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A COUNTEREXAMPLE FOR CS-RINGS

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Dedicated to Professor Klaus W. Roggenkamp on his sixtieth birthday

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Abstract. A module M is called a *CS-module* or an *extending module* if every submodule is essential in a direct summand of M. A ring R is called a right *CS-ring* or a right *extending ring* if R_R is a CS-module. For several types of right CS-rings it is known that either all right ideals or some large class of right ideals inherit the CS property. For example, by a result of Dung-Smith or Vanaja-Purav, a ring R is (right and left) Artinian, serial, and $J(R)^2 = 0$ if and only if every R-module is CS. In particular, if R is a QF-ring and $J(R)^2 = 0$ (hence R is serial), then every R-module is CS. However we exhibit a finite, serial, strongly bounded QF group algebra R with $J(R)^3 = 0$ for which there is a *principal* right ideal which is a right essential extension of a CS-module and essential in R_R but *not* CS itself.

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Throughout this paper R will denote an associative ring with unity, $J(R)$ is its Jacobson radical, and all modules will be unital right R-modules (unless otherwise indicated). A submodule N of a module M is called a *closed submodule* if there is no proper essential extension of N inside M. A module M is called a *CS-module* or an *extending module* if every submodule of M is essential in a direct summand of M. Thus a module M is CS if and only if every closed submodule of M is a direct summand of M. The class of CS-modules includes all injective modules, all quasi-continuous modules, and all uniform modules. A ring R is called *right* (resp. *left*) *CS* or *extending* if R_R (resp. ${}_RR$) is CS or extending. The usefulness of the CS concept is well documented in [**10**] or more recently [**3**].

A module is called *uniserial* if it has a unique composition series of finite length. A ring R is called *right* (resp. *left*) *serial* if R_R (resp. ${}_RR$) is a finite direct sum of uniserial right (resp. left) ideals. A ring R is called *serial* if it is both right and left serial.

Recall that a ring R is called *quasi-Frobenius* (or simply QF) if R is right or left Artinian and right or left self-injective. For a QF-ring R, it is well known that the right socle, $\mathrm{Soc}(R_R)$, of R is equal to the left socle, $\mathrm{Soc}({}_RR)$, of R. Without ambiguity, when R is a QF-ring, we simply denote the right or left socle of R by $\mathrm{Soc}(R)$. Also note that if a ring R is QF, then $\ell(J(R)) = r(J(R)) = \mathrm{Soc}(R)$ and $\ell(\mathrm{Soc}(R)) = r(\mathrm{Soc}(R)) = J(R)$, where $J(R)$, $\ell(-)$, and $r(-)$ are the Jacobson radical of R, the left annihilator, and the right annihilator, respectively.

A ring R is called *strongly right* (resp. *left*) *bounded* if every nonzero right (resp. left) ideal of R contains a nonzero ideal. A ring R is called *strongly bounded* if R is both strongly right and strongly left bounded. Observe that if R is strongly right bounded then every nonzero right ideal of R is an essential extension of an ideal of R.

Unfortunately, the class of CS-modules is somewhat pathological in that it is not closed under homomorphic images, finite direct sums, or extensions. We shall show that this pathology extends to closure with respect to submodules. When investigating a class $\mathfrak{K}$ of modules it is natural to ask: for $M \in \mathfrak{K}$ under what conditions are the submodules (or some distinguished set of submodules) of M also in $\mathfrak{K}$? In particular, we ask: if R_R is CS, when are all right ideals (or some distinguished set of right ideals) of R also CS? The following results provide some answers to this question.

(1) [**3**, p.134]. *If R_R is nonsingular and (finitely, countably) Σ-extending, then every (finitely, countably generated) right ideal is CS.*

(2) [**4**, **13**]. *Every module is CS if and only if R is (right and left) Artinian, serial, and $J(R)^2 = 0$. From* [**6**, 25.4.3], *if R is a QF-ring with $J(R)^2 = 0$, then R is serial. Thus in this case every R-module is CS.*

(3) [**1**, Corollaries 1.3 and 2.2]. *Let R_R be CS. We have the following:*

(i) *every ideal is CS;*

(ii) *if every idempotent is central, then every right ideal is CS;*

(iii) *if R_R is nonsingular, then every principal right ideal is CS.*

A serial QF-ring R with $J(R) \neq 0$ can arguably be considered a quintessential example of a CS-module which is not nonsingular. Observe that a QF-ring is Σ-injective and Σ-extending [**3**, p.95 and p.170]. From the above results one would expect that the CS condition will be inherited by a large class of right ideals of a serial QF-ring. In particular, in light of (1), (2), and (3), one could reasonably conjecture that *if R is a stongly bounded, PI, serial QF-ring with $J(R)^3 = 0$, then every principal right ideal is CS*. Surprisingly, this is not the case as is illustrated in the following example of a finite group algebra.

Not only is the choice of this group algebra counterintuitive but the calculations involve a degree of judgement (e.g., in Step 3 of Claim 4 there are three choices for the appropriate injective hull of $(2+\tau)R \cap (1+\sigma+\tau)R$ in the group algebra R).

Example. We consider the group algebra $R = \mathbb{Z}_3[S_3]$ of the symmetric group S_3 on three symbols $\{1, 2, 3\}$ over the field $\mathbb{Z}_3$ of three elements. Denote $\sigma = (123)$ and $\tau = (12)$ in S_3.

Note that the ring R is right self-injective by [**11**, Theorem 2.8, p.79] and so R is QF. Since R is a finite ring, it is a PI-ring. We shall show that R is serial, strongly bounded, and $J(R)^3 = 0$ such that there exists a principal right ideal of R that is a right essential extension of a CS-module and it is essential in R_R, but it is not CS itself.

Claim 1. *$J(R)^3 = 0$ and R is strongly bounded.*

Proof. By [**11**, Exercise 8, p.106] $J(R)$ is $\omega(\mathbb{Z}_3[N])R$, where

$$\omega(\mathbb{Z}_3[N]) = \{a + b\sigma + c\sigma^2 \mid a + b + c = 0, a, b, c \in \mathbb{Z}_3\},$$

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which is the augmentation ideal of $\mathbb{Z}_3[N]$ and $N = <\sigma>$. Thus by direct calculation $J(R) = (2+\sigma)R$, which is

$$\{a_0 + a_1\sigma + 2(a_0 + a_1)\sigma^2 + b_0\tau + b_1\sigma\tau + 2(b_0 + b_1)\sigma^2\tau \mid a_0, a_1, b_0, b_1 \in \mathbb{Z}_3\}.$$

Hence it can be easily checked that $J(R)^3 = 0$.

The vector space dimension of $J(R)$ over the field $\mathbb{Z}_3$ is four, and so the vector space dimension of $R/J(R)$ over $\mathbb{Z}_3$ is two. Therefore the number of elements of the ring $R/J(R)$ is nine. Since R is QF, the ring $R/J(R)$ is semisimple Artinian. Note that the ring R is not a local ring because there is a nontrivial idempotent in R, for example $2+\tau$. Hence, by Wedderburn-Artin, $R/J(R) \cong \mathbb{Z}_3 \bigoplus \mathbb{Z}_3$ as rings. Thus R is basic. Now, by [**7**, Theorem 1.7B], R is a strongly bounded ring.

According to [**2**, Definition 7.11, p.480], recall that an algebra A over a field F is called a *separable algebra* over F if $A \bigotimes_F H$ is a semisimple algebra over H, for every extension field H of F. Following [**12**], for a prime number p, recall that a group is called *p-solvable* if each of its composition factors is either a p-group or has order prime to p.

Claim 2. *R is a serial ring.*

Proof. Now in our situation, since $R/J(R) \cong \mathbb{Z}_3 \bigoplus \mathbb{Z}_3$, it follows that the $\mathbb{Z}_3$-algebra $R/J(R)$ is separable over the field $\mathbb{Z}_3$. Let $\overline{\mathbb{Z}}_3$ be the algebraic closure of the field $\mathbb{Z}_3$. Then the group ring $\overline{\mathbb{Z}}_3[S_3] = \overline{\mathbb{Z}}_3 \bigotimes_{\mathbb{Z}_3} R$ is serial by [**12**, Theorem 3] because the group S_3 is 3-solvable with a 3-Sylow subgroup. Since $R/J(R)$ is separable over $\mathbb{Z}_3$, the ring R is serial by [**5**, Theorem 4.1].

Remark 3. Explicitly, by direct calculation, $(2+\tau)R$ has the unique composition series

$$0 \subsetneq \operatorname{Soc}((2+\tau)R) \subsetneq (2+\tau)R \cap (1+\sigma+\tau)R \subsetneq (2+\tau)R$$

and also $(2+2\tau)R$ has the unique composition series

$$0 \subsetneq \operatorname{Soc}((2+2\tau)R) \subsetneq (2+2\tau)R \cap (1+\sigma+\tau)R \subsetneq (2+2\tau)R.$$

Thus $R = (2+\tau)R \bigoplus (2+2\tau)R$ is right serial. Similarly, $R = R(2+\tau) \bigoplus R(2+2\tau)$ is left serial. Consequently, R is a serial ring.

Claim 4. *There is a principal right ideal of R that is a right essential extension of a CS-module and essential in R_R, but it is not CS itself. In fact, we have that $(1+\sigma+\tau)R$ is such a principal right ideal.*

Step 1. *The right uniform dimension of R is two.*

Proof of Step 1. $\operatorname{Soc}(R) = \ell(J(R)) = (1+\sigma+\sigma^2)R$ because R is QF. Hence

$$\operatorname{Soc}(R) = \{a(1+\sigma+\sigma^2) + b(1+\sigma+\sigma^2)\tau \mid a, b \in \mathbb{Z}_3\}.$$

Thus the vector space dimension of $\operatorname{Soc}(R)$ over the field $\mathbb{Z}_3$ is two. Since R has a nontrivial idempotent, the right uniform dimension of $\operatorname{Soc}(R)$ is greater than one.

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Therefore the right uniform dimension of R is two because $\mathrm{Soc}(R)$ is an essential R-submodule of R_R.

STEP 2. *The form of elements from* $(2+\tau)R \cap (1+\sigma+\tau)R$ *is*

$$a + b\sigma + (2a + 2b)\sigma^2 + 2a\tau + (a + b)\sigma\tau + 2b\sigma^2\tau,$$

for some $a, b \in \mathbb{Z}_3$.

Proof of Step 2. Note that

$$(1+\sigma+\tau)R = \{(a_0 + a_2 + b_0) + (a_0 + a_1 + b_2)\sigma + (a_1 + a_2 + b_1)\sigma^2 + (a_0 + b_0 + b_2)\tau + (a_2 + b_0 + b_1)\sigma\tau + (a_1 + b_1 + b_2)\sigma^2\tau \mid a_i, b_i \in \mathbb{Z}_3, i = 0, 1, 2\}$$

and

$$(2+\tau)R = \{a + b\sigma + c\sigma^2 + 2a\tau + 2c\sigma\tau + 2b\sigma^2\tau \mid a, b, c \in \mathbb{Z}_3\}.$$

Direct calculation yields the desired form of elements in $(2+\tau)R \cap (1+\sigma+\tau)R$.

STEP 3. $(2+\tau)R \cap (1+\sigma+\tau)R$ *has no proper essential extension in* $(1+\sigma+\tau)R$. *Moreover,* $(2+\tau)R \cap (1+\sigma+\tau)R$ *has a unique injective hull in* R_R.

Proof of Step 3. For our convenience, let $f = 2+\tau$. Let K be a maximal essential extension of $fR \cap (1+\sigma+\tau)R$ in $(1+\sigma+\tau)R$. Then there is an injective hull $E(fR \cap (1+\sigma+\tau)R) = gR$ of $fR \cap (1+\sigma+\tau)R$ with $g = g^2 \in R$ such that $K \subseteq gR$ and so $K \subseteq gR \cap (1+\sigma+\tau)R$. Since $fR \cap (1+\sigma+\tau)R$ is essential in $gR \cap (1+\sigma+\tau)R$, it follows that $K = gR \cap (1+\sigma+\tau)R$. Furthermore since gR is an injective hull of $fR \cap (1+\sigma+\tau)R$, we have that $fR \cong gR$.

Since $fR \cap (1+\sigma+\tau)R$ is essential in K and $(fR \cap (1+\sigma+\tau)R) \cap (1-f)R = 0$, it follows that $K \cap (1-f)R = 0$. Also since K is essential in gR and $K \cap (1-f)R = 0$, we have that $gR \cap (1-f)R = 0$. Since $gR \cap (1-f)R = 0$ and, by Step 1, the right uniform dimension of R is two, it follows that $R = gR \bigoplus (1-f)R$. We *claim* that

$$\overline{R} = \overline{g}\,\overline{R} \bigoplus (\overline{1} - \overline{f})\overline{R},$$

where $\overline{R} = R/J(R)$. Obviously $\overline{R} = \overline{g}\,\overline{R} + (\overline{1} - \overline{f})\overline{R}$. Now if $\overline{g}\,\overline{R} \cap (\overline{1} - \overline{f})\overline{R} \neq \overline{0}$, then, since $\overline{R} \cong \mathbb{Z}_3 \bigoplus \mathbb{Z}_3$, $\overline{g}\,\overline{R} \cap (\overline{1} - \overline{f})\overline{R}$ is a minimal ideal of $\overline{R}$. If the uniform dimension of $\overline{g}\,\overline{R}$ is two, then $\overline{g}\,\overline{R} = \overline{R}$ and so $\overline{g}$ is invertible and hence g is invertible in R, which is a contradiction. Thus the uniform dimension of $\overline{g}\,\overline{R}$ is one, and so $\overline{g}\,\overline{R}$ is a minimal right ideal of $\overline{R}$. Thus $\overline{g}\,\overline{R} \cap (\overline{1} - \overline{f})\overline{R} = \overline{0}$, so that

$$\overline{R} = \overline{g}\,\overline{R} \bigoplus (\overline{1} - \overline{f})\overline{R} = \overline{f}\overline{R} \bigoplus (\overline{1} - \overline{f})\overline{R}.$$

Since $\overline{R}$ is commutative, $\overline{g}\overline{f}$ is an idempotent. If $\overline{g}\overline{f} = \overline{0}$, then $\overline{g}\overline{R} \subseteq (\overline{1} - \overline{f})\overline{R}$. Thus $\overline{g}\overline{f} \neq \overline{0}$. Also $\overline{g}\overline{f}\,\overline{R} \subseteq \overline{f}\,\overline{R}$ and $\overline{g}\overline{f}\,\overline{R} \subseteq \overline{g}\overline{R}$. But both $\overline{f}\,\overline{R}$ and $\overline{g}\overline{R}$ have the vector space dimension one over $\mathbb{Z}_3$. Thus $\overline{f}\,\overline{R} = \overline{g}\overline{f}\,\overline{R} = \overline{g}\overline{R}$. Consequently, since $\overline{R}$ is commutative, $\overline{f} = \overline{g}$ and so $g = f + \gamma$, for some $\gamma \in J(R) = (2+\sigma)R$.

Therefore in the proof of Claim 1,

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$$\gamma = a_0 + a_1\sigma + 2(a_0 + a_1)\sigma^2 + b_0\tau + b_1\sigma\tau + 2(b_0 + b_1)\sigma^2\tau,$$

for some $a_i, b_i \in \mathbb{Z}_3$, $i = 0, 1$; hence we have

$$g = (2 + a_0) + a_1\sigma + 2(a_0 + a_1)\sigma^2 + (1 + b_0)\tau + b_1\sigma\tau + 2(b_0 + b_1)\sigma^2\tau.$$

We now compute the coefficients of g^2 and compare them with those of g.

(i) The coefficient of 1 in g^2 is

$$(2 + a_0)^2 + 2a_1(a_0 + a_1) + 2(a_0 + a_1)a_1 + (1 + b_0)^2 + b_1^2 + 4(b_0 + b_1)^2,$$

which is equal to $2 + a_0$. Thus we have the relation

$$a_0^2 + a_0a_1 + a_1^2 + 2b_0^2 + 2b_0b_1 + 2b_1^2 + 2b_0 = 0.$$

(ii) The coefficient of τ in g^2 is

$$\begin{aligned}&(2 + a_0)(1 + b_0) + 2a_1(b_0 + b_1) + 2(a_0 + a_1)b_1\\&+ (1 + b_0)(2 + a_0) + a_1b_1 + 4(a_0 + a_1)(b_0 + b_1),\end{aligned}$$

which is equal to $1 + b_0$. It follows that $2a_0 = 0$ and so $a_0 = 0$.

(iii) The coefficient of $\sigma\tau$ in g^2 is

$$\begin{aligned}&(2 + a_0)b_1 + a_1(1 + b_0) + 4(a_0 + a_1)(b_0 + b_1)\\&+ 2(b_0 + b_1)(a_0 + a_1) + b_1(2 + a_0) + 2(b_0 + b_1)a_1,\end{aligned}$$

which is equal to b_1. Thus $a_0 + 2a_1 = 0$. Since $a_0 = 0$, we have that $a_1 = 0$.

From (i), (ii) and (iii), it follows that $2b_0^2 + 2b_0b_1 + 2b_1^2 + 2b_0 = 0$, and so

$$b_0^2 + b_0b_1 + b_1^2 + b_0 = 0.$$

Thus there are only three possibilities for b_0 and b_1: $b_0 = b_1 = 0$; $b_0 = 2, b_1 = 0$; and $b_0 = 2, b_1 = 1$. Therefore $g = f$, $g = 2 + \sigma\tau$ or $g = 2 + \sigma^2\tau$. Hence all candidates for maximal essential extensions of $(2 + \tau)R \cap (1 + \sigma + \tau)R$ in $(1 + \sigma + \tau)R$ are $(2 + \tau)R \cap (1 + \sigma + \tau)R$, $(2 + \sigma\tau)R \cap (1 + \sigma + \tau)R$, and $(2 + \sigma^2\tau)R \cap (1 + \sigma + \tau)R$.

Note that $1 + 2\sigma + 2\tau + \sigma^2\tau = (2 + \tau)(2 + \sigma) = (1 + \sigma + \tau)(1 + \sigma + 2\sigma^2 + \tau)$, and so we have $1 + 2\sigma + 2\tau + \sigma^2\tau \in (2 + \tau)R \cap (1 + \sigma + \tau)R$. However $1 + 2\sigma + 2\tau + \sigma^2\tau \notin (2 + \sigma\tau)R$ and $1 + 2\sigma + 2\tau + \sigma^2\tau \notin (2 + \sigma^2\tau)R$. Therefore it follows that $(2 + \tau)R \cap (1 + \sigma + \tau)R \nsubseteq (2 + \sigma\tau)R$ and $(2 + \tau)R \cap (1 + \sigma + \tau)R \nsubseteq (2 + \sigma^2\tau)R$. Consequently, $(2 + \tau)R \cap (1 + \sigma + \tau)R$ is a closed submodule in $(1 + \sigma + \tau)R$. Also $(2 + \tau)R$ is the unique injective hull of $(2 + \tau)R \cap (1 + \sigma + \tau)R$ in R_R.

STEP 4. *$(1 + \sigma + \tau)R$ is not CS as a right R-module.*

Proof of Step 4. Assume to the contrary that $(1 + \sigma + \tau)R$ is CS as a right R-module. Then, since $(2 + \tau)R \cap (1 + \sigma + \tau)R$ is a closed submodule in $(1 + \sigma + \tau)R$

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by Step 3, it is a direct summand of $(1+\sigma+\tau)R$; and so we have a projection π from $(1+\sigma+\tau)R$ onto $(2+\tau)R \cap (1+\sigma+\tau)R$. Since R is right self-injective, there is an R-homomorphism θ from R to R that extends the R-homomorphism $i \circ \pi$, where i is the inclusion from $(2+\tau)R \cap (1+\sigma+\tau)R$ to R. Hence $(i \circ \pi)(1+\sigma+\tau) = \pi(1+\sigma+\tau) = \theta(1+\sigma+\tau) = \theta(1)(1+\sigma+\tau) = x_0(1+\sigma+\tau)$, where $x_0 = \theta(1) \in R$. Thus we have in this case

$$x_0(1+\sigma+\tau) \in (2+\tau)R \cap (1+\sigma+\tau)R.$$

Say $x_0 = a_0 + a_1\sigma + a_2\sigma^2 + b_0\tau + b_1\sigma\tau + b_2\sigma^2\tau$, for some $a_i, b_i \in \mathbb{Z}_3$ for $i = 0, 1, 2$. Then $(a_0 + a_1\sigma + a_2\sigma^2 + b_0\tau + b_1\sigma\tau + b_2\sigma^2\tau)(1+\sigma+\tau) = (a_0 + a_2 + b_0) + (a_0 + a_1 + b_1)\sigma + (a_1 + a_2 + b_2)\sigma^2 + (a_0 + b_0 + b_1)\tau + (a_1 + b_1 + b_2)\sigma\tau + (a_2 + b_0 + b_2)\,\sigma^2\tau$ is in $(2+\tau)R \cap (1+\sigma+\tau)R$. Since

$$(2+\tau)R \cap (1+\sigma+\tau)R \subseteq \{a + b\sigma + (2a+2b)\sigma^2 + 2a\tau + (a+b)\sigma\tau + 2b\sigma^2\tau \mid a, b \in \mathbb{Z}_3\}$$

by Step 2, we have that

$$\begin{aligned} a_0 + a_2 + b_0 &= a, \\ a_0 + a_1 + b_1 &= b, \\ a_1 + a_2 + b_2 &= 2a + 2b \end{aligned} \tag{1}$$

and

$$\begin{aligned} a_0 + b_0 + b_1 &= 2a, \\ a_1 + b_1 + b_2 &= a + b, \\ a_2 + b_0 + b_2 &= 2b, \end{aligned} \tag{2}$$

for some $a, b \in \mathbb{Z}_3$.

From the equations $a_0 + b_0 + b_1 = 2a$, $a_1 + b_1 + b_2 = a + b$, and $a_2 + b_0 + b_2 = 2b$ in (2), we have $a_0 = 2a + 2b_0 + 2b_1$, $a_1 = a + b + 2b_1 + 2b_2$, and $a_2 = 2b + 2b_0 + 2b_2$, respectively. By substituting these equations into (1), we get that $2b_0 + 2b_1 + 2b_2 = 2a + b$, $b_0 + b_1 + b_2 = 0$, and $2b_0 + 2b_1 + 2b_2 = a + 2b$. Thus we have that $a + 2b = 0$, or equivalently, $a = b$. Therefore, it follows that

$$x_0(1+\sigma+\tau) = a + a\sigma + a\sigma^2 + 2a\tau + 2a\sigma\tau + 2a\sigma^2\tau = a(1+\sigma+\sigma^2) + 2a(1+\sigma+\sigma^2)\tau$$

and so it is in $(1+\sigma+\sigma^2)R$. Thus $\pi(1+\sigma+\tau) \in (1+\sigma+\sigma^2)R$, and hence

$$(2+\tau)R \cap (1+\sigma+\tau)R = \pi(1+\sigma+\tau)R \subseteq (1+\sigma+\sigma^2)R.$$

Finally as we previously noted, $1 + 2\sigma + 2\tau + \sigma^2\tau \in (2+\tau)R \cap (1+\sigma+\tau)R$. However $1 + 2\sigma + 2\tau + \sigma^2\tau \notin (1+\sigma+\sigma^2)R$, and so we have a contradiction. Consequently, $(1+\sigma+\tau)R$ is *not* CS as a right R-module.

STEP 5. *There exists an ideal I of R such that I is right CS and right essential in $(1+\sigma+\tau)R$ and $(1+\sigma+\tau)R$ is essential in R_R.*

Proof of Step 5. By Claim 1, since R is strongly bounded, there exsits an ideal I of R such that $I \subseteq (1+\sigma+\tau)R$ and I is right essential in $(1+\sigma+\tau)R$. In this situation, since R_R is CS, I is CS by [**1**, Corollary 1.3 (ii)]. Since

$\mathrm{Soc}((2+\tau)R) \bigoplus \mathrm{Soc}((2+2\tau)R) \subseteq (1+\sigma+\tau)R$ by Remark 3 and the right uniform dimension of R is two by Step 1, the uniform dimension of $(1+\sigma+\tau)R$ is two. Hence $(1+\sigma+\tau)R$ is essential in R_R.

REMARK 5. It is well known that an Artinian CS-module is a finite direct sum of uniform modules. But, in the proof of Example, we can show that the converse does not hold even for a finite QF-ring. Since $2+\sigma^2\tau \in (1+\sigma+\tau)R$ and $2+\sigma^2\tau$ is an idempotent, $(1+\sigma+\tau)R = (2+\sigma^2\tau)R \bigoplus Y$, for some submodule Y of $(1+\sigma+\tau)R$. Since $(2+\sigma^2\tau)R$ is injective and uniform, and by Steps 1 and 5 of Claim 4 the uniform dimension of $(1+\sigma+\tau)R$ is two, it follows that Y is also uniform. Hence the direct sum of two Artinian uniform modules is not necessarily CS. For more on finite direct sums of CS-modules, see [**8**]. However in [**9**] it was proved that if R is a QF-ring which is serial with $J(R)^3 = 0$, then every right R-module is a direct sum of a projective module and a CS-module. It follows that every principal right ideal of $\mathbb{Z}_3[S_3]$ is a direct sum of an injective module and a CS-module.

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