Bull. Aust. Math. Soc. (First published online 2024), page 1 of 13*
doi:10.1017/S0004972724000650
*Provisional—final page numbers to be inserted when paper edition is published

POINTS ON $x^4 + y^4 = z^4$ OVER QUADRATIC EXTENSIONS OF $\mathbb{Q}(\zeta_8)(T_1, T_2, \ldots, T_n)$

NGUYEN XUAN THO

(Received 23 March 2024; accepted 21 June 2024)

Dedicated to Professor Andrew Bremner

Abstract

Ishitsuka *et al.* ['Explicit calculation of the mod 4 Galois representation associated with the Fermat quartic', *Int. J. Number Theory* **16**(4) (2020), 881–905] found all points on the Fermat quartic $F_4\colon x^4 + y^4 = z^4$ over quadratic extensions of $\mathbb{Q}(\zeta_8)$, where ζ_8 is the eighth primitive root of unity $e^{i\pi/4}$. Using Mordell's technique, we give an alternative proof for the result of Ishitsuka *et al.* and extend it to the rational function field $\mathbb{Q}(\zeta_8)(T_1, T_2, \ldots, T_n)$.

2020 *Mathematics subject classification*: primary 11D25; secondary 11G05.

Keywords and phrases: Diophantine geometry, Diophantine equations, Fermat quartics, quadratic points.

1. Introduction

The problem of finding points on the Fermat quartic

$$F_4\colon x^4 + y^4 = z^4$$

over number fields has been studied by several authors. Fermat showed that F_4 only has trivial points over the rational numbers, where a trivial point on F_4 is a point $[x : y : z]$ with $xyz = 0$. Aigner [1] showed that if F_4 has nontrivial points in a quadratic number field $\mathbb{Q}(\sqrt{d})$, then $d = -7$. Faddeev [5] later found all points on F_4 over all quadratic number fields and all cubic number fields. Bremner and Choudhry [3] showed that F_4 only has trivial points in any cyclic cubic number field. Recently, Ishitsuka *et al.* found all points on F_4 over quadratic extensions of $\mathbb{Q}(\zeta_8)$.

THEOREM 1.1 (Ishitsuka *et al.* [6, Theorem 7.3]). *There are* 188 *points on* F_4 *defined over quadratic extensions of* $\mathbb{Q}(\zeta_8)$*:*

The author is supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) (grant number 101.04-2023.21).

(A) 12 *trivial points defined over* $\mathbb{Q}(\zeta_8)$*:* $[1 : \zeta_8^j : 0], [0 : \pm 1 : 1], [0 : \pm{\zeta_8}^2 : 1]$*, and* $[\pm 1 : 0 : 1], [\pm{\zeta_8}^2 : 0 : 1], j = 1, 3, 5, 7$*;*
(B) 48 *points defined over* $\mathbb{Q}(2^{1/4}\zeta_8)$*:* $[2^{1/4}\zeta_8^{2i} : \zeta_8^{1+2j} : 1], [\zeta_8^{1+2j} : 2^{1/4}\zeta_8^{2i} : 1]$, $[2^{-1/4}\zeta_8^{2i} : 2^{-1/4}\zeta_8^{2j} : 1], 0 \le i, j \le 3$*;*
(C) 32 *points defined over* $\mathbb{Q}(\zeta_3, \zeta_8)$*:* $[\zeta_3\zeta_8^{1+2i} : \zeta_3^2\zeta_8^{1+2j} : 1], [\zeta_3^2\zeta_8^{1+2i} : \zeta_3\zeta_8^{1+2j} : 1]$, $0 \le i, j \le 3$*;*
(D) 96 *points defined over* $\mathbb{Q}(\sqrt{-7}, \zeta_8)$*:* $[\alpha\zeta_8^{2i} : \overline{\alpha}\zeta_8^{2j} : 1], [\overline{\alpha}\zeta_8^{2i} : \alpha\zeta_8^{2j} : 1]$, $[\overline{\alpha}\zeta_8^{7+2j} : \zeta_8^6 : \alpha\zeta_8^{2i}]$, $[\alpha\zeta_8^{7+2j} : \zeta_8^6 : \overline{\alpha}\zeta_8^{2i}]$, $[1 : \alpha\zeta_8^{1+2i} : \overline{\alpha}\zeta_8^{2+2j}]$, $[1 : \overline{\alpha}\zeta_8^{1+2i} : \alpha\zeta_8^{2+2j}]$, $0 \le i, j \le 3$,

where $\zeta_3 = e^{2i\pi/3}$*,* $\alpha = (1 + \sqrt{-7})/2$ *and* $\overline{\alpha} = (1 - \sqrt{-7})/2$.

In this paper, we give an alternative proof for Theorem 1.1 and its extension to rational function fields.

THEOREM 1.2. *There are* 188 *points on* F_4 *defined over quadratic extensions of* $\mathbb{Q}(\zeta_8)(T_1, T_2, \ldots, T_n)$*. These points are the* 188 *points in Theorem 1.1.*

Before moving on to the proof of Theorem 1.2, we note that our approach is completely different from the approach of Ishitsuka *et al.* In [6], the authors use techniques from Galois representation theory and their proof relies on Rohrlich's result [11, Corollary 1, page 117] and Faddeev's result [5, Section 3, page 1150]. This paper is modelled on Mordell's paper [9], where he reproved Faddeev's result [5], see also [10, Theorem 4, pages 116–118]. The advantage of Mordell's approach is that it is simple, easy to use and concrete in calculations. For some other applications of this approach, see Li [7] and Manley [8].

2. Some preliminary results

LEMMA 2.1. *Let* k *be a field of characteristic not* 2*. Let* $K = k(T_1, T_2, \ldots, T_n)$ *be the function field over* k *generated by* n *algebraically independent variables* $T_1, T_2, \ldots, T_n$*. Let* E *be the elliptic curve over* K *given by* $y^2 = x^3 + Ax + B$*, where* $A, B \in k$*. Then,* $E(K) = E(k)$*.*

PROOF. The following proof of Lemma 2.1 is due to Professor Andrew Bremner (personal communication). We use induction on n. When $n = 1$, see Cohen [4, Proposition 7.3.2, pages 487–488]. Assume that Lemma 2.1 is true for n. Let $K = k(T_1, T_2, \ldots, T_n)$. Consider the elliptic curve E over $L = k(T_1, T_2, \ldots, T_{n+1})$ given by $y^2 = x^3 + Ax + B$, where $A, B \in k$. Since $L = K(T_{n+1})$ and $k \subset K$, by induction,

$$E(L) = E(K(T_{n+1})) = E(K) = E(k).$$

The proof is complete. □

Lemma 2.1 enables us to verify Lemmas 2.2, 2.3 and 2.4 using MAGMA [2].

https://doi.org/10.1017/S0004972724000650 Published online by Cambridge University Press

LEMMA 2.2. *All* $\mathbb{Q}(\zeta_8)(T_1, T_2, \ldots, T_n)$*-points on* $y^2 = 2x(x^2+1)$ *are*

$$\infty,\ (0,0),\ (1,\pm 2),\ (\pm{\zeta_8}^2, 0),\ (-1, \pm 2{\zeta_8}^2),$$

$$({\zeta_8}^3 + {\zeta_8}^2 + \zeta_8, \pm(2{\zeta_8}^3 + 2{\zeta_8}^2 - 2)),\ ({\zeta_8}^3 - {\zeta_8}^2 + \zeta_8, \pm(2{\zeta_8}^2 - 2\zeta_8 + 2)),$$

$$(-({\zeta_8}^3 + {\zeta_8}^2 + \zeta_8), \pm(2{\zeta_8}^2 + 2\zeta_8 + 2)),\ (-({\zeta_8}^3 - {\zeta_8}^2 + \zeta_8)), \pm(2{\zeta_8}^3 - 2{\zeta_8}^2 + 2)).$$

LEMMA 2.3. *All* $\mathbb{Q}(\zeta_8)(T_1, T_2, \ldots, T_n)$*-points on* $y^2 = 2x(x^2-1)$ *are*

$$\infty,\ (0,0),\ (\pm 1, 0),\ ({\zeta_8}^2, \pm 2{\zeta_8}^3),\ (-{\zeta_8}^2, \pm 2\zeta_8),$$

$$({\zeta_8}^3 - \zeta_8 - 1, \pm 2({\zeta_8}^3 + 2{\zeta_8}^2 + 2\zeta_8)),\ ({\zeta_8}^3 - \zeta_8 + 1, \pm(2{\zeta_8}^3 - 2\zeta_8 + 2)),$$

$$(-{\zeta_8}^3 + \zeta_8 + 1, \pm(2{\zeta_8}^3 - 2\zeta_8 - 2)),\ (-{\zeta_8}^3 + \zeta_8 - 1, \pm(2{\zeta_8}^3 - 2{\zeta_8}^2 + 2\zeta_8)).$$

LEMMA 2.4. *All* $\mathbb{Q}(\zeta_8)(T_1, T_2, \ldots, T_n)$*-points on* $s^2 = t^4 - 1$ *are*

$$\infty,\ (0, \pm{\zeta_8}^2),\ (\pm 1, 0),\ (\pm{\zeta_8}^2, 0),\ (\pm\zeta_8, \pm({\zeta_8}^3 + \zeta_8)),\ (\pm{\zeta_8}^3, \pm({\zeta_8}^3 + \zeta_8)).$$

3. Proof of Theorem 1.2

Let $K = \mathbb{Q}(\zeta_8)(T_1, T_2, \ldots, T_n)$. Let $L = K(\sqrt{d})$ be a quadratic extension of K, where $d \in K$ and $\sqrt{d} \notin K$. Assume $[x : y : z] \in F_4(L)$.

If $x = 0$, then $y^4 = z^4$. Hence,

$$[x : y : z] = [0 : \pm 1 : 1], [0 : \pm\zeta_8^2 : 1]. \tag{3.1}$$

Similarly, if $y = 0$, then

$$[x : y : z] = [\pm 1 : 0 : 1], [\pm\zeta_8^2 : 0 : 1]. \tag{3.2}$$

If $z = 0$, then $x^4 + y^4 = 0$. Hence,

$$[x : y : z] = [{\zeta_8}^j : 1 : 0], \quad j = 1, 3, 5, 7. \tag{3.3}$$

Note that the 12 points in (3.1), (3.2) and (3.3) are the 12 points in Theorem 1.1(A).

Assume $xyz \neq 0$. Let $z = 1$. Then,

$$x^4 + y^4 = 1. \tag{3.4}$$

Since $x \neq 0$, $y^2 \neq \pm 1$. From (3.4), $(1 + y^2)/x^2 = x^2/(1 - y^2)$. Let

$$t = \frac{1 + y^2}{x^2} \left(= \frac{x^2}{1 - y^2} \right). \tag{3.5}$$

Then, $t \neq 0$. If $t = \pm\zeta_8^2$, by (3.5),

$$-1 = t^2 = \frac{1 + y^2}{x^2} \cdot \frac{x^2}{1 - y^2} = \frac{1 + y^2}{1 - y^2}.$$

https://doi.org/10.1017/S0004972724000650 Published online by Cambridge University Press

Hence, $1 + y^2 = y^2 - 1$, which is impossible. If $t = \pm 1$, by (3.5),

$$1 = t^2 = \frac{1+y^2}{x^2} \cdot \frac{x^2}{1-y^2} = \frac{1+y^2}{1-y^2}.$$

Hence, $y = 0$, which is impossible. Therefore,

$$t \notin \{0, \pm 1, \pm \zeta_8^2\}. \tag{3.6}$$

It follows from (3.4) and (3.5) that

$$x^2 = \frac{2t}{t^2+1}, \quad y^2 = \frac{t^2-1}{t^2+1}, \tag{3.7}$$

Let $X = x(t^2+1)$ and $Y = xy(t^2+1)$. From (3.7),

$$X^2 = 2t(t^2+1), \tag{3.8}$$

$$Y^2 = 2t(t^2-1). \tag{3.9}$$

Case I: $t \in K$. Let $X = u + v\sqrt{d}$ and $Y = u_1 + v_1\sqrt{d}$, where $u, v, u_1, v_1 \in K$. From (3.8), $X^2, Y^2 \in K$. Since $\sqrt{d} \notin K$, $uv = u_1 v_1 = 0$.

Case I.1: $v = 0$. Then, $X = u \in K$. Thus, (t, u) is a K-point on (3.8). Therefore, by Lemma 2.2, $t = \pm({\zeta_8}^3 + {\zeta_8}^2 + \zeta_8), \pm({\zeta_8}^3 - {\zeta_8}^2 + \zeta_8)$. If $t = {\zeta_8}^3 + {\zeta_8}^2 + \zeta_8$, then

$$[x : y : z] = [\pm{\zeta_8}^3 : \pm\sqrt{\zeta_8 - {\zeta_8}^3} : 1]. \tag{3.10}$$

If $t = -({\zeta_8}^3 + {\zeta_8}^2 + \zeta_8)$, then

$$[x : y : z] = [\pm\zeta_8 : \pm\sqrt{\zeta_8 - {\zeta_8}^3} : 1]. \tag{3.11}$$

If $t = {\zeta_8}^3 - {\zeta_8}^2 + \zeta_8$, then

$$[x : y : z] = [\pm\zeta_8 : \pm\sqrt{{\zeta_8}^3 - \zeta_8} : 1]. \tag{3.12}$$

If $t = -({\zeta_8}^3 - {\zeta_8}^2 + \zeta_8)$, then

$$[x : y : z] = [\pm{\zeta_8}^3 : \pm\sqrt{{\zeta_8}^3 - \zeta_8} : 1]. \tag{3.13}$$

Case I.2: $v_1 = 0$. Then, (t, u_1) is a K-point on (3.9). Therefore, by Lemma 2.3, $t = \pm({\zeta_8}^3 - \zeta_8 - 1), \pm({\zeta_8}^3 - \zeta_8 + 1)$. If $t = {\zeta_8}^3 - \zeta_8 - 1$, then

$$[x : y : z] = \left[\pm\sqrt{\frac{{\zeta_8}^3 - \zeta_8}{2}} : \pm\sqrt{\frac{\zeta_8 - {\zeta_8}^3}{2}} : 1\right]. \tag{3.14}$$

If $t = -({\zeta_8}^3 - \zeta_8 - 1)$, then

$$[x : y : z] = \left[\pm\sqrt{\frac{\zeta_8 - {\zeta_8}^3}{2}} : \pm\sqrt{\frac{\zeta_8 - {\zeta_8}^3}{2}} : 1\right]. \tag{3.15}$$

https://doi.org/10.1017/S0004972724000650 Published online by Cambridge University Press

If $t = {\zeta_8}^3 - \zeta_8 + 1$, then

$$[x : y : z] = \left[\pm\sqrt{\frac{{\zeta_8}^3 - \zeta_8}{2}} : \pm\sqrt{\frac{{\zeta_8}^3 - \zeta_8}{2}} : 1\right]. \tag{3.16}$$

If $t = -({\zeta_8}^3 - \zeta_8 + 1)$, then

$$[x : y : z] = \left[\pm\sqrt{\frac{\zeta_8 - {\zeta_8}^3}{2}} : \pm\sqrt{\frac{{\zeta_8}^3 - \zeta_8}{2}} : 1\right]. \tag{3.17}$$

Case I.3: $vv_1 \neq 0$. Since $X^2, Y^2 \in K$ and $X, Y \notin K$, we have $u = u_1 = 0$. It follows from (3.8) and (3.9) that $dv^2 = 2t(t^2 + 1)$ and $dv_1^2 = 2t(t^2 - 1)$. Hence,

$$s^2 = (t^2 + 1)(t^2 - 1), \tag{3.18}$$

where $s = dvv_1/(2t) \in K$. By Lemma 2.4, $t = \pm\zeta_8, \pm\zeta_8^3$. If $t = {\zeta_8}^3$, then

$$[x : y : z] = [\pm\sqrt{{\zeta_8}^3 - \zeta_8} : \pm{\zeta_8}^3 : 1]. \tag{3.19}$$

If $t = -{\zeta_8}^3$, then

$$[x : y : z] = [\pm\sqrt{\zeta_8 - {\zeta_8}^3} : \pm{\zeta_8}^3 : 1]. \tag{3.20}$$

If $t = \zeta_8$, then

$$[x : y : z] = [\pm\sqrt{\zeta_8 - {\zeta_8}^3} : \pm\zeta_8 : 1]. \tag{3.21}$$

If $t = -\zeta_8$, then

$$[x : y : z] = [\pm\sqrt{{\zeta_8}^3 - \zeta_8} : \pm\zeta_8 : 1]. \tag{3.22}$$

Note that the 48 points in (3.10), (3.11), (3.12), (3.13), (3.14), (3.15), (3.16), (3.17), (3.19), (3.20), (3.21) and (3.22) are the 48 points in Theorem (1.1)(B).

Case II: $t \notin K$. Let $P(T) \in K[T]$ be the monic minimal polynomial of t over K. Then, $\deg P(T) = 2$.

Step 1: There exist $a, b \in K$ such that $X = at + b$. By (3.8), the polynomial $2T(T^2 + 1) - (aT + b)^2$ has a root $T = t$. Therefore, there exist $c, d \in K$ such that

$$2T(T^2 + 1) - (aT + b)^2 = P(T)(cT + d). \tag{3.23}$$

Then, $c = 2$ and $(-d/c, -ad/c + b)$ is a K-point on (3.8). Hence, by Lemma 2.3,

$$-d/c \in \{0, \pm 1, \pm{\zeta_8}^2, \pm({\zeta_8}^3 + {\zeta_8}^2 + \zeta_8), \pm({\zeta_8}^3 - {\zeta_8}^2 + \zeta_8)\}.$$

Case 1.1: $-d/c = 0$. Then, $b = d = 0$. From (3.23),

$$P(T) = T^2 - \frac{a^2}{2}T + 1. \tag{3.24}$$

https://doi.org/10.1017/S0004972724000650 Published online by Cambridge University Press

Case 1.2: $-d/c = 1$. Then, $d = -2$ and $a + b = \pm 2$. By changing the signs of a and b, we can assume that $a + b = -2$. From (3.23),

$$P(T) = T^2 + \Big(-\frac{a^2}{2} + 1\Big)T + \frac{a^2}{2} + 2a + 2. \tag{3.25}$$

Case 1.3: $-d/c = -1$. Then, $d = 2$ and $a - b = \pm 2{\zeta_8}^2$. We can assume that $a - b = 2{\zeta_8}^2$. From (3.23),

$$P(T) = T^2 + \Big(-\frac{a^2}{2} - 1\Big)T - \frac{a^2}{2} - 2{\zeta_8}^2 a + 2. \tag{3.26}$$

Case 1.4: $-d/c = {\zeta_8}^2$. Then, $d = -2{\zeta_8}^2$ and $b = -a{\zeta_8}^2$. From (3.23),

$$P(T) = T^2 + \Big(-\frac{a^2}{2} + {\zeta_8}^2\Big)T + \frac{1}{2}{\zeta_8}^2 a^2. \tag{3.27}$$

Case 1.5: $-d/c = -{\zeta_8}^2$. Then, $d = 2{\zeta_8}^2$ and $b = a{\zeta_8}^2$. From (3.23),

$$P(T) = T^2 + \Big(-\frac{a^2}{2} - {\zeta_8}^2\Big)T - \frac{1}{2}{\zeta_8}^2 a^2. \tag{3.28}$$

Case 1.6: $-d/c = {\zeta_8}^3 + {\zeta_8}^2 + \zeta_8$. Then, $d = -2({\zeta_8}^3 + {\zeta_8}^2 + \zeta_8)$. It follows that

$$({\zeta_8}^3 + {\zeta_8}^2 + \zeta_8)a + b = \pm 2({\zeta_8}^3 + {\zeta_8}^2 - 1).$$

We can assume that $b = 2({\zeta_8}^3 + {\zeta_8}^2 - 1) - ({\zeta_8}^3 + {\zeta_8}^2 + \zeta_8)a$. From (3.23),

$$\begin{aligned} P(T) = T^2 &+ \Big(-\frac{a^2}{2} + ({\zeta_8}^3 + {\zeta_8}^2 + \zeta_8)\Big)T + \frac{1}{2}({\zeta_8}^3 + {\zeta_8}^2 + \zeta_8)a^2 \\ &+ (-2{\zeta_8}^3 - 2{\zeta_8}^2 + 2)a + 2{\zeta_8}^3 - 2\zeta_8 - 2. \end{aligned} \tag{3.29}$$

Case 1.7 $-d/c = -({\zeta_8}^3 + {\zeta_8}^2 + \zeta_8)$. Then, $d = 2({\zeta_8}^3 + {\zeta_8}^2 + \zeta_8)$. Then,

$$-({\zeta_8}^3 + {\zeta_8}^2 + \zeta_8)a + b = \pm 2({\zeta_8}^2 + \zeta_8 + 1).$$

We can assume that $b = 2({\zeta_8}^2 + \zeta_8 + 1) + ({\zeta_8}^3 + {\zeta_8}^2 + \zeta_8)a$. From (3.23),

$$\begin{aligned} P(T) = T^2 &+ \Big(-\frac{a^2}{2} - {\zeta_8}^3 - {\zeta_8}^2 - \zeta_8\Big)T + \frac{1}{2}(-{\zeta_8}^3 - {\zeta_8}^2 - \zeta_8)a^2 \\ &+ (-2{\zeta_8}^2 - 2\zeta_8 - 2)a + 2{\zeta_8}^3 - 2\zeta_8 - 2. \end{aligned} \tag{3.30}$$

Case 1.8: $-d/c = {\zeta_8}^3 - {\zeta_8}^2 + \zeta_8$. Then, $d = -2({\zeta_8}^3 - {\zeta_8}^2 + \zeta_8)$ and

$$({\zeta_8}^3 - {\zeta_8}^2 + \zeta_8)a + b = \pm 2({\zeta_8}^2 - \zeta_8 + 1).$$

We can assume that

$$b = 2({\zeta_8}^2 - \zeta_8 + 1) - ({\zeta_8}^3 - {\zeta_8}^2 + \zeta_8)a.$$

https://doi.org/10.1017/S0004972724000650 Published online by Cambridge University Press

From (3.23),

$$P(T) = T^2 + \Big(-\frac{a^2}{2} + ({\zeta_8}^3 - {\zeta_8}^2 + \zeta_8)\Big)T + \frac{1}{2}({\zeta_8}^3 - {\zeta_8}^2 + \zeta_8)a^2 \\ + (-2{\zeta_8}^2 + 2\zeta_8 - 2)a - 2{\zeta_8}^3 + 2\zeta_8 - 2. \tag{3.31}$$

Case 1.9: $-d/c = -({\zeta_8}^3 - {\zeta_8}^2 + \zeta_8)$. Then, $d = 2({\zeta_8}^3 - {\zeta_8}^2 + \zeta_8)$ and

$$-({\zeta_8}^3 - {\zeta_8}^2 + \zeta_8)a + b = \pm 2({\zeta_8}^3 - {\zeta_8}^2 + 1).$$

We can assume that $b = 2({\zeta_8}^3 - {\zeta_8}^2 + 1) + ({\zeta_8}^3 - {\zeta_8}^2 + \zeta_8)a$. From (3.23),

$$P(T) = T^2 + \Big(-\frac{a^2}{2} - {\zeta_8}^3 + {\zeta_8}^2 - \zeta_8\Big)T + \frac{1}{2}(-{\zeta_8}^3 + {\zeta_8}^2 - \zeta_8)a^2 \\ + (-2{\zeta_8}^3 + 2{\zeta_8}^2 - 2)a - 2{\zeta_8}^3 + 2\zeta_8 - 2. \tag{3.32}$$

Step 2: There exist $a_1, b_1 \in K$ such that $Y = a_1 t + b_1$. Then, (3.9) shows that the polynomial $2T(T^2 - 1) - (a_1 T + b_1)^2$ has a root $T = t$. Hence, there exist $c_1, d_1 \in K$ such that

$$2T(T^2 - 1) - (a_1 T + b_1)^2 = P(T)(c_1 T + d_1). \tag{3.33}$$

Thus, $c_1 = 2$ and $(-d_1/c_1, -a_1 d_1/c_1 + b_1)$ is a finite K-point on (3.9). Hence

$$-d_1/c_1 \in \{0, \pm 1, \pm {\zeta_8}^2, \pm({\zeta_8}^3 - \zeta_8 - 1), \pm({\zeta_8}^3 - \zeta_8 + 1)\}.$$

Case 2.1: $-d_1/c_1 = 0$. Then, $b_1 = d_1 = 0$. From (3.33),

$$P(T) = T^2 - \frac{a_1^2}{2}T - 1. \tag{3.34}$$

Case 2.2: $-d_1/c_1 = 1$. Then, $d_1 = -2$ and $b_1 = -a_1$. From (3.33),

$$P(T) = T^2 + \Big(-\frac{a_1^2}{2} + 1\Big)T + \frac{a_1^2}{2}. \tag{3.35}$$

Case 2.3: $-d_1/c_1 = -1$. Then, $d_1 = 2$ and $b_1 = a_1$. From (3.33),

$$P(T) = T^2 + \Big(-\frac{a_1^2}{2} - 1\Big)T - \frac{a_1^2}{2}. \tag{3.36}$$

Case 2.4: $-d_1/c_1 = {\zeta_8}^3 - \zeta_8 - 1$. Then, $d_1 = -2({\zeta_8}^3 - \zeta_8 - 1)$ and

$$({\zeta_8}^3 - \zeta_8 - 1)a_1 + b_1 = \pm 2({\zeta_8}^3 + {\zeta_8}^2 + \zeta_8).$$

By changing the signs of a_1 and b_1, we can assume that

$$b_1 = 2({\zeta_8}^3 + {\zeta_8}^2 + \zeta_8) - ({\zeta_8}^3 - \zeta_8 - 1)a_1.$$

From (3.33),

$$P(T) = T^2 + \Big(-\frac{a_1^2}{2} + ({\zeta_8}^3 - \zeta_8 - 1)\Big)T + \frac{1}{2}({\zeta_8}^3 - \zeta_8 - 1)a_1^2 \\ + (-2{\zeta_8}^3 - 2{\zeta_8}^2 - 2\zeta_8)a_1 - 2{\zeta_8}^3 + 2\zeta_8 + 2. \tag{3.37}$$

https://doi.org/10.1017/S0004972724000650 Published online by Cambridge University Press

Case 2.5: $-d_1/c_1 = -(\zeta_8{}^3 - \zeta_8 - 1)$. Then, $d_1 = 2(\zeta_8{}^3 - \zeta_8 - 1)$ and

$$-(\zeta_8{}^3 - \zeta_8 - 1)a_1 + b_1 = \pm 2(\zeta_8{}^3 - \zeta_8 - 1).$$

We can assume that $b_1 = 2(\zeta_8{}^3 - \zeta_8 - 1) + (\zeta_8{}^3 - \zeta_8 - 1)a_1$. From (3.33),

$$P(T) = T^2 + \Big(-\frac{a_1^2}{2} - \zeta_8{}^3 + \zeta_8 + 1\Big)T + \frac{1}{2}(-\zeta_8{}^3 + \zeta_8 + 1)a_1^2 + (-2\zeta_8{}^3 + 2\zeta_8 + 2)a_1 - 2\zeta_8{}^3 + 2\zeta_8 + 2. \tag{3.38}$$

Case 2.6: $-d_1/c_1 = \zeta_8{}^2$. Then, $d_1 = -2\zeta_8{}^2$ and $\zeta_8{}^2 a_1 + b_1 = \pm 2\zeta_8{}^3$. We can assume that $b_1 = 2\zeta_8{}^3 - a_1\zeta_8{}^2$. From (3.33),

$$P(T) = T^2 + \Big(-\frac{a_1^2}{2} + \zeta_8{}^2\Big)T + \frac{1}{2}\zeta_8{}^2 a_1^2 - 2\zeta_8{}^3 a_1 - 2. \tag{3.39}$$

Case 2.7: $-d_1/c_1 = -\zeta_8{}^2$. Then, $d_1 = 2\zeta_8{}^2$ and $-\zeta_8{}^2 a_1 + b_1 = \pm 2\zeta_8$. We can assume that $b_1 = 2\zeta_8 + \zeta_8{}^2 a_1$. From (3.33),

$$P(T) = T^2 - \Big(\frac{a_1^2}{2} + \zeta_8{}^2\Big)T - \frac{1}{2}\zeta_8{}^2 a_1^2 - 2\zeta_8 a_1 - 2. \tag{3.40}$$

Case 2.8: $-d_1/c_1 = \zeta_8{}^3 - \zeta_8 + 1$. Then, $d_1 = -2(\zeta_8{}^3 - \zeta_8 + 1)$ and

$$(\zeta_8{}^3 - \zeta_8 + 1)a_1 + b_1 = \pm 2(\zeta_8{}^3 - \zeta_8 + 1).$$

We can assume that $b_1 = 2(\zeta_8{}^3 - \zeta_8 + 1) - (\zeta_8{}^3 - \zeta_8 + 1)a_1$. From (3.33),

$$P(T) = T^2 + \Big(-\frac{a_1^2}{2} + \zeta_8{}^3 - \zeta_8 + 1\Big)T + \frac{1}{2}(\zeta_8{}^3 - \zeta_8 + 1)a_1^2 + (-2\zeta_8{}^3 + 2\zeta_8 - 2)a_1 + 2\zeta_8{}^3 - 2\zeta_8 + 2. \tag{3.41}$$

Case 2.9: $-d_1/c_1 = -(\zeta_8{}^3 - \zeta_8 + 1)$. Then, $d_1 = 2(\zeta_8{}^3 - \zeta_8 + 1)$ and

$$-(\zeta_8{}^3 - \zeta_8 + 1)a_1 + b_1 = \pm 2(\zeta_8{}^3 - \zeta_8{}^2 + \zeta_8).$$

We can assume that $b_1 = 2(\zeta_8{}^3 - \zeta_8{}^2 + \zeta_8) + (\zeta_8{}^3 - \zeta_8 + 1)a_1$. From (3.33),

$$P(T) = T^2 + \Big(-\frac{a_1^2}{2} - \zeta_8{}^3 + \zeta_8 - 1\Big)T + \frac{1}{2}(-\zeta_8{}^3 + \zeta_8 - 1)a_1^2 + (-2\zeta_8{}^3 + 2\zeta_8{}^2 - 2\zeta_8)a_1 + 2\zeta_8{}^3 - 2\zeta_8 + 2. \tag{3.42}$$

Step 3: One polynomial from (3.24), (3.25), (3.26), (3.27), (3.28), (3.29), (3.30), (3.31) and (3.32) needs to match with one polynomial from (3.34), (3.35), (3.36), (3.37), (3.38), (3.39), (3.40), (3.41) and (3.42), resulting in 81 systems of equations in a, a_1. For each of these systems, MAGMA [2] is used to find a and a_1. MAGMA codes are available from the author on request. Even though $a, a_1 \in K$, each of these 81 systems of equations has coefficients in $\mathbb{Q}(\zeta_8)$, so if a solution with $a, a_1 \in K$ exists, then $a, a_1 \in \mathbb{Q}(\zeta_8)$. Our computation shows that only 20 of these 81 systems have

https://doi.org/10.1017/S0004972724000650 Published online by Cambridge University Press

solutions and only 16 of these 20 systems give an irreducible polynomial $P(T)$. All of the remaining 61 systems have no solutions $a, a_1 \in \mathbb{Q}(\zeta_8)$.

Case 3.1: (3.24) and (3.35). Then,

$$-\frac{a^2}{2} = -\frac{a_1^2}{2} + 1, \quad 1 = \frac{a_1^2}{2}.$$

Thus, $(a, a_1) = (0, \pm\sqrt{2})$. Hence $P(T) = T^2 + 1 = (T + \zeta_8{}^2)(T - \zeta_8{}^2)$, which is reducible in $K[T]$.

Case 3.2: (3.24) and (3.36). Then,

$$-\frac{a^2}{2} = -\frac{a_1^2}{2} - 1, \quad 1 = -\frac{a_1^2}{2}.$$

Thus, $(a, a_1) = (0, \pm\sqrt{-2})$. Hence, $P(T) = T^2 + 1 = (T + \zeta_8{}^2)(T - \zeta_8{}^2)$, which is reducible in $K[T]$.

Case 3.3: (3.25) and (3.35). Then,

$$-\frac{a^2}{2} + 1 = -\frac{a_1^2}{2} + 1, \quad \frac{a^2}{2} + 2a + 2 = \frac{a_1^2}{2}.$$

Thus, $(a, a_1) = (-1, \pm 1)$. Hence, $P(T) = T^2 + \frac{1}{2}T + \frac{1}{2}$. So $t^2 + \frac{1}{2}t + \frac{1}{2} = 0$. Therefore,

$$[x : y : z] = [\pm(2t + 1) : \pm 2t : 1]. \tag{3.43}$$

Case 3.4: (3.25) and (3.36). Then,

$$-\frac{a^2}{2} + 1 = -\frac{a_1^2}{2} - 1, \quad \frac{a^2}{2} + 2a + 2 = -\frac{a_1^2}{2}.$$

Hence, $(a, a_1) = (-2, 0), (0, \pm 2\zeta_8{}^2)$. The first solution gives $P(T) = T^2 - T$, which is reducible in $K[T]$. The second solution gives $P(T) = T^2 + T + 2$. So $t^2 + t + 2 = 0$. Therefore,

$$[x : y : z] = [\pm t : \pm(t + 1)\zeta_8{}^2 : 1]. \tag{3.44}$$

Case 3.5: (3.25) and (3.39). Then,

$$-\frac{a^2}{2} + 1 = -\frac{a_1^2}{2} + \zeta_8{}^2, \quad \frac{a^2}{2} + 2a + 2 = \frac{1}{2}\zeta_8{}^2 a_1^2 - 2\zeta_8{}^3 a_1 - 2.$$

Hence, $(a, a_1) = (\zeta_8{}^2 - 1, \zeta_8 + \zeta_8{}^3), (-(\zeta_8{}^2 + 3), 3\zeta_8 - \zeta_8{}^3)$. The first solution gives

$$P(T) = T^2 + (\zeta_8{}^2 + 1)T + \zeta_8{}^2 = (T + 1)(T + \zeta_8{}^2),$$

which is reducible in $K[T]$. The second solution gives $P(T) = T^2 - (3\zeta_8{}^2 + 3)T + \zeta_8{}^2$. So $t^2 - (3\zeta_8{}^2 + 3)t + \zeta_8{}^2 = 0$. Therefore,

$$[x : y : z] = \left[\pm\frac{(\zeta_8{}^2 - 1)t + \zeta_8{}^2 + 3}{4} : \pm\frac{(\zeta_8{}^3 + \zeta_8)(t - 3)}{4} : 1\right]. \tag{3.45}$$

https://doi.org/10.1017/S0004972724000650 Published online by Cambridge University Press

Case 3.6: (3.25) and (3.40). Then,

$$-\frac{a^2}{2}+1=-\frac{a_1^2}{2}-{\zeta_8}^2,\quad \frac{a^2}{2}+2a+2=-\frac{1}{2}{\zeta_8}^2a_1^2-2\zeta_8a_1-2.$$

Hence, $(a,a_1)=(-{\zeta_8}^2-1,\zeta_8+{\zeta_8}^3)$, $({\zeta_8}^2-3,3{\zeta_8}^3-\zeta_8)$. The first solution gives

$$P(T)=T^2+(-{\zeta_8}^2+1)T-{\zeta_8}^2=(T+1)(T-{\zeta_8}^2),$$

which is reducible in $K[T]$. The second solution gives $P(T)=T^2+(3{\zeta_8}^2-3)T-{\zeta_8}^2$. So $t^2+(3{\zeta_8}^2-3)t-{\zeta_8}^2=0$. Therefore,

$$[x:y:z]=\left[\pm\frac{({\zeta_8}^2+1)t+{\zeta_8}^2-3}{4}:\pm\frac{({\zeta_8}^3+\zeta_8)(t-3)}{4}:1\right]. \tag{3.46}$$

Case 3.7: (3.26) and (3.35). Then,

$$-\frac{a^2}{2}-1=-\frac{a_1^2}{2}+1,\quad -\frac{a^2}{2}-2{\zeta_8}^2a+2=\frac{a_1^2}{2}.$$

Hence, $(a,a_1)=(2{\zeta_8}^2,0)$, $(0,\pm2)$. The first solution gives $P(T)=T^2+T=T(T+1)$, which is reducible in $K[T]$. The second solution gives $P(T)=T^2-T+2$. So $t^2-t+2=0$. Therefore,

$$[x:y:z]=[\pm t{\zeta_8}^2:(t-1){\zeta_8}^2:1]. \tag{3.47}$$

Case 3.8: (3.26) and (3.36). Then,

$$-\frac{a^2}{2}-1=-\frac{a_1^2}{2}-1,\quad -\frac{a^2}{2}-2{\zeta_8}^2a+2=-\frac{a_1^2}{2}.$$

Hence, $(a,a_1)=({\zeta_8}^2,\pm{\zeta_8}^2)$. Therefore, $P(T)=T^2-\frac{1}{2}T+\frac{1}{2}$. So $t^2-\frac{1}{2}t+\frac{1}{2}=0$. Therefore,

$$[x:y:z]=[\pm(2t-1){\zeta_8}^2:\pm2t:1]. \tag{3.48}$$

Case 3.9: (3.26) and (3.39). Then,

$$-\frac{a^2}{2}-1=-\frac{a_1^2}{2}+{\zeta_8}^2,\quad -\frac{a^2}{2}+2{\zeta_8}^2a+2=\frac{1}{2}{\zeta_8}^2a_1^2-2{\zeta_8}^3a_1-2.$$

Hence, $(a,a_1)=({\zeta_8}^2-1,\zeta_8-{\zeta_8}^3)$, $(3{\zeta_8}^2+1,3\zeta_8+{\zeta_8}^3)$. The first solution gives

$$P(T)=T^2+({\zeta_8}^2-1)T-{\zeta_8}^2=(T-1)(T+{\zeta_8}^2),$$

which is reducible in $K[T]$. The second solution gives $P(T)=T^2+(-3{\zeta_8}^2+3)T-{\zeta_8}^2$. So $t^2+(-3{\zeta_8}^2+3)t-{\zeta_8}^2=0$. Therefore,

$$[x:y:z]=\left[\pm\frac{({\zeta_8}^2-1)t+3{\zeta_8}^2+1}{4}:\pm\frac{({\zeta_8}^3+\zeta_8)(t+3)}{4}:1\right]. \tag{3.49}$$

Case 3.10: (3.26) and (3.40). Then,

$$-\frac{a^2}{2} - 1 = -\frac{a_1^2}{2} - {\zeta_8}^2, \quad -\frac{a^2}{2} + 2{\zeta_8}^2 a + 2 = -\frac{1}{2}{\zeta_8}^2 a_1^2 - 2\zeta_8 a_1 - 2.$$

Hence, $(a, a_1) = ({\zeta_8}^2 + 1, \zeta_8 - {\zeta_8}^3)$, $(3{\zeta_8}^2 - 1, \zeta_8 + 3{\zeta_8}^3)$. The first solution gives

$$P(T) = T^2 + (-{\zeta_8}^2 - 1)T + {\zeta_8}^2 = (T - 1)(T - {\zeta_8}^2),$$

which is reducible in $K[T]$. The second solution gives $P(T) = T^2 + (3{\zeta_8}^2 + 3)T + {\zeta_8}^2$. So $t^2 + (3{\zeta_8}^2 + 3)t + {\zeta_8}^2 = 0$. Therefore,

$$[x : y : z] = \left[\pm\frac{({\zeta_8}^2 + 1)t + (3{\zeta_8}^2 - 1)}{4} : \pm\frac{({\zeta_8}^3 + \zeta_8)(t + 3)}{4} : 1\right]. \tag{3.50}$$

Case 3.11: (3.27) and (3.34). Then,

$$-\frac{a^2}{2} + {\zeta_8}^2 = -\frac{a_1^2}{2}, \quad \frac{1}{2}{\zeta_8}^2 a^2 = -1.$$

Hence, $(a, a_1) = (a, a_1) = (\pm\sqrt{2}\zeta_8, 0))$. Therefore, $P(T) = T^2 - 1$, which is reducible in $K[T]$.

Case 3.12: (3.27) and (3.35). Then,

$$-\frac{a^2}{2} + {\zeta_8}^2 = -\frac{a_1^2}{2} + 1, \quad \frac{1}{2}{\zeta_8}^2 a^2 = \frac{a_1^2}{2}.$$

Hence, $(a, a_1) = (\pm(\zeta_8 + {\zeta_8}^3), \pm({\zeta_8}^2 - 1))$. Therefore, $P(T) = T^2 + ({\zeta_8}^2 + 1)T - {\zeta_8}^2$. So $t^2 + ({\zeta_8}^2 + 1)t - {\zeta_8}^2 = 0$. Therefore,

$$[x : y : z] = \left[\pm\frac{({\zeta_8}^3 - \zeta_8)(t + 1)}{2} : \pm\frac{({\zeta_8}^3 - \zeta_8)t - {\zeta_8}^3 - \zeta_8)}{2} : 1\right]. \tag{3.51}$$

Case 3.13: (3.27) and (3.36). Then,

$$-\frac{a^2}{2} + {\zeta_8}^2 = -\frac{a_1^2}{2} - 1, \quad \frac{1}{2}{\zeta_8}^2 a^2 = -\frac{a_1^2}{2}.$$

Hence, $(a, a_1) = (\pm({\zeta_8}^3 - \zeta_8), \pm({\zeta_8}^2 - 1))$. Therefore, $P(T) = T^2 + ({\zeta_8}^2 - 1)T + {\zeta_8}^2$. So $t^2 + ({\zeta_8}^2 - 1)t + {\zeta_8}^2 = 0$. Therefore,

$$[x : y : z] = \left[\pm\left(\frac{({\zeta_8}^3 + \zeta_8)(1 - t)}{2}\right) : \pm\frac{({\zeta_8}^3 - \zeta_8)t - {\zeta_8}^3 - \zeta_8}{2} : 1\right]. \tag{3.52}$$

Case 3.14: (3.27) and (3.39). Then,

$$-\frac{a^2}{2} + {\zeta_8}^2 = -\frac{a_1^2}{2} + {\zeta_8}^2, \quad \frac{1}{2}{\zeta_8}^2 a^2 = \frac{1}{2}{\zeta_8}^2 a_1^2 - 2{\zeta_8}^3 a_1 - 2.$$

Hence, $(a, b) = (\pm\zeta_8, \zeta_8)$. Therefore, $P(T) = T^2 + \frac{1}{2}{\zeta_8}^2 T - \frac{1}{2}$. So $t^2 + \frac{1}{2}{\zeta_8}^2 t - \frac{1}{2} = 0$. Therefore,

$$[x : y : z] = \left[\pm\left(\zeta_8 t - \frac{{\zeta_8}^3}{2}\right) : \pm\left({\zeta_8}^2 t - \frac{1}{2}\right) : 1\right]. \tag{3.53}$$

https://doi.org/10.1017/S0004972724000650 Published online by Cambridge University Press

Case 3.15: (3.27) and (3.40). Then,

$$-\frac{a^2}{2}+\zeta_8{}^2=-\frac{a_1^2}{2}-\zeta_8{}^2,\quad \frac{1}{2}\zeta_8{}^2a^2=-\frac{1}{2}\zeta_8{}^2a_1^2-2\zeta_8a_1-2.$$

Hence, $(a,b)=(0,2\zeta_8{}^3),(\pm 2\zeta_8,0)$. The first solution gives $P(T)=T^2+\zeta_8{}^2T=T(T+\zeta_8{}^2)$, which is reducible in $K[T]$. The second solution gives $P(T)=T^2-\zeta_8{}^2T-2$. So $t^2-\zeta_8{}^2t-2=0$. Therefore,

$$[x:y:z]=\Big[\pm\Big(\frac{1}{2}\zeta_8t-\zeta_8{}^3\Big):\pm\frac{t}{2}:1\Big]. \tag{3.54}$$

Case 3.16: (3.28) and (3.34). Then,

$$-\frac{a^2}{2}-\zeta_8{}^2=-\frac{a_1^2}{2},\quad -\frac{1}{2}\zeta_8{}^2a^2=-1.$$

Hence, $(a,a_1)=(\pm\sqrt{2}\zeta_8^3,0)$. Therefore, $P(T)=T^2-1$, which is reducible in $K[T]$.

Case 3.17: (3.28) and (3.35). Then,

$$-\frac{a^2}{2}-\zeta_8{}^2=-\frac{a_1^2}{2}+1,\quad -\frac{1}{2}\zeta_8{}^2a^2=\frac{a_1^2}{2}.$$

Hence, $(a,a_1)=(\pm(\zeta_8+\zeta_8{}^3),\pm(1+\zeta_8{}^2))$. Thus, $P(T)=T^2+(1-\zeta_8{}^2)T+\zeta_8{}^2$. So $t^2+(1-\zeta_8{}^2)T+\zeta_8{}^2=0$. Therefore,

$$[x:y:z]=\Big[\pm\frac{(\zeta_8{}^3-\zeta_8)(t+1)}{2}\ :\ \pm\frac{(\zeta_8{}^3-\zeta_8)t+\zeta_8{}^3+\zeta_8}{2}:1\Big]. \tag{3.55}$$

Case 3.18: (3.28) and (3.36). Then,

$$-\frac{a^2}{2}-\zeta_8{}^2=-\frac{a_1^2}{2}-1,\quad -\frac{1}{2}\zeta_8{}^2a^2=-\frac{a_1^2}{2}.$$

Hence, $(a,a_1)=(\pm(\zeta_8{}^3-\zeta_8),\pm(\zeta_8{}^2+1))$. Thus, $P(T)=T^2-(\zeta_8{}^2+1)T-\zeta_8{}^2$. So $t^2-(\zeta_8{}^2+1)t-\zeta_8{}^2=0$. Therefore,

$$[x:y:z]=\Big[\pm\frac{(\zeta_8+\zeta_8{}^3)(t-1)}{2}\ :\ \pm\frac{(\zeta_8{}^3-\zeta_8)t+\zeta_8{}^3+\zeta_8}{2}:1\Big]. \tag{3.56}$$

Case 3.19: (3.28) and (3.39). Then,

$$-\frac{a^2}{2}-\zeta_8{}^2=-\frac{a_1^2}{2}+\zeta_8{}^2,\quad -\frac{1}{2}\zeta_8{}^2a^2=\frac{1}{2}\zeta_8{}^2a_1^2-2\zeta_8{}^3a_1-2.$$

Hence, $(a,a_1)=(0,2\zeta_8),(\pm 2\zeta_8{}^3,0)$. The first solution gives $P(T)=T^2-\zeta_8{}^2T$, which is reducible in $K[T]$. The second solution gives $P(T)=T^2+\zeta_8{}^2T-2$. So $t^2+\zeta_8{}^2t-2=0$. Therefore,

$$[x:y:z]=\Big[\pm\Big(\frac{1}{2}\zeta_8{}^3t-\zeta_8\Big):\pm\frac{t}{2}:1\Big]. \tag{3.57}$$

https://doi.org/10.1017/S0004972724000650 Published online by Cambridge University Press

Case 3.20: (3.28) and (3.40). Then,

$$-\frac{a^2}{2} - {\zeta_8}^2 = -\frac{a_1^2}{2} - {\zeta_8}^2, \quad -\frac{1}{2}{\zeta_8}^2 a^2 = -\frac{1}{2}{\zeta_8}^2 a_1^2 - 2\zeta_8 a_1 - 2.$$

Hence, $(a, a_1) = (\pm{\zeta_8}^3, {\zeta_8}^3)$. Thus, $P(T) = T^2 - \frac{1}{2}{\zeta_8}^2 T - \frac{1}{2}$. So $t^2 - \frac{1}{2}{\zeta_8}^2 t - \frac{1}{2} = 0$. Therefore,

$$[x : y : z] = \left[\pm\left({\zeta_8}^3 t - \frac{\zeta_8}{2}\right) : \pm\left({\zeta_8}^2 t + \frac{1}{2}\right) : 1\right]. \tag{3.58}$$

Note that the 32 points in (3.51), (3.52), (3.55) and (3.56) are the 32 points in Theorem 1.1(C) and the 96 points in (3.43), (3.44), (3.45), (3.46), (3.47), (3.48), (3.49), (3.50), (3.53), (3.54), (3.57) and (3.58) are the 96 points in Theorem 1.1(D).

The proof of Theorem 1.2 is complete.

Acknowledgement

The author is grateful to the referee for many suggestions to improve the presentation of the paper.

References

[1] A. Aigner, 'Über die Möglichkeit von $x^4 + y^4 = z^4$ in quadratische Körper', *J. Math. Verein.* **43** (1934), 226–228.

[2] W. Bosma, J. Cannon and C. Playoust, 'The MAGMA algebra system. I. The user language', *J. Symbolic Comput.* **24**(3–4) (1997), 235–265.

[3] A. Bremner and A. Choudhry, 'The Fermat cubic and quartic curves over cyclic fields', *Period. Math. Hungar.* **80** (2019), 147–157.

[4] H. Cohen, *Number Theory, Volume I: Tools and Diophantine Equations*, Graduate Texts in Mathematics, 239 (Springer, New York, 2007).

[5] D. K. Faddeev, 'Group of divisor classes on the curve defined by the equation $x^4 + y^4 = 1$', *Soviet Math. Dokl.* **1** (1960), 1149–1151; *Dokl. Akad. Nauk SSSR* **134** (1960), 776–777 (in Russian).

[6] Y. Ishitsuka, T. Ito and T. Oshita, 'Explicit calculation of the mod 4 Galois representation associated with the Fermat quartic', *Int. J. Number Theory* **16**(4) (2020), 881–905.

[7] A. Li, 'The Diophantine equation $x^4 + 2^n y^4 = 1$ in quadratic number fields', *Bull. Aust. Math. Soc.* **104** (2021), 21–28.

[8] E. D. Manley, 'On quadratic solutions of $x^4 + py^4 = z^4$', *Rocky Mountain J. Math.* **36**(3) (2006), 1027–1031.

[9] L. J. Mordell, 'The Diophantine equation $x^4 + y^4 = 1$ in algebraic number fields', *Acta Arith.* **14** (1967/1968), 347–355.

[10] L. J. Mordell, *Diophantine Equations*, Pure and Applied Mathematics, 30 (Academic Press, London–New York, 1969).

[11] D. E. Rohrlich, 'Points at infinity on the Fermat curves', *Invent. Math.* **39**(2) (1977), 95–127.

NGUYEN XUAN THO, School of Applied Mathematics and Informatics,
Hanoi University of Science and Technology, Hanoi, Vietnam
e-mail: tho.nguyenxuan1@hust.edu.vn