

POINTS ON $x^4 + y^4 = z^4$ OVER QUADRATIC EXTENSIONS OF $\mathbb{Q}(\zeta_8)(T_1, T_2, \dots, T_n)$

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Dedicated to Professor Andrew Bremner

Abstract

Ishitsuka *et al.* [‘Explicit calculation of the mod 4 Galois representation associated with the Fermat quartic’, *Int. J. Number Theory* **16**(4) (2020), 881–905] found all points on the Fermat quartic $F_4: x^4 + y^4 = z^4$ over quadratic extensions of $\mathbb{Q}(\zeta_8)$, where ζ_8 is the eighth primitive root of unity $e^{i\pi/4}$. Using Mordell’s technique, we give an alternative proof for the result of Ishitsuka *et al.* and extend it to the rational function field $\mathbb{Q}(\zeta_8)(T_1, T_2, \dots, T_n)$.

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1. Introduction

The problem of finding points on the Fermat quartic

$$F_4: x^4 + y^4 = z^4$$

over number fields has been studied by several authors. Fermat showed that F_4 only has trivial points over the rational numbers, where a trivial point on F_4 is a point $[x : y : z]$ with $xyz = 0$. Aigner [1] showed that if F_4 has nontrivial points in a quadratic number field $\mathbb{Q}(\sqrt{d})$, then $d = -7$. Faddeev [5] later found all points on F_4 over all quadratic number fields and all cubic number fields. Bremner and Choudhry [3] showed that F_4 only has trivial points in any cyclic cubic number field. Recently, Ishitsuka *et al.* found all points on F_4 over quadratic extensions of $\mathbb{Q}(\zeta_8)$.

THEOREM 1.1 (Ishitsuka *et al.* [6, Theorem 7.3]). *There are 188 points on F_4 defined over quadratic extensions of $\mathbb{Q}(\zeta_8)$:*

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- (A) 12 trivial points defined over $\mathbb{Q}(\zeta_8)$: $[1 : \zeta_8^j : 0], [0 : \pm 1 : 1], [0 : \pm \zeta_8^2 : 1]$, and $[\pm 1 : 0 : 1], [\pm \zeta_8^2 : 0 : 1], j = 1, 3, 5, 7$;
- (B) 48 points defined over $\mathbb{Q}(2^{1/4}\zeta_8)$: $[2^{1/4}\zeta_8^{2i} : \zeta_8^{1+2j} : 1], [\zeta_8^{1+2j} : 2^{1/4}\zeta_8^{2i} : 1], [2^{-1/4}\zeta_8^{2i} : 2^{-1/4}\zeta_8^{2j} : 1], 0 \leq i, j \leq 3$;
- (C) 32 points defined over $\mathbb{Q}(\zeta_3, \zeta_8)$: $[\zeta_3\zeta_8^{1+2i} : \zeta_3^2\zeta_8^{1+2j} : 1], [\zeta_3^2\zeta_8^{1+2i} : \zeta_3\zeta_8^{1+2j} : 1], 0 \leq i, j \leq 3$;
- (D) 96 points defined over $\mathbb{Q}(\sqrt{-7}, \zeta_8)$: $[\alpha\zeta_8^{2i} : \bar{\alpha}\zeta_8^{2j} : 1], [\bar{\alpha}\zeta_8^{2i} : \alpha\zeta_8^{2j} : 1], [\bar{\alpha}\zeta_8^{7+2j} : \zeta_8^6 : \alpha\zeta_8^{2i}], [\alpha\zeta_8^{7+2j} : \zeta_8^6 : \bar{\alpha}\zeta_8^{2i}], [1 : \alpha\zeta_8^{1+2i} : \bar{\alpha}\zeta_8^{2+2j}], [1 : \bar{\alpha}\zeta_8^{1+2i} : \alpha\zeta_8^{2+2j}], 0 \leq i, j \leq 3$,

where $\zeta_3 = e^{2\pi i/3}$, $\alpha = (1 + \sqrt{-7})/2$ and $\bar{\alpha} = (1 - \sqrt{-7})/2$.

In this paper, we give an alternative proof for Theorem 1.1 and its extension to rational function fields.

THEOREM 1.2. *There are 188 points on F_4 defined over quadratic extensions of $\mathbb{Q}(\zeta_8)(T_1, T_2, \dots, T_n)$. These points are the 188 points in Theorem 1.1.*

Before moving on to the proof of Theorem 1.2, we note that our approach is completely different from the approach of Ishitsuka *et al.* In [6], the authors use techniques from Galois representation theory and their proof relies on Rohrlich’s result [11, Corollary 1, page 117] and Faddeev’s result [5, Section 3, page 1150]. This paper is modelled on Mordell’s paper [9], where he reproved Faddeev’s result [5], see also [10, Theorem 4, pages 116–118]. The advantage of Mordell’s approach is that it is simple, easy to use and concrete in calculations. For some other applications of this approach, see Li [7] and Manley [8].

2. Some preliminary results

LEMMA 2.1. *Let k be a field of characteristic not 2. Let $K = k(T_1, T_2, \dots, T_n)$ be the function field over k generated by n algebraically independent variables T_1, T_2, \dots, T_n . Let E be the elliptic curve over K given by $y^2 = x^3 + Ax + B$, where $A, B \in k$. Then, $E(K) = E(k)$.*

PROOF. The following proof of Lemma 2.1 is due to Professor Andrew Bremner (personal communication). We use induction on n . When $n = 1$, see Cohen [4, Proposition 7.3.2, pages 487–488]. Assume that Lemma 2.1 is true for n . Let $K = k(T_1, T_2, \dots, T_n)$. Consider the elliptic curve E over $L = k(T_1, T_2, \dots, T_{n+1})$ given by $y^2 = x^3 + Ax + B$, where $A, B \in k$. Since $L = K(T_{n+1})$ and $k \subset K$, by induction,

$$E(L) = E(K(T_{n+1})) = E(K) = E(k).$$

The proof is complete. □

Lemma 2.1 enables us to verify Lemmas 2.2, 2.3 and 2.4 using MAGMA [2].

LEMMA 2.2. All $\mathbb{Q}(\zeta_8)(T_1, T_2, \dots, T_n)$ -points on $y^2 = 2x(x^2 + 1)$ are

$$\begin{aligned} & \infty, (0, 0), (1, \pm 2), (\pm \zeta_8^2, 0), (-1, \pm 2\zeta_8^2), \\ & (\zeta_8^3 + \zeta_8^2 + \zeta_8, \pm(2\zeta_8^3 + 2\zeta_8^2 - 2)), (\zeta_8^3 - \zeta_8^2 + \zeta_8, \pm(2\zeta_8^2 - 2\zeta_8 + 2)), \\ & (-\zeta_8^3 + \zeta_8^2 + \zeta_8, \pm(2\zeta_8^2 + 2\zeta_8 + 2)), (-\zeta_8^3 - \zeta_8^2 + \zeta_8), \pm(2\zeta_8^3 - 2\zeta_8^2 + 2)). \end{aligned}$$

LEMMA 2.3. All $\mathbb{Q}(\zeta_8)(T_1, T_2, \dots, T_n)$ -points on $y^2 = 2x(x^2 - 1)$ are

$$\begin{aligned} & \infty, (0, 0), (\pm 1, 0), (\zeta_8^2, \pm 2\zeta_8^3), (-\zeta_8^2, \pm 2\zeta_8), \\ & (\zeta_8^3 - \zeta_8 - 1, \pm 2(\zeta_8^3 + 2\zeta_8^2 + 2\zeta_8)), (\zeta_8^3 - \zeta_8 + 1, \pm(2\zeta_8^3 - 2\zeta_8 + 2)), \\ & (-\zeta_8^3 + \zeta_8 + 1, \pm(2\zeta_8^3 - 2\zeta_8 - 2)), (-\zeta_8^3 + \zeta_8 - 1, \pm(2\zeta_8^3 - 2\zeta_8^2 + 2\zeta_8)). \end{aligned}$$

LEMMA 2.4. All $\mathbb{Q}(\zeta_8)(T_1, T_2, \dots, T_n)$ -points on $s^2 = t^4 - 1$ are

$$\infty, (0, \pm \zeta_8^2), (\pm 1, 0), (\pm \zeta_8^2, 0), (\pm \zeta_8, \pm(\zeta_8^3 + \zeta_8)), (\pm \zeta_8^3, \pm(\zeta_8^3 + \zeta_8)).$$

3. Proof of Theorem 1.2

Let $K = \mathbb{Q}(\zeta_8)(T_1, T_2, \dots, T_n)$. Let $L = K(\sqrt{d})$ be a quadratic extension of K , where $d \in K$ and $\sqrt{d} \notin K$. Assume $[x : y : z] \in F_4(L)$.

If $x = 0$, then $y^4 = z^4$. Hence,

$$[x : y : z] = [0 : \pm 1 : 1], [0 : \pm \zeta_8^2 : 1]. \quad (3.1)$$

Similarly, if $y = 0$, then

$$[x : y : z] = [\pm 1 : 0 : 1], [\pm \zeta_8^2 : 0 : 1]. \quad (3.2)$$

If $z = 0$, then $x^4 + y^4 = 0$. Hence,

$$[x : y : z] = [\zeta_8^j : 1 : 0], \quad j = 1, 3, 5, 7. \quad (3.3)$$

Note that the 12 points in (3.1), (3.2) and (3.3) are the 12 points in Theorem 1.1(A).

Assume $xyz \neq 0$. Let $z = 1$. Then,

$$x^4 + y^4 = 1. \quad (3.4)$$

Since $x \neq 0$, $y^2 \neq \pm 1$. From (3.4), $(1 + y^2)/x^2 = x^2/(1 - y^2)$. Let

$$t = \frac{1 + y^2}{x^2} \left(= \frac{x^2}{1 - y^2} \right). \quad (3.5)$$

Then, $t \neq 0$. If $t = \pm \zeta_8^2$, by (3.5),

$$-1 = t^2 = \frac{1 + y^2}{x^2} \cdot \frac{x^2}{1 - y^2} = \frac{1 + y^2}{1 - y^2}.$$

Hence, $1 + y^2 = y^2 - 1$, which is impossible. If $t = \pm 1$, by (3.5),

$$1 = t^2 = \frac{1 + y^2}{x^2} \cdot \frac{x^2}{1 - y^2} = \frac{1 + y^2}{1 - y^2}.$$

Hence, $y = 0$, which is impossible. Therefore,

$$t \notin \{0, \pm 1, \pm \zeta_8^2\}. \tag{3.6}$$

It follows from (3.4) and (3.5) that

$$x^2 = \frac{2t}{t^2 + 1}, \quad y^2 = \frac{t^2 - 1}{t^2 + 1}, \tag{3.7}$$

Let $X = x(t^2 + 1)$ and $Y = xy(t^2 + 1)$. From (3.7),

$$X^2 = 2t(t^2 + 1), \tag{3.8}$$

$$Y^2 = 2t(t^2 - 1). \tag{3.9}$$

Case I: $t \in K$. Let $X = u + v\sqrt{d}$ and $Y = u_1 + v_1\sqrt{d}$, where $u, v, u_1, v_1 \in K$. From (3.8), $X^2, Y^2 \in K$. Since $\sqrt{d} \notin K$, $uv = u_1v_1 = 0$.

Case I.1: $v = 0$. Then, $X = u \in K$. Thus, (t, u) is a K -point on (3.8). Therefore, by Lemma 2.2, $t = \pm(\zeta_8^3 + \zeta_8^2 + \zeta_8), \pm(\zeta_8^3 - \zeta_8^2 + \zeta_8)$. If $t = \zeta_8^3 + \zeta_8^2 + \zeta_8$, then

$$[x : y : z] = [\pm \zeta_8^3 : \pm \sqrt{\zeta_8 - \zeta_8^3} : 1]. \tag{3.10}$$

If $t = -(\zeta_8^3 + \zeta_8^2 + \zeta_8)$, then

$$[x : y : z] = [\pm \zeta_8 : \pm \sqrt{\zeta_8 - \zeta_8^3} : 1]. \tag{3.11}$$

If $t = \zeta_8^3 - \zeta_8^2 + \zeta_8$, then

$$[x : y : z] = [\pm \zeta_8 : \pm \sqrt{\zeta_8^3 - \zeta_8} : 1]. \tag{3.12}$$

If $t = -(\zeta_8^3 - \zeta_8^2 + \zeta_8)$, then

$$[x : y : z] = [\pm \zeta_8^3 : \pm \sqrt{\zeta_8^3 - \zeta_8} : 1]. \tag{3.13}$$

Case I.2: $v_1 = 0$. Then, (t, u_1) is a K -point on (3.9). Therefore, by Lemma 2.3, $t = \pm(\zeta_8^3 - \zeta_8 - 1), \pm(\zeta_8^3 - \zeta_8 + 1)$. If $t = \zeta_8^3 - \zeta_8 - 1$, then

$$[x : y : z] = \left[\pm \sqrt{\frac{\zeta_8^3 - \zeta_8}{2}} : \pm \sqrt{\frac{\zeta_8 - \zeta_8^3}{2}} : 1 \right]. \tag{3.14}$$

If $t = -(\zeta_8^3 - \zeta_8 - 1)$, then

$$[x : y : z] = \left[\pm \sqrt{\frac{\zeta_8 - \zeta_8^3}{2}} : \pm \sqrt{\frac{\zeta_8^3 - \zeta_8}{2}} : 1 \right]. \tag{3.15}$$

If $t = \zeta_8^3 - \zeta_8 + 1$, then

$$[x : y : z] = \left[\pm\sqrt{\frac{\zeta_8^3 - \zeta_8}{2}} : \pm\sqrt{\frac{\zeta_8^3 - \zeta_8}{2}} : 1 \right]. \tag{3.16}$$

If $t = -(\zeta_8^3 - \zeta_8 + 1)$, then

$$[x : y : z] = \left[\pm\sqrt{\frac{\zeta_8 - \zeta_8^3}{2}} : \pm\sqrt{\frac{\zeta_8^3 - \zeta_8}{2}} : 1 \right]. \tag{3.17}$$

Case I.3: $vv_1 \neq 0$. Since $X^2, Y^2 \in K$ and $X, Y \notin K$, we have $u = u_1 = 0$. It follows from (3.8) and (3.9) that $dv^2 = 2t(t^2 + 1)$ and $dv_1^2 = 2t(t^2 - 1)$. Hence,

$$s^2 = (t^2 + 1)(t^2 - 1), \tag{3.18}$$

where $s = dvv_1/(2t) \in K$. By Lemma 2.4, $t = \pm\zeta_8, \pm\zeta_8^3$. If $t = \zeta_8^3$, then

$$[x : y : z] = [\pm\sqrt{\zeta_8^3 - \zeta_8} : \pm\zeta_8^3 : 1]. \tag{3.19}$$

If $t = -\zeta_8^3$, then

$$[x : y : z] = [\pm\sqrt{\zeta_8 - \zeta_8^3} : \pm\zeta_8^3 : 1]. \tag{3.20}$$

If $t = \zeta_8$, then

$$[x : y : z] = [\pm\sqrt{\zeta_8 - \zeta_8^3} : \pm\zeta_8 : 1]. \tag{3.21}$$

If $t = -\zeta_8$, then

$$[x : y : z] = [\pm\sqrt{\zeta_8^3 - \zeta_8} : \pm\zeta_8 : 1]. \tag{3.22}$$

Note that the 48 points in (3.10), (3.11), (3.12), (3.13), (3.14), (3.15), (3.16), (3.17), (3.19), (3.20), (3.21) and (3.22) are the 48 points in Theorem (1.1)(B).

Case II: $t \notin K$. Let $P(T) \in K[T]$ be the monic minimal polynomial of t over K . Then, $\deg P(T) = 2$.

Step 1: There exist $a, b \in K$ such that $X = at + b$. By (3.8), the polynomial $2T(T^2 + 1) - (aT + b)^2$ has a root $T = t$. Therefore, there exist $c, d \in K$ such that

$$2T(T^2 + 1) - (aT + b)^2 = P(T)(cT + d). \tag{3.23}$$

Then, $c = 2$ and $(-d/c, -ad/c + b)$ is a K -point on (3.8). Hence, by Lemma 2.3,

$$-d/c \in \{0, \pm 1, \pm\zeta_8^2, \pm(\zeta_8^3 + \zeta_8^2 + \zeta_8), \pm(\zeta_8^3 - \zeta_8^2 + \zeta_8)\}.$$

Case 1.1: $-d/c = 0$. Then, $b = d = 0$. From (3.23),

$$P(T) = T^2 - \frac{a^2}{2}T + 1. \tag{3.24}$$

Case 1.2: $-d/c = 1$. Then, $d = -2$ and $a + b = \pm 2$. By changing the signs of a and b , we can assume that $a + b = -2$. From (3.23),

$$P(T) = T^2 + \left(-\frac{a^2}{2} + 1\right)T + \frac{a^2}{2} + 2a + 2. \tag{3.25}$$

Case 1.3: $-d/c = -1$. Then, $d = 2$ and $a - b = \pm 2\zeta_8^2$. We can assume that $a - b = 2\zeta_8^2$. From (3.23),

$$P(T) = T^2 + \left(-\frac{a^2}{2} - 1\right)T - \frac{a^2}{2} - 2\zeta_8^2 a + 2. \tag{3.26}$$

Case 1.4: $-d/c = \zeta_8^2$. Then, $d = -2\zeta_8^2$ and $b = -a\zeta_8^2$. From (3.23),

$$P(T) = T^2 + \left(-\frac{a^2}{2} + \zeta_8^2\right)T + \frac{1}{2}\zeta_8^2 a^2. \tag{3.27}$$

Case 1.5: $-d/c = -\zeta_8^2$. Then, $d = 2\zeta_8^2$ and $b = a\zeta_8^2$. From (3.23),

$$P(T) = T^2 + \left(-\frac{a^2}{2} - \zeta_8^2\right)T - \frac{1}{2}\zeta_8^2 a^2. \tag{3.28}$$

Case 1.6: $-d/c = \zeta_8^3 + \zeta_8^2 + \zeta_8$. Then, $d = -2(\zeta_8^3 + \zeta_8^2 + \zeta_8)$. It follows that

$$(\zeta_8^3 + \zeta_8^2 + \zeta_8)a + b = \pm 2(\zeta_8^3 + \zeta_8^2 - 1).$$

We can assume that $b = 2(\zeta_8^3 + \zeta_8^2 - 1) - (\zeta_8^3 + \zeta_8^2 + \zeta_8)a$. From (3.23),

$$\begin{aligned} P(T) &= T^2 + \left(-\frac{a^2}{2} + (\zeta_8^3 + \zeta_8^2 + \zeta_8)\right)T + \frac{1}{2}(\zeta_8^3 + \zeta_8^2 + \zeta_8)a^2 \\ &\quad + (-2\zeta_8^3 - 2\zeta_8^2 + 2)a + 2\zeta_8^3 - 2\zeta_8 - 2. \end{aligned} \tag{3.29}$$

Case 1.7 $-d/c = -(\zeta_8^3 + \zeta_8^2 + \zeta_8)$. Then, $d = 2(\zeta_8^3 + \zeta_8^2 + \zeta_8)$. Then,

$$-(\zeta_8^3 + \zeta_8^2 + \zeta_8)a + b = \pm 2(\zeta_8^2 + \zeta_8 + 1).$$

We can assume that $b = 2(\zeta_8^2 + \zeta_8 + 1) + (\zeta_8^3 + \zeta_8^2 + \zeta_8)a$. From (3.23),

$$\begin{aligned} P(T) &= T^2 + \left(-\frac{a^2}{2} - \zeta_8^3 - \zeta_8^2 - \zeta_8\right)T + \frac{1}{2}(-\zeta_8^3 - \zeta_8^2 - \zeta_8)a^2 \\ &\quad + (-2\zeta_8^2 - 2\zeta_8 - 2)a + 2\zeta_8^3 - 2\zeta_8 - 2. \end{aligned} \tag{3.30}$$

Case 1.8: $-d/c = \zeta_8^3 - \zeta_8^2 + \zeta_8$. Then, $d = -2(\zeta_8^3 - \zeta_8^2 + \zeta_8)$ and

$$(\zeta_8^3 - \zeta_8^2 + \zeta_8)a + b = \pm 2(\zeta_8^2 - \zeta_8 + 1).$$

We can assume that

$$b = 2(\zeta_8^2 - \zeta_8 + 1) - (\zeta_8^3 - \zeta_8^2 + \zeta_8)a.$$

From (3.23),

$$P(T) = T^2 + \left(-\frac{a^2}{2} + (\zeta_8^3 - \zeta_8^2 + \zeta_8)\right)T + \frac{1}{2}(\zeta_8^3 - \zeta_8^2 + \zeta_8)a^2 + (-2\zeta_8^2 + 2\zeta_8 - 2)a - 2\zeta_8^3 + 2\zeta_8 - 2. \tag{3.31}$$

Case 1.9: $-d/c = -(\zeta_8^3 - \zeta_8^2 + \zeta_8)$. Then, $d = 2(\zeta_8^3 - \zeta_8^2 + \zeta_8)$ and

$$-(\zeta_8^3 - \zeta_8^2 + \zeta_8)a + b = \pm 2(\zeta_8^3 - \zeta_8^2 + 1).$$

We can assume that $b = 2(\zeta_8^3 - \zeta_8^2 + 1) + (\zeta_8^3 - \zeta_8^2 + \zeta_8)a$. From (3.23),

$$P(T) = T^2 + \left(-\frac{a^2}{2} - \zeta_8^3 + \zeta_8^2 - \zeta_8\right)T + \frac{1}{2}(-\zeta_8^3 + \zeta_8^2 - \zeta_8)a^2 + (-2\zeta_8^3 + 2\zeta_8^2 - 2)a - 2\zeta_8^3 + 2\zeta_8 - 2. \tag{3.32}$$

Step 2: There exist $a_1, b_1 \in K$ such that $Y = a_1t + b_1$. Then, (3.9) shows that the polynomial $2T(T^2 - 1) - (a_1T + b_1)^2$ has a root $T = t$. Hence, there exist $c_1, d_1 \in K$ such that

$$2T(T^2 - 1) - (a_1T + b_1)^2 = P(T)(c_1T + d_1). \tag{3.33}$$

Thus, $c_1 = 2$ and $(-d_1/c_1, -a_1d_1/c_1 + b_1)$ is a finite K -point on (3.9). Hence

$$-d_1/c_1 \in \{0, \pm 1, \pm \zeta_8^2, \pm(\zeta_8^3 - \zeta_8 - 1), \pm(\zeta_8^3 - \zeta_8 + 1)\}.$$

Case 2.1: $-d_1/c_1 = 0$. Then, $b_1 = d_1 = 0$. From (3.33),

$$P(T) = T^2 - \frac{a_1^2}{2}T - 1. \tag{3.34}$$

Case 2.2: $-d_1/c_1 = 1$. Then, $d_1 = -2$ and $b_1 = -a_1$. From (3.33),

$$P(T) = T^2 + \left(-\frac{a_1^2}{2} + 1\right)T + \frac{a_1^2}{2}. \tag{3.35}$$

Case 2.3: $-d_1/c_1 = -1$. Then, $d_1 = 2$ and $b_1 = a_1$. From (3.33),

$$P(T) = T^2 + \left(-\frac{a_1^2}{2} - 1\right)T - \frac{a_1^2}{2}. \tag{3.36}$$

Case 2.4: $-d_1/c_1 = \zeta_8^3 - \zeta_8 - 1$. Then, $d_1 = -2(\zeta_8^3 - \zeta_8 - 1)$ and

$$(\zeta_8^3 - \zeta_8 - 1)a_1 + b_1 = \pm 2(\zeta_8^3 + \zeta_8^2 + \zeta_8).$$

By changing the signs of a_1 and b_1 , we can assume that

$$b_1 = 2(\zeta_8^3 + \zeta_8^2 + \zeta_8) - (\zeta_8^3 - \zeta_8 - 1)a_1.$$

From (3.33),

$$P(T) = T^2 + \left(-\frac{a_1^2}{2} + (\zeta_8^3 - \zeta_8 - 1)\right)T + \frac{1}{2}(\zeta_8^3 - \zeta_8 - 1)a_1^2 + (-2\zeta_8^3 - 2\zeta_8^2 - 2\zeta_8)a_1 - 2\zeta_8^3 + 2\zeta_8 + 2. \tag{3.37}$$

Case 2.5: $-d_1/c_1 = -(\zeta_8^3 - \zeta_8 - 1)$. Then, $d_1 = 2(\zeta_8^3 - \zeta_8 - 1)$ and

$$-(\zeta_8^3 - \zeta_8 - 1)a_1 + b_1 = \pm 2(\zeta_8^3 - \zeta_8 - 1).$$

We can assume that $b_1 = 2(\zeta_8^3 - \zeta_8 - 1) + (\zeta_8^3 - \zeta_8 - 1)a_1$. From (3.33),

$$P(T) = T^2 + \left(-\frac{a_1^2}{2} - \zeta_8^3 + \zeta_8 + 1\right)T + \frac{1}{2}(-\zeta_8^3 + \zeta_8 + 1)a_1^2 + (-2\zeta_8^3 + 2\zeta_8 + 2)a_1 - 2\zeta_8^3 + 2\zeta_8 + 2. \tag{3.38}$$

Case 2.6: $-d_1/c_1 = \zeta_8^2$. Then, $d_1 = -2\zeta_8^2$ and $\zeta_8^2 a_1 + b_1 = \pm 2\zeta_8^3$. We can assume that $b_1 = 2\zeta_8^3 - a_1\zeta_8^2$. From (3.33),

$$P(T) = T^2 + \left(-\frac{a_1^2}{2} + \zeta_8^2\right)T + \frac{1}{2}\zeta_8^2 a_1^2 - 2\zeta_8^3 a_1 - 2. \tag{3.39}$$

Case 2.7: $-d_1/c_1 = -\zeta_8^2$. Then, $d_1 = 2\zeta_8^2$ and $-\zeta_8^2 a_1 + b_1 = \pm 2\zeta_8$. We can assume that $b_1 = 2\zeta_8 + \zeta_8^2 a_1$. From (3.33),

$$P(T) = T^2 - \left(\frac{a_1^2}{2} + \zeta_8^2\right)T - \frac{1}{2}\zeta_8^2 a_1^2 - 2\zeta_8 a_1 - 2. \tag{3.40}$$

Case 2.8: $-d_1/c_1 = \zeta_8^3 - \zeta_8 + 1$. Then, $d_1 = -2(\zeta_8^3 - \zeta_8 + 1)$ and

$$(\zeta_8^3 - \zeta_8 + 1)a_1 + b_1 = \pm 2(\zeta_8^3 - \zeta_8 + 1).$$

We can assume that $b_1 = 2(\zeta_8^3 - \zeta_8 + 1) - (\zeta_8^3 - \zeta_8 + 1)a_1$. From (3.33),

$$P(T) = T^2 + \left(-\frac{a_1^2}{2} + \zeta_8^3 - \zeta_8 + 1\right)T + \frac{1}{2}(\zeta_8^3 - \zeta_8 + 1)a_1^2 + (-2\zeta_8^3 + 2\zeta_8 - 2)a_1 + 2\zeta_8^3 - 2\zeta_8 + 2. \tag{3.41}$$

Case 2.9: $-d_1/c_1 = -(\zeta_8^3 - \zeta_8 + 1)$. Then, $d_1 = 2(\zeta_8^3 - \zeta_8 + 1)$ and

$$-(\zeta_8^3 - \zeta_8 + 1)a_1 + b_1 = \pm 2(\zeta_8^3 - \zeta_8^2 + \zeta_8).$$

We can assume that $b_1 = 2(\zeta_8^3 - \zeta_8^2 + \zeta_8) + (\zeta_8^3 - \zeta_8 + 1)a_1$. From (3.33),

$$P(T) = T^2 + \left(-\frac{a_1^2}{2} - \zeta_8^3 + \zeta_8 - 1\right)T + \frac{1}{2}(-\zeta_8^3 + \zeta_8 - 1)a_1^2 + (-2\zeta_8^3 + 2\zeta_8^2 - 2\zeta_8)a_1 + 2\zeta_8^3 - 2\zeta_8 + 2. \tag{3.42}$$

Step 3: One polynomial from (3.24), (3.25), (3.26), (3.27), (3.28), (3.29), (3.30), (3.31) and (3.32) needs to match with one polynomial from (3.34), (3.35), (3.36), (3.37), (3.38), (3.39), (3.40), (3.41) and (3.42), resulting in 81 systems of equations in a, a_1 . For each of these systems, MAGMA [2] is used to find a and a_1 . MAGMA codes are available from the author on request. Even though $a, a_1 \in K$, each of these 81 systems of equations has coefficients in $\mathbb{Q}(\zeta_8)$, so if a solution with $a, a_1 \in K$ exists, then $a, a_1 \in \mathbb{Q}(\zeta_8)$. Our computation shows that only 20 of these 81 systems have

solutions and only 16 of these 20 systems give an irreducible polynomial $P(T)$. All of the remaining 61 systems have no solutions $a, a_1 \in \mathbb{Q}(\zeta_8)$.

Case 3.1: (3.24) and (3.35). Then,

$$-\frac{a^2}{2} = -\frac{a_1^2}{2} + 1, \quad 1 = \frac{a_1^2}{2}.$$

Thus, $(a, a_1) = (0, \pm\sqrt{2})$. Hence $P(T) = T^2 + 1 = (T + \zeta_8^2)(T - \zeta_8^2)$, which is reducible in $K[T]$.

Case 3.2: (3.24) and (3.36). Then,

$$-\frac{a^2}{2} = -\frac{a_1^2}{2} - 1, \quad 1 = -\frac{a_1^2}{2}.$$

Thus, $(a, a_1) = (0, \pm\sqrt{-2})$. Hence, $P(T) = T^2 + 1 = (T + \zeta_8^2)(T - \zeta_8^2)$, which is reducible in $K[T]$.

Case 3.3: (3.25) and (3.35). Then,

$$-\frac{a^2}{2} + 1 = -\frac{a_1^2}{2} + 1, \quad \frac{a^2}{2} + 2a + 2 = \frac{a_1^2}{2}.$$

Thus, $(a, a_1) = (-1, \pm 1)$. Hence, $P(T) = T^2 + \frac{1}{2}T + \frac{1}{2}$. So $t^2 + \frac{1}{2}t + \frac{1}{2} = 0$. Therefore,

$$[x : y : z] = [\pm(2t + 1) : \pm 2t : 1]. \tag{3.43}$$

Case 3.4: (3.25) and (3.36). Then,

$$-\frac{a^2}{2} + 1 = -\frac{a_1^2}{2} - 1, \quad \frac{a^2}{2} + 2a + 2 = -\frac{a_1^2}{2}.$$

Hence, $(a, a_1) = (-2, 0), (0, \pm 2\zeta_8^2)$. The first solution gives $P(T) = T^2 - T$, which is reducible in $K[T]$. The second solution gives $P(T) = T^2 + T + 2$. So $t^2 + t + 2 = 0$. Therefore,

$$[x : y : z] = [\pm t : \pm(t + 1)\zeta_8^2 : 1]. \tag{3.44}$$

Case 3.5: (3.25) and (3.39). Then,

$$-\frac{a^2}{2} + 1 = -\frac{a_1^2}{2} + \zeta_8^2, \quad \frac{a^2}{2} + 2a + 2 = \frac{1}{2}\zeta_8^2 a_1^2 - 2\zeta_8^3 a_1 - 2.$$

Hence, $(a, a_1) = (\zeta_8^2 - 1, \zeta_8 + \zeta_8^3), (-\zeta_8^2 + 3), 3\zeta_8 - \zeta_8^3)$. The first solution gives

$$P(T) = T^2 + (\zeta_8^2 + 1)T + \zeta_8^2 = (T + 1)(T + \zeta_8^2),$$

which is reducible in $K[T]$. The second solution gives $P(T) = T^2 - (3\zeta_8^2 + 3)T + \zeta_8^2$. So $t^2 - (3\zeta_8^2 + 3)t + \zeta_8^2 = 0$. Therefore,

$$[x : y : z] = \left[\pm \frac{(\zeta_8^2 - 1)t + \zeta_8^2 + 3}{4} : \pm \frac{(\zeta_8^3 + \zeta_8)(t - 3)}{4} : 1 \right]. \tag{3.45}$$

Case 3.6: (3.25) and (3.40). Then,

$$-\frac{a^2}{2} + 1 = -\frac{a_1^2}{2} - \zeta_8^2, \quad \frac{a^2}{2} + 2a + 2 = -\frac{1}{2}\zeta_8^2 a_1^2 - 2\zeta_8 a_1 - 2.$$

Hence, $(a, a_1) = (-\zeta_8^2 - 1, \zeta_8 + \zeta_8^3), (\zeta_8^2 - 3, 3\zeta_8^3 - \zeta_8)$. The first solution gives

$$P(T) = T^2 + (-\zeta_8^2 + 1)T - \zeta_8^2 = (T + 1)(T - \zeta_8^2),$$

which is reducible in $K[T]$. The second solution gives $P(T) = T^2 + (3\zeta_8^2 - 3)T - \zeta_8^2$. So $t^2 + (3\zeta_8^2 - 3)t - \zeta_8^2 = 0$. Therefore,

$$[x : y : z] = \left[\pm \frac{(\zeta_8^2 + 1)t + \zeta_8^2 - 3}{4} : \pm \frac{(\zeta_8^3 + \zeta_8)(t - 3)}{4} : 1 \right]. \tag{3.46}$$

Case 3.7: (3.26) and (3.35). Then,

$$-\frac{a^2}{2} - 1 = -\frac{a_1^2}{2} + 1, \quad -\frac{a^2}{2} - 2\zeta_8^2 a + 2 = \frac{a_1^2}{2}.$$

Hence, $(a, a_1) = (2\zeta_8^2, 0), (0, \pm 2)$. The first solution gives $P(T) = T^2 + T = T(T + 1)$, which is reducible in $K[T]$. The second solution gives $P(T) = T^2 - T + 2$. So $t^2 - t + 2 = 0$. Therefore,

$$[x : y : z] = [\pm t\zeta_8^2 : (t - 1)\zeta_8^2 : 1]. \tag{3.47}$$

Case 3.8: (3.26) and (3.36). Then,

$$-\frac{a^2}{2} - 1 = -\frac{a_1^2}{2} - 1, \quad -\frac{a^2}{2} - 2\zeta_8^2 a + 2 = -\frac{a_1^2}{2}.$$

Hence, $(a, a_1) = (\zeta_8^2, \pm\zeta_8^2)$. Therefore, $P(T) = T^2 - \frac{1}{2}T + \frac{1}{2}$. So $t^2 - \frac{1}{2}t + \frac{1}{2} = 0$. Therefore,

$$[x : y : z] = [\pm(2t - 1)\zeta_8^2 : \pm 2t : 1]. \tag{3.48}$$

Case 3.9: (3.26) and (3.39). Then,

$$-\frac{a^2}{2} - 1 = -\frac{a_1^2}{2} + \zeta_8^2, \quad -\frac{a^2}{2} + 2\zeta_8^2 a + 2 = \frac{1}{2}\zeta_8^2 a_1^2 - 2\zeta_8^3 a_1 - 2.$$

Hence, $(a, a_1) = (\zeta_8^2 - 1, \zeta_8 - \zeta_8^3), (3\zeta_8^2 + 1, 3\zeta_8 + \zeta_8^3)$. The first solution gives

$$P(T) = T^2 + (\zeta_8^2 - 1)T - \zeta_8^2 = (T - 1)(T + \zeta_8^2),$$

which is reducible in $K[T]$. The second solution gives $P(T) = T^2 + (-3\zeta_8^2 + 3)T - \zeta_8^2$. So $t^2 + (-3\zeta_8^2 + 3)t - \zeta_8^2 = 0$. Therefore,

$$[x : y : z] = \left[\pm \frac{(\zeta_8^2 - 1)t + 3\zeta_8^2 + 1}{4} : \pm \frac{(\zeta_8^3 + \zeta_8)(t + 3)}{4} : 1 \right]. \tag{3.49}$$

Case 3.10: (3.26) and (3.40). Then,

$$-\frac{a^2}{2} - 1 = -\frac{a_1^2}{2} - \zeta_8^2, \quad -\frac{a^2}{2} + 2\zeta_8^2 a + 2 = -\frac{1}{2}\zeta_8^2 a_1^2 - 2\zeta_8 a_1 - 2.$$

Hence, $(a, a_1) = (\zeta_8^2 + 1, \zeta_8 - \zeta_8^3), (3\zeta_8^2 - 1, \zeta_8 + 3\zeta_8^3)$. The first solution gives

$$P(T) = T^2 + (-\zeta_8^2 - 1)T + \zeta_8^2 = (T - 1)(T - \zeta_8^2),$$

which is reducible in $K[T]$. The second solution gives $P(T) = T^2 + (3\zeta_8^2 + 3)T + \zeta_8^2$. So $t^2 + (3\zeta_8^2 + 3)t + \zeta_8^2 = 0$. Therefore,

$$[x : y : z] = \left[\pm \frac{(\zeta_8^2 + 1)t + (3\zeta_8^2 - 1)}{4} : \pm \frac{(\zeta_8^3 + \zeta_8)(t + 3)}{4} : 1 \right]. \tag{3.50}$$

Case 3.11: (3.27) and (3.34). Then,

$$-\frac{a^2}{2} + \zeta_8^2 = -\frac{a_1^2}{2}, \quad \frac{1}{2}\zeta_8^2 a^2 = -1.$$

Hence, $(a, a_1) = (a, a_1) = (\pm\sqrt{2}\zeta_8, 0)$. Therefore, $P(T) = T^2 - 1$, which is reducible in $K[T]$.

Case 3.12: (3.27) and (3.35). Then,

$$-\frac{a^2}{2} + \zeta_8^2 = -\frac{a_1^2}{2} + 1, \quad \frac{1}{2}\zeta_8^2 a^2 = \frac{a_1^2}{2}.$$

Hence, $(a, a_1) = (\pm(\zeta_8 + \zeta_8^3), \pm(\zeta_8^2 - 1))$. Therefore, $P(T) = T^2 + (\zeta_8^2 + 1)T - \zeta_8^2$. So $t^2 + (\zeta_8^2 + 1)t - \zeta_8^2 = 0$. Therefore,

$$[x : y : z] = \left[\pm \frac{(\zeta_8^3 - \zeta_8)(t + 1)}{2} : \pm \frac{(\zeta_8^3 - \zeta_8)t - \zeta_8^3 - \zeta_8}{2} : 1 \right]. \tag{3.51}$$

Case 3.13: (3.27) and (3.36). Then,

$$-\frac{a^2}{2} + \zeta_8^2 = -\frac{a_1^2}{2} - 1, \quad \frac{1}{2}\zeta_8^2 a^2 = -\frac{a_1^2}{2}.$$

Hence, $(a, a_1) = (\pm(\zeta_8^3 - \zeta_8), \pm(\zeta_8^2 - 1))$. Therefore, $P(T) = T^2 + (\zeta_8^2 - 1)T + \zeta_8^2$. So $t^2 + (\zeta_8^2 - 1)t + \zeta_8^2 = 0$. Therefore,

$$[x : y : z] = \left[\pm \left(\frac{(\zeta_8^3 + \zeta_8)(1 - t)}{2} \right) : \pm \frac{(\zeta_8^3 - \zeta_8)t - \zeta_8^3 - \zeta_8}{2} : 1 \right]. \tag{3.52}$$

Case 3.14: (3.27) and (3.39). Then,

$$-\frac{a^2}{2} + \zeta_8^2 = -\frac{a_1^2}{2} + \zeta_8^2, \quad \frac{1}{2}\zeta_8^2 a^2 = \frac{1}{2}\zeta_8^2 a_1^2 - 2\zeta_8^3 a_1 - 2.$$

Hence, $(a, b) = (\pm\zeta_8, \zeta_8)$. Therefore, $P(T) = T^2 + \frac{1}{2}\zeta_8^2 T - \frac{1}{2}$. So $t^2 + \frac{1}{2}\zeta_8^2 t - \frac{1}{2} = 0$. Therefore,

$$[x : y : z] = \left[\pm \left(\zeta_8 t - \frac{\zeta_8^3}{2} \right) : \pm \left(\zeta_8^2 t - \frac{1}{2} \right) : 1 \right]. \tag{3.53}$$

Case 3.15: (3.27) and (3.40). Then,

$$-\frac{a^2}{2} + \zeta_8^2 = -\frac{a_1^2}{2} - \zeta_8^2, \quad \frac{1}{2}\zeta_8^2 a^2 = -\frac{1}{2}\zeta_8^2 a_1^2 - 2\zeta_8 a_1 - 2.$$

Hence, $(a, b) = (0, 2\zeta_8^3), (\pm 2\zeta_8, 0)$. The first solution gives $P(T) = T^2 + \zeta_8^2 T = T(T + \zeta_8^2)$, which is reducible in $K[T]$. The second solution gives $P(T) = T^2 - \zeta_8^2 T - 2$. So $t^2 - \zeta_8^2 t - 2 = 0$. Therefore,

$$[x : y : z] = \left[\pm \left(\frac{1}{2}\zeta_8 t - \zeta_8^3 \right) : \pm \frac{t}{2} : 1 \right]. \tag{3.54}$$

Case 3.16: (3.28) and (3.34). Then,

$$-\frac{a^2}{2} - \zeta_8^2 = -\frac{a_1^2}{2}, \quad -\frac{1}{2}\zeta_8^2 a^2 = -1.$$

Hence, $(a, a_1) = (\pm\sqrt{2}\zeta_8^3, 0)$. Therefore, $P(T) = T^2 - 1$, which is reducible in $K[T]$.

Case 3.17: (3.28) and (3.35). Then,

$$-\frac{a^2}{2} - \zeta_8^2 = -\frac{a_1^2}{2} + 1, \quad -\frac{1}{2}\zeta_8^2 a^2 = \frac{a_1^2}{2}.$$

Hence, $(a, a_1) = (\pm(\zeta_8 + \zeta_8^3), \pm(1 + \zeta_8^2))$. Thus, $P(T) = T^2 + (1 - \zeta_8^2)T + \zeta_8^2$. So $t^2 + (1 - \zeta_8^2)T + \zeta_8^2 = 0$. Therefore,

$$[x : y : z] = \left[\pm \frac{(\zeta_8^3 - \zeta_8)(t + 1)}{2} : \pm \frac{(\zeta_8^3 - \zeta_8)t + \zeta_8^3 + \zeta_8}{2} : 1 \right]. \tag{3.55}$$

Case 3.18: (3.28) and (3.36). Then,

$$-\frac{a^2}{2} - \zeta_8^2 = -\frac{a_1^2}{2} - 1, \quad -\frac{1}{2}\zeta_8^2 a^2 = -\frac{a_1^2}{2}.$$

Hence, $(a, a_1) = (\pm(\zeta_8^3 - \zeta_8), \pm(\zeta_8^2 + 1))$. Thus, $P(T) = T^2 - (\zeta_8^2 + 1)T - \zeta_8^2$. So $t^2 - (\zeta_8^2 + 1)t - \zeta_8^2 = 0$. Therefore,

$$[x : y : z] = \left[\pm \frac{(\zeta_8 + \zeta_8^3)(t - 1)}{2} : \pm \frac{(\zeta_8^3 - \zeta_8)t + \zeta_8^3 + \zeta_8}{2} : 1 \right]. \tag{3.56}$$

Case 3.19: (3.28) and (3.39). Then,

$$-\frac{a^2}{2} - \zeta_8^2 = -\frac{a_1^2}{2} + \zeta_8^2, \quad -\frac{1}{2}\zeta_8^2 a^2 = \frac{1}{2}\zeta_8^2 a_1^2 - 2\zeta_8^3 a_1 - 2.$$

Hence, $(a, a_1) = (0, 2\zeta_8), (\pm 2\zeta_8^3, 0)$. The first solution gives $P(T) = T^2 - \zeta_8^2 T$, which is reducible in $K[T]$. The second solution gives $P(T) = T^2 + \zeta_8^2 T - 2$. So $t^2 + \zeta_8^2 t - 2 = 0$. Therefore,

$$[x : y : z] = \left[\pm \left(\frac{1}{2}\zeta_8^3 t - \zeta_8 \right) : \pm \frac{t}{2} : 1 \right]. \tag{3.57}$$

Case 3.20: (3.28) and (3.40). Then,

$$-\frac{a^2}{2} - \zeta_8^2 = -\frac{a_1^2}{2} - \zeta_8^2, \quad -\frac{1}{2}\zeta_8^2 a^2 = -\frac{1}{2}\zeta_8^2 a_1^2 - 2\zeta_8 a_1 - 2.$$

Hence, $(a, a_1) = (\pm\zeta_8^3, \zeta_8^3)$. Thus, $P(T) = T^2 - \frac{1}{2}\zeta_8^2 T - \frac{1}{2}$. So $t^2 - \frac{1}{2}\zeta_8^2 t - \frac{1}{2} = 0$. Therefore,

$$[x : y : z] = \left[\pm \left(\zeta_8^3 t - \frac{\zeta_8}{2} \right) : \pm \left(\zeta_8^2 t + \frac{1}{2} \right) : 1 \right]. \quad (3.58)$$

Note that the 32 points in (3.51), (3.52), (3.55) and (3.56) are the 32 points in Theorem 1.1(C) and the 96 points in (3.43), (3.44), (3.45), (3.46), (3.47), (3.48), (3.49), (3.50), (3.53), (3.54), (3.57) and (3.58) are the 96 points in Theorem 1.1(D).

The proof of Theorem 1.2 is complete.

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