By simple calculation we get that M is isometric to (a component of) $M_1 \times S^{n-2i-j-1}(r)$ if $p \le 2i+j$ or $M_1 \times S_{p-2i-j}^{n-2i-j-1}(r)$ or $M_1 \times H_{p-2i-j-1}^{n-2i-j-1}(r)$ if $p > 2i+j$ where M_1 is the quadratic isoparametric hypersurface in $V = \{x : x \in \mathbb{R}_p^n,\ x_{2i+j+1} = \ldots = x_n = 0\}$ obtained by the quadratic isoparametric function $g(y) = \langle By, y\rangle + 2\langle b, y\rangle\ \forall y \in V$ where

$$B = \begin{bmatrix} \begin{array}{cc|} 0 & l_1 \\ 0 & 0 \\ \hline \end{array} & & & & \\ & \ddots & & & \\ & & \begin{array}{|cc|} \hline 0 & l_1 \\ 0 & 0 \\ \hline \end{array} & & \\ & & & \begin{array}{|c} 0 \end{array} & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix},$$

$b = (a_1, 0, a_3, \ldots, a_{2i-1}, 0, a_{2i+1}, \ldots, a_{2i+j})$.

By the same method used in Proposition 11 we see that M_1 is extrinsically symmetric in V. Thus M is homogeneous in $\mathbb{R}_p^n$.

For the system (d) when $i = 1$ the submanifold M obtained from the system is either some $(n-2)$-dimensional sphere or a component of a pseudo sphere or pseudo hyperbolic space or a product of some K-dimensional plane with some $(n-K-2)$ dimensional (pseudo) sphere. So in any case M is homogeneous.

Now we come to the cases for which we find enough Killing vector fields on M. The integral curves of such vector fields give us isometries in $I(\mathbb{R}_2^4)$ which leave M invariant and act transitively on M, so M is homogeneous. We should mention that in these cases we restrict ourselves to $n = 4$.

PROPOSITION 14. *The submanifold M obtained by the system*

$$A_1 = I_4,\ A_2 = \left[\begin{array}{cc|cc} \lambda & 1 & & \\ 0 & \lambda & \multicolumn{2}{c}{0} \\ \hline & & \lambda & 1 \\ \multicolumn{2}{c|}{0} & 0 & \lambda \end{array}\right],\ \lambda \in \mathbb{R},\ J = \left[\begin{array}{cc|cc} 0 & 1 & & \\ 1 & 0 & & \\ \hline & & 0 & 1 \\ & & 1 & 0 \end{array}\right],$$

$a^1 = (a_1, \ldots, a_4)$, $a^2 = (\lambda a_1 + a_2,\ \lambda a_2,\ \lambda a_3 + a_4,\ \lambda a_4)$ is homogeneous.

Proof. If we put $\begin{cases} f_1 = c_1 \\ f_2 = c_2 \end{cases}$, $y = x + a^1$ then an easy calculation shows that $M = \{x : x = y - a^1, 2(y_1 y_2 + y_3 y_4) = c_1 + a_1 a_2 + a_3 a_4 = d_1$ and $y_2^2 + y_4^2 = c_2 - \lambda c_1 + a_2^2 + a_4^2 = d_2\}$. Thus if we show that $M' = \{y : 2(y_1 y_2 + y_3 y_4) = d_1,\ y_2^2 + y_4^2 = d_2\}$ is homogeneous, then certainly M is homogeneous, since it is obtained from M' just with a translation by vector a^1. So for simplicity we can assume that $a^1 = 0$ which implies that $a^2 = 0$. We find that $S_1^*(x) = x_1 e_1 - x_2 e_2$, $S_2^*(x) = x_3 e_1 - x_2 e_4$, $S_3^*(x) = x_4 e_1 - x_2 e_3$, $S_4^*(x) = x_3 e_2 - x_1 e_4$, $S_5^*(x) =$

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$x_4e_2 - x_1e_3$, $S_6^*(x) = x_3e_3 - x_4e_4$ are Killing vector fields on $\mathbb{R}_2^4$, $e = \{e_1, e_2, e_3, e_4\}$ is obtained by the parallel transport of a basis of $\mathbb{R}_2^4$ such that with respect to it A_2 is in the given form and $\langle e_i, e_j\rangle = [J]_{ij}$. $x_1, \ldots, x_4$ are the coordinates of $x \in \mathbb{R}_p^n$ with respect to e. We observe that $X = S_3^*$, $Y = S_2^* + S_5^*$ are tangent to M at all points of M, thus $X \mid M$, $Y \mid M$ are Killing vector fields on M. If $x(t) = x_1(t)e_1 + x_2(t)e_2 + x_3(t)e_3 + x_4(t)e_4$ is the integral curve of X with the initial condition $x(0) = \mathring{x}_1e_1 + \mathring{x}_2e_2 + \mathring{x}_3e_3 + \mathring{x}_4e_4$ then we have $x_1(t) = \mathring{x}_4t + \mathring{x}_1$, $x_2(t) = \mathring{x}_2$, $x_3(t) = -\mathring{x}_2t + x_3$, $x_4(t) = \mathring{x}_4$.

Similarly for the integral curve of Y we have

$$y_1(t) = \mathring{x}_1 \cos t + \mathring{x}_3 \sin t, \; y_2(t) = \mathring{x}_2 \cos t + \mathring{x}_4 \sin t,$$

$$y_3(t) = \mathring{x}_3 \cos t - \mathring{x}_1 \sin t, \; y_4(t) = \mathring{x}_4 \cos t - \mathring{x}_2 \sin t.$$

If $\mathring{x}_1e_1 + \mathring{x}_2e_2 + \mathring{x}_3e_3 + \mathring{x}_4e_4 = \mathring{x}$, $x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 = x$ are two points of M, we look for a $g \in I(\mathbb{R}_2^4)$ such that $g(\mathring{x}) = x$ and $g(M) = M$. Since M is nondegenerate x_2, x_4 cannot be zero simultaneously for $x \in M$ thus we can pass from $\mathring{x}$ to the point $y_1e_1 + x_2e_2 + y_3e_3 + x_4e_4 \in M$ by the integral curve of Y and pass from this point to the point x by the integral curve of X. Hence M is homogeneous.

PROPOSITION 15. *The submanifold M obtained from the system*

$$A_1 = I_4,\; A_2 = \left[\begin{array}{cc|cc} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ \hline 0 & 0 & a & b \\ 0 & 0 & -b & a \end{array}\right],\; b \neq 0,\; J = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right],$$

$a^1 = (a_1, \ldots, a_4)$, $a^2 = (aa_1 + ba_2, -ba_1 + aa_2, aa_3 + ba_4, -ba_3 + aa_4)$, *is homogeneous.*

Proof. The same argument as in the proof of Proposition 14 shows that we can assume $a^1 = a^2 = 0$. We see that $S_1^*(x) = x_2e_1 + x_1e_2$, $S_2^*(x) = x_3e_1 - x_1e_3$, $S_3^*(x) = x_4e_1 + x_1e_4$, $S_4^*(x) = x_3e_2 + x_2e_3$, $S_5^*(x) = x_4e_2 - x_2e_4$, $S_6^*(x) = x_4e_3 + x_3e_4$, are Killing vector fields on $\mathbb{R}_2^4$, $\{e_1, \ldots, e_4\}$ is obtained by parallel translation of the basis of $\mathbb{R}_2^4$ such that with respect to it A_2 is in the given form and $\langle e_i, e_j\rangle = [J]_{ij}$. We also observe that $X(x) = S_2^* + S_5^*$, $Y(x) = S_3^* - S_4^*$ are in T_xM for each $x \in M$, thus $X|_M$, $Y|_M$ are Killing vector fields on M. Since $[x, y] = 0$, the general form of an element of the algebra generated by X, Y is $aX + bY$, $a, b \in \mathbb{R}$. By solving the related system of differential equations for the integral curve of $aX + bY$ with initial condition $x(0) = (\mathring{x}_1, \mathring{x}_2, \mathring{x}_3, \mathring{x}_4)$ we get the system

$$\left.\begin{aligned} x_1(t) &= (\mathring{x}_1 \cosh bt + \mathring{x}_4 \sinh bt)\cos at + (\mathring{x}_3 \cosh bt - \mathring{x}_2 \sinh bt)\sin at \\ x_2(t) &= (\mathring{x}_2 \cosh bt - \mathring{x}_3 \sinh bt)\cos at + (\mathring{x}_1 \sinh bt + \mathring{x}_4 \cosh bt)\sin at \\ x_3(t) &= (\mathring{x}_3 \cosh bt - \mathring{x}_2 \sinh bt)\cos at + (\mathring{x}_1 \sinh bt + \mathring{x}_4 \cosh bt)\sin at \\ x_4(t) &= (\mathring{x}_4 \cosh bt + \mathring{x}_1 \sinh bt)\cos at + (\mathring{x}_3 \sinh bt - \mathring{x}_2 \cosh bt)\sin at \end{aligned}\right\} \qquad (*)$$

Let G be the isometry group generated by the integral curves of elements of the algebra generated by X, Y. We prove that $G\mathring{x}$ is open and closed in M for some special point $\mathring{x} = \mathring{x}_3e_3 + \mathring{x}_4e_4 \in M$ and $\mathring{x}_3 \neq 0$. Since M is connected we see that $M = G\mathring{x}$, i.e. M is homogeneous.

We prove that G is a closed subgroup of $I(\mathbb{R}_2^4)$ by taking a sequence $\{g_n\}$ in G which converges to some $g \in I(\mathbb{R}_2^4)$ and showing that $g \in G$. This is easy, but messy. As a simple corollary we see that $G\mathring{x}$ is closed in M.

Now we prove that $G\mathring{x}$ is open in M. Let $\mathscr{A}_G(M)$ be the Lie algebra of G and K be the algebra of Killing vector fields generated by X, Y. Since all elements of K are complete, by using [**4**, Theorem 9.32 and Proposition 9.33] for $\mathscr{A}_G(M)$ and K we conclude that there is an isomorphism of $\mathscr{A}_G(M)$ onto K.

Since $\dim K = 2$ we have $\dim \mathscr{A}_G(M) = \dim T_eG = \dim G = \dim K = 2$.

Next we prove that there is a diffeomorphism ϕ between G and $G\mathring{x}$ as it gives us $\dim G\mathring{x} = \dim G = 2$ (in fact, we can define ϕ by $\phi(g) = g\mathring{x}$). The relation $G\mathring{x} \subset M$ with $\dim G\mathring{x} = \dim M = 2$ gives that $G\mathring{x}$ is open in M. So the proof is complete.

REMARK 16. For the system

$$A_1 = I_4,\ A_2 = \left[\begin{array}{cc|cc} a & b & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -b & a & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & a & b \\ & & -b & a \end{array}\right],\quad J$$

as before, we see, by the same method used in Proposition 15, that the submanifold M is homogeneous.

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