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IRREDUCIBILITY OF THE ANALYTIC CONTINUATION OF THE PRINCIPAL SERIES OF A FREE GROUP

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Abstract

In this paper it is proved that the principal series of representations of $\Gamma = \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$ may be analytically continued to give uniformly bounded representations on the *same* Hilbert space, and that these representations are *irreducible*. Further, the reducibility of the restrictions to $\Gamma \subset SL(2, \mathbf{Q}_p)$ of the irreducible unitary representations of $SL(2, \mathbf{Q}_p)$ is examined.

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1. Introduction

In this paper we solve some open problems on the analytic continuation of the principal series of the free product $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$, $(p + 1$ times), and we examine the irreducibility of the restrictions to Γ of the irreducible unitary representations of $PGL(2, \mathbf{Q}_p)$, the two by two projective linear group over a p-adic field $\mathbf{Q}_p$.

The theory of uniformly bounded representations was initiated by R. A. Kunze and E. M. Stein. In [9] they constructed, for the real 2×2 unimodular group, a family of representations characterized as the analytic continuation of the principal series; such representations act on a fixed Hilbert space H and are uniformly bounded and irreducible on H. Several authors have subsequently extended this construction to more general groups. We mention here the work of

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Kunze and Stein [10], [11]; P. Sally [21], [22]; R. L. Lipsman [12], [13], [14], [15]; E. N. Wilson [26]; N. Lohoué [16]; L. Bamazi [1] and M. G. Cowling [3].

In this paper we construct, for a free product $\Gamma = \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$, $(p+1)$-times, a family of representations satisfying properties like those described above.

Another natural question is whether irreducibility of the unitary representations of $G = PGL(2, \mathbb{Q}_p)$ is preserved under restriction to Γ, considered as a discrete subgroup of G [7], [2]. A positive answer has been given by A. Figà-Talamanca and A. M. Picardello for the class of spherical unitary representations of G [7]. In this paper we make a contribution to this problem, by analyzing other irreducible unitary representations of G.

In Section 2 we consider the analytic family of spherical representations π_z of Γ, described in [5], [6] and in [17], [18], which act on different Hilbert spaces $H_{\mathrm{Re}\,z}$, depending on z. We construct a family of continuous invertible linear operators $W(z, 1/2)$ from $H_{\mathrm{Re}\,z}$ onto a fixed Hilbert space H. Then we check that the representations $U_z = W(z, 1/2)\pi_2 W(z, 1/2)^{-1}$, for $0 < \mathrm{Re}\,z < 1$, are uniformly bounded on H and they are, up to unitary equivalence, the principal series, for $\mathrm{Re}\,z = 1/2$, and the complementary series of Γ, for $\mathrm{Im}\,z = h\pi/\log p$, $h \in \mathbb{Z}$.

In Section 3 we prove the irreducibility of the representation U_z on the space H, for any $0 < \mathrm{Re}\,z < 1$. We use here some of the techniques and ideas contained in T. Steger's Ph. D. dissertation. All the results in this section were obtained in collaboration with T. Steger, while he was visiting the University of Genoa, in July 1985.

In Section 4 we examine the irreducible unitary representations of G according to the classification of A. J. Silberger [23]. We observe that there are representations of the principal and complementary series of G, other than the spherical ones, which remain irreducible representations of Γ, when restricted to this discrete subgroup. On the other hand, the representations of the discrete series and of the special series become reducible while passing from G to Γ.

We wish to point out that all results of this paper hold for any discrete group acting faithfully on a homogeneous tree.

2. Intertwining operators

We refer to [6] for notations and unexplained results. We denote by Ω the Poisson boundary of the homogeneous tree associated to Γ, with respect to a simple random walk given by the probability distribution

$$\mu(x) = \begin{cases} (p+1)^{-1}, & |x| = 1, \\ 0, & \text{otherwise, for any } x \in \Gamma. \end{cases}$$

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The probability measure ν on Ω is defined by

$$\nu(E(x)) = p^{1-n}(p+1)^{-1} \quad \text{if } |x| = n,$$

where $E(x) = \{\omega \in \Omega, \omega \text{ starts with } x\}$. This measure is quasi-invariant with respect to the action of Γ on Ω and the Radon-Nikodym derivative $d\nu_x/d\nu$ is the Poisson kernel $P(x,\omega)$ of Γ associated to μ. We denote by $K(\Omega)$ the space of cylindrical functions on Ω and by $K'(\Omega)$ the space of martingales on Ω: we can formally write any $\mathbf{f} = (f_n) \in K'(\Omega)$ as

$$\mathbf{f} = \sum_{n=0}^{\infty} \Delta_n \mathbf{f}, \qquad \text{where } \Delta_n \mathbf{f} = f_n - f_{n-1},\ n \in \mathbb{N}.$$

Let π_z, $z \in \mathbb{C}$, $0 \leqslant \operatorname{Re} z \leqslant 1$, be the spherical representation defined by

$$\pi_z(x)f(\omega) = P^z(x,\omega)f(x^{-1}\omega), \qquad f \in K(\Omega),\ x \in \Gamma,\ \omega \in \Omega,$$

and extended by duality to $K'(\Omega)$. For z in the strip $S = \{z = s + it \in \mathbb{C}, 0 < s < 1\}$, the representation π_z is uniformly bounded on the space H_s defined as the completion of $K(\Omega)$ with respect to the norm

$$\|f\|_s = \langle f, I_s f\rangle,$$

I_z being the intertwining operator defined in [17]. We note that for $f = \sum_{n=0}^{M} \Delta_n f$, then

$$I_z(f) = \Delta_0 f + \theta(z) \sum_{n=1}^{M} p^{(1-2z)n} \Delta_n f,$$

and

$$\|f\|_s^2 = \|\Delta_0 f\|_2^2 + \theta(s) \sum_{n=1}^{M} p^{(1-2s)n} \|\Delta_n f\|_2^2,$$

where $\theta(z) = (1 - p^{-2(1-z)})(1 - p^{-2z})^{-1}$.

In order to construct a family of representations of Γ which act on the same Hilbert space, we define a family of continuous linear operators $W(z,z')$ on $K'(\Omega)$ as follows. Let z, $z' \in S$; we set for $f \in K(\Omega)$, $f = \sum_{n=0}^{M} \Delta_n f$,

$$W(z,z')f = \Delta_0 f + d(z,z') \sum_{n=1}^{M} p^{(z'-z)n} \Delta_n f,$$

where $d(z,z')$ is the principal values of the square root of $\theta(z)/\theta(z')$. Obviously $W(z,z')$ maps $K(\Omega)$ into $K(\Omega)$ and $W(z,z')\Delta_n f = \Delta_n W(z,z')f$, for any $n \geqslant 0$. These operators can be extended as operators from $K'(\Omega)$ into $K'(\Omega)$ by setting

$$W(z,z')\mathbf{f} = \left(W(z,z')f_n\right) \quad \text{for any } \mathbf{f} = (f_n) \in K'(\Omega).$$

We remark that, for all $z \in S$, $W(z, 1-z) = I_z$.

Starting from the formula

$$\Delta_n f(\omega) = \int_\Omega \delta_n(\omega, \omega') f(\omega')\, d\nu(\omega'), \qquad n \geqslant 0,\ f \in K(\Omega),$$

and recalling the expression of I_z as an integral operator [17],

$$I_z f(\omega) = \lim_{n\to\infty} \int_{N(\omega,\omega')\leqslant n} k_z(\omega, \omega') f(\omega')\, d\nu(\omega'),$$

for $f \in K(\Omega)$, we can find an integral kernel for the operator $W(z, z')$.

Proposition 2.1. *For every z, $z' \in S$, $s \neq s'$ and for every $f \in K(\Omega)$, we have*

$$W(z, z') f(\omega) = \lim_{n\to\infty} \int_{N(\omega,\omega')\leqslant n} L(z, z')(\omega, \omega') f(\omega')\, d\nu(\omega'), \qquad \omega \in \Omega,$$

where for $\omega \neq \omega'$

$$\begin{aligned} L(z, z')(\omega, \omega') &= 1 + d(z, z') \sum_{n=1}^{\infty} p^{(z'-z)n} \delta_n(\omega, \omega') \\ &= 1 + d(z, z')\left[p^{(1+z-z')N(\omega,\omega')} - \theta^{-1}((1 + z - z')/2)\right]. \end{aligned}$$

Proof. $L(\omega, \omega')$ is well defined for $\omega \neq \omega'$ because $\delta_n(\omega, \omega') = 0$ if $N(\omega, \omega') < n - 1$. Furthermore we note that for z, $z' \in S$, $s \neq s'$,

$$d(z, z')\left(I_{(1+z-z')/2} f - \Delta_0 f\right) = \theta((1 + z - z')/2)\left(W(z, z') f - \Delta_0 f\right),$$

and

$$d(z, z')\left(k((1 + z - z')/2) - \delta_0\right) = \theta((1 + z - z')/2)\left(L(z, z') - \delta_0\right).$$

The proof follows by the expression of I_z as an integral operator on Ω.

Combining the representations π_z with the operators $W(z, z')$, we define a family of representations U_z with the required properties.

Theorem 2.2. (i) *For z, $z' \in S$, $W(z, z')$ is a bounded invertible linear operator from $H_s(\Omega)$ onto $H_{s'}(\Omega)$.*

(ii) *For $z \in S$, the representations U_z of Γ on $H = L^2(\Omega)$, defined by*

$$U_z = W(z, 1/2)\pi_z W(1/2, z), \qquad z \in S,$$

are an analytic family of continuous representations in the strip S.

(iii) *$U_{1/2+it}$ is unitarily equivalent to the representation $\pi_{1/2+it}$ of the principal series and U_{s+it}, $t = h\pi/\log p$, $h \in \mathbf{Z}$, is unitarily equivalent to the representation π_{s+it} of the complementary series.*

(iv) *$U_z(x)$ is uniformly bounded with respect to $x \in \Gamma$.*

PROOF. Since $K(\Omega)$ is a dense subspace of $H_s(\Omega)$, $0 < s < 1$, it is enough to consider only functions $f \in K(\Omega)$. Then, for $f = \sum_{n=0}^{M} \Delta_n f$,

$$\begin{aligned} \| W(z, z')f \|_s^2 &= \| \Delta_0 W(z, z')f \|_2^2 + \theta(s') \sum_{n=1}^{M} p^{(1-2s')n} \| W(z, z')\Delta_n f \|_2^2 \\ &= \| \Delta_0 f \|_2^2 + \theta(s') |d(z, z')|^2 \sum_{n=1}^{M} p^{(1-2s')n} \| \Delta_n f \|_2^2 \\ &\leqslant \eta(z, z') \| f \|_s^2, \end{aligned}$$

where $\eta(z, z') = \max\{1, \theta(s')|d(z, z')|^2\}$. This proves that $W(z, z')$ is bounded from H_s into $H_{s'}$. To conclude (i) it is enough to observe that $W(z', z)$ is the inverse of $W(z, z')$.

To prove (iii) we observe that $\|W(z, 1/2)f\|_2 = \|f\|_s$, where $s = 1/2$ or $t = h\pi/\log p$, $h \in \mathbb{Z}$, since $\theta(s')|d(z, z')|^2 = \theta(s)$ for every z, $z' \in S$. (ii) and (iv) follow from the analogous properties of π_z.

3. Irreducibility of U_z

The irreducibility of the representations of the principal and of the complementary series of Γ was proved in [5] (see also [25], [20]). We prove that every uniformly bounded representation U_z is irreducible on H; actually we consider the equivalent representation π_z on $H_s(\Omega)$. To prove the irreducibility of π_z we only have to show, as for the principal series [5], [25], that the eigenspace of the operator

$$\pi_z(\mu) = (p + 1)^{-1} \sum_{|y|=1} \pi_z(y),$$

associated to the eigenvalue $\gamma(z) = (p^z + p^{1-z})/(p + 1)$ is one dimensional. We cannot apply here the functional calculus used in [25] because $\pi_z(\mu)$ is not selfadjoint; however we can obtain the same result, by considering $\pi_z(\mu)$ as a small perturbation of a selfadjoint operator.

First of all we compute the adjoint operator of $\pi_z(\mu)$.

LEMMA 3.1. *For $z \in S$,*

$$\pi_z(\mu)^* = I_{1-s} \pi_{1-z}(\mu) I_s.$$

PROOF. By definition, for all f, $g \in H_s(\Omega)$, $x \in \Gamma$,

$$\begin{aligned}(f, \pi_z(x)^* g)_s &= (\pi_z(x) f, g)_s \\ &= \int_\Omega \pi_z(x) f(\omega) \overline{I_s g(\omega)}\, d\nu(\omega) \\ &= \int_\Omega P^z(x, \omega) f(x^{-1}\omega) \overline{I_s g(\omega)}\, d\nu(\alpha) \\ &= \int_\Omega f(\omega) \overline{P^{1-\bar{z}}(x^{-1}, \omega) I_s g(x\omega)}\, d\nu(\omega);\end{aligned}$$

moreover

$$(f, \pi_z(x)^* g)_s = \int_\Omega f(\omega) \overline{I_s(\pi_z(x)^* g)(\omega)}\, dnu(\omega);$$

so,

$$\pi_z(x)^* g = I_s^{-1} \pi_{1-\bar{z}}(x^{-1}) I_s g.$$

We compute now explicitly the adjoint operator $\pi_z(\mu)^*$ on a particular family of generators of $K(\Omega)$. Let

$$F_e = 1, \quad F_{xa} = \nu^{-1}(E(xa)) \chi_{E(xa)} - \nu^{-1}(E(x)) \chi_{E(x)},$$

when a is a generator of Γ, $x \in \Gamma$, and $|x| + 1 = |xa|$. It is proved in [25] that these functions generate $K(\Omega)$ and that, for every $0 < s < 1$ and for every F_y, with $|y| \geqslant 1$

$$I_s F_y = \theta(s) p^{(1-2s)(|x|+1)} F_y.$$

LEMMA 3.2. *For every $y \in \Gamma$, with $|y| \geqslant 2$,*

$$\pi_z(\mu)^* F_y = \pi_z(\mu) F_y,$$

while, for every y, with $|y| = 1$,

$$\pi_z(\mu)^* F_y = \left(\pi_z(\mu) - 2ip^{-1} \operatorname{Im} \gamma(z)\right) F_y.$$

PROOF. We omit the proof because it is based on direct calculations which, though they require some precision, follow straight from the definitions and Lemma 2.1.

Let $H_s^0(\Omega)$ be the closure with respect to the norm $\| \ \|_s$ of the space generated by $\{F_y\colon y \in \Gamma, |y| \geqslant 1\}$. It can be proved, by a direct computation, that $H_s^0(\Omega)$ is invariant with respect to the operator $\pi_z(\mu)$; let $\pi_z^0(\mu)$ denote the restriction of $\pi_z(\mu)$ to $H_s^0(\Omega)$.

Proposition 3.3. *For every $z \in S$,*

(1) $$\pi_z^0(\mu) = A + iB,$$

where A and B are selfadjoint operators on $H_s^0(\Omega)$.

Proof. We define $B = \frac{i}{2}[\pi_z^0(\mu) - \pi_z^0(\mu)^*]$ and $A = \pi_z^0(\mu) - iB$. Then (1) holds and, by Lemma 3.2, A and B are selfadjoint.

It is known that $\pi_z(\mu)\mathbf{1} = \gamma(z)\mathbf{1}$; we prove now that $\gamma(Oz)$ is an isolated point of the spectrum of $\pi_z(\mu)$, if π_z is a uniformly bounded nonunitary representation.

Proposition 3.4. *Suppose $z \in S$, and $s \neq \frac{1}{2}$ or $t \neq h\pi/\log p$, $h \in \mathbb{Z}$. Then* $\mathrm{sp}(\pi_z^0(\mu))$, *the spectrum of $\pi_z^0(\mu)$, is a compact set contained in the strip*

$$|\mathrm{Im}\, w| \leqslant 2p^{-1}|\mathrm{Im}\,\gamma(z)|.$$

Proof. Let $w = \sigma + i\tau$, $\tau \neq 0$. We can formally write

$$\left(w - \pi_z^0(\mu)\right)^{-1} = \left(I - (w - A)^{-1}iB\right)^{-1}(w - A)^{-1}.$$

Since A is selfadjoint and $\tau \neq 0$, then $(w - A)^{-1}$ is well defined and $\|(w - A)^{-1}\| \leqslant |\tau|^{-1}$. Therefore, since $\|B\| = 2p^{-1}|\mathrm{Im}\,\gamma(z)|$, it follows that for any τ, such that $|\tau| > 2p^{-1}|\mathrm{Im}(z)|$,

$$\left\|(w - A)^{-1}iB\right\| < 1,$$

so $(w - \pi_z^0(\mu))^{-1}$ exists.

We conclude that the eigenspace of $\pi_z(\mu)$ associated to $\gamma(z)$ is one dimensional.

Proposition 3.5. *Let $z \in S$, $s \neq 1/2$ or $t \neq h\pi/\log p$, $h \in \mathbb{Z}$. There exists a sequence of polynomials $\{P_n\}_{n \geqslant 0}$ such that for every $f \in H_s(\Omega)$,*

(2) $$\left\|P_n(\pi_z(\mu))f - (f,\mathbf{1})_s\mathbf{1}\right\|_s \to 0 \quad \textit{as } n \to \infty.$$

Proof. By Proposition 3.4, $\gamma(z) \in \mathrm{sp}(\pi_z^0(\mu))$, so it is possible to construct a sequence of polynomials on $\mathbb{C}$, $\{P_n\}_{n \geqslant 0}$, such that

(i) $P_n(\gamma(z)) = 1$, $\forall n \geqslant 0$,

(ii) $P_n(w) \to 0$, as $n \to \infty$, uniformly on a neighbourhood of $\mathrm{sp}(\pi_z^0(\mu))$. By [4, Lemma 13, page 571], $P_n(\pi_z^0(\mu)) \to 0$, as $n \to \infty$, in the operator norm. So, if $(f,\mathbf{1})_s = 0$, then $P_n(\pi_z(\mu))f \to 0$, as $n \Rightarrow r\infty$. On the other hand, $P_n(\pi_z(\mu) -)\mathbf{1} = P_n(\gamma(z))\mathbf{1} = \mathbf{1}$, $\forall n \geqslant 0$, and (2) holds also for every $f \in H_s(\Omega)$, $(f,\mathbf{1})_s \neq 0$.

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The irreducibility of the uniformly bounded representation follows now by a standard argument and by the next lemma.

LEMMA 3.6. *The vector* $\mathbb{1}$ *is a cyclic vector for the operators* π_z *and* π_z^*, *for every* $z \in S$.

PROOF. In Proposition 1.1 [6, page 52] it is checked that $\mathbb{1}$ is cyclic for π_z. To prove that $\mathbb{1}$ is cyclic also for π_z^*, we exploit the expression of π_z^* given in Lemma 3.1 and the surjectivity of the operator I_s.

THEOREM 3.7. *Every uniformly bounded representation* π_z *is irreducible on the space* $H_s(\Omega)$.

PROOF. Let M be a closed invariant subspace of $H_s(\Omega)$ with respect to π_z and let $f \in M$. Then, by Proposition 3.5,

$$\lim_{n \to \infty} P_n(\pi_z(\mu))f = (f, \mathbb{1})_s \mathbb{1}$$

belongs to M. If there exists an element $f \in M$ such that $(f, \mathbb{1})_s \neq 0$, then $\mathbb{1} \in M$ and $M = H_s(\Omega)$, by Lemma 3.6. If $(f, \mathbb{1})_s = 0$ for every $f \in M$, then $0 = (\pi_z(x)f, \mathbb{1})_s = (f, \pi_z^*(x)\mathbb{1})_s$, for all $x \in \Gamma$, so $M = \{0\}$, by Lemma 3.6.

The reader may wish to compare this theorem with Theorem 1 of Pytlik and Szwarc in [20], relative to a free group with infinitely many generators.

4. Restriction of unitary irreducible representations of $PGL(2, \mathbb{Q}_p)$ to Γ

It is known that $G = PGL(2, \mathbb{Q}_p)$, p prime, can be realised as a group of isometries of a homogeneous tree T of order $p + 1$ [24] and it contains discrete subgroups isomorphic to free groups or free products. In [7] Figà-Talamanca and Picardello exhibited a discrete subgroup Γ of G with compact quotient, which is isomorphic to $\mathbb{Z}_2 * \cdots * \mathbb{Z}_2$, $(p + 1)$-times, and which act in exactly the same way on its associated graph and on the tree of G. In [2] Choucroun extended this result to a free group of order $(p + 1)/2$, $p > 2$, and to any group acting faithfully on the homogeneous tree of $PGL(2, \mathbb{Q}_p)$. This inclusion suggests an investigation of the connection between the unitary irreducible representations of G and those of Γ. Silberger in [23], gives a complete classification of all irreducible unitary representations of G (see also [8]), (with no reference to trees) into the principal, the complementary, the discrete, the special and the degenerate series.

The principal series consists of the representations T_χ, where χ is a unitary multiplicative character of $\mathbf{Q}_p$; the representation space is $L^2(\mathbf{Q}_p)$ and for any $g = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in G$ and for any $f \in L^2(\mathbf{Q}_p)$,

$$T_\chi(g)f(x) = \chi\left(\frac{\det g}{(\beta x+\delta)^2}\right)\frac{|\det g|^{1/2}}{|\beta x+\delta|}f\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right).$$

In particular the spherical representations of the principal series are those associated to the characters $\chi(x) = |x|^{it}$, $t \in \mathbf{R}$. We note that $|\det g|/|\beta x + \delta|^2$ is the Radon-Nikodym derivative of the measure $d\mu_g(x) = d\mu(xg)$, where $d\mu$ is the Lebesgue measure on $\mathbf{Q}_p$ and $x \to xg$ is the usual action of G on $\mathbf{Q}_p$:

$$xg = \frac{\alpha x+\gamma}{\beta x+\delta}.$$

If we realize G as a group of isometries on T, and Ω is the usual Poisson boundary of T, then it is possible to identify [17] the measure space $\mathbf{Q}_p$, with respect to the Lebesgue measure dt, with the measure space $(\Omega, d\nu)$; the action of G on $\mathbf{Q}_p$ coincide with the action of this group on Ω and

$$P(g,x) = \frac{|\det g|}{|\beta x+\delta|^2}.$$

We also note that if, for all $g \in G$, and for $x \in \mathbf{Q}_p$ with $\beta x + \delta \neq 0$, we can decompose $xg = \tau x'$, where $\tau = \left(\begin{smallmatrix}\tau_{11} & \tau_{12}\\ 0 & 1\end{smallmatrix}\right)$, $x' = \left(\begin{smallmatrix}1 & 0\\ x' & 1\end{smallmatrix}\right)$, then

$$x' = \frac{\alpha x+\gamma}{\beta x+\delta} \quad\text{and}\quad \tau_{11} = \frac{\det g}{(\beta x+\delta)^2}.$$

We deduce that the spherical representations of the principal series of G are

$$T_t(g)f(x) = p^{1/2+it}(g,x)f(xg), \qquad \forall t \in \mathbf{R},$$

and their restrictions to Γ are the representations $\pi_{1/2+it}$ of the principal series of Γ [7].

We recall that, for $p > 2$, the functions sgn_τ, where $\tau = \varepsilon,\ p,\ \varepsilon p$, together with the function $\mathbf{1}$ form a complete system of multiplicative characters of the factor group $\mathbf{Q}_p^*/(\mathbf{Q}_p^*)^2$ [see [8] for notations]. Then

PROPOSITION 4.1. *The representation* T_χ, *where* $\chi(x) = |x|^{it}\operatorname{sgn}_\tau(x)$, $x \in \mathbf{Q}_p$, $t \in \mathbf{R}$, $\tau = \varepsilon,\ p,\ \varepsilon p$, *restricts to the representation of* Γ

$$T_\chi(g) = (\operatorname{sgn}_\tau p)^{|g|}\pi_{\frac12+it}(g), \quad \textit{for any } g \in \Gamma.$$

PROOF. First we observe that $\operatorname{sgn}_\tau((\det g)(\beta x+\delta)^{-2}) = \operatorname{sgn}_\tau(\det g)$, for all $g \in G$ and $x \in \mathbf{Q}_p$. Then we write every $g \in \Gamma$ as a free product of the generators g_j, $j = 0, 1, \ldots, p$, where

$$g_0 = \begin{pmatrix}0 & -p\\ 1 & 0\end{pmatrix} \quad\text{and}\quad g_j = \begin{pmatrix}j & -1\\ p+j^2 & -j\end{pmatrix}, \qquad j = 1,\ldots,p,$$

(see [7]). Since $\det g_j = p$ for all $j = 0, \dots, p$, then $\operatorname{sgn}_\tau(\det g) = (\operatorname{sgn}_\tau p)^{|g|}$, where $|g|$ denotes the length of the word $g \in \Gamma$.

The complementary series of G consists, for $p > 2$, of the representations $V_{\chi^*,z}$ where $\chi^* = \mathbb{1}$ or $\chi^* = \operatorname{sgn}_\tau$, $\tau = \varepsilon,\ p,\ \varepsilon p$ and $z = s + it$ is a complex number such that $0 < s < 1$ and $t = h\pi/\log p$, $h \in \mathbf{Z}$. The representation space consists of the functions on $\mathbf{Q}_p$ with the inner product

$$(f_1, f_2)_s = \iint f_1(x)\overline{f_2(y)}\,|x - y|^{2(s-1)}\,dx\,dy,$$

and the representation operators act on this space according to the formula

$$V_{\chi^*,z}(g)f(x) = \chi^*(\det g)\left(\frac{|\det g|}{|\beta x + \delta|^2}\right)^{s+it} f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right),$$

for all $g \in G$. In particular for $\chi^* \equiv 1$ we obtain the spherical representations of the complementary series of G. By the same argument used in Proposition 4.1 we can prove that, for any $z \in \mathbf{C}$ considered and for any χ^*, the restriction of $V_{\chi^*,z}$ to Γ coincides with the representation

$$V_{\chi^*,z}(g) = (\operatorname{sgn}_\tau p)^{|g|}\pi_{s+it}(g), \quad \forall g \in \Gamma,$$

where π_{s+it} belongs to the complementary series of Γ.

Since the irreducibility of the representation π_{s+it} implies that of the representation π'_{s+it}, where $\pi'_{s+it}(g) = (-1)^{|g|}\pi_{s+it}(g)$, $g \in \Gamma$, we have proved the following result.

PROPOSITION 4.2. *The irreducible representations of the principal series associated to a character* $\chi(x) = |x|^{it}\operatorname{sgn}_\tau(x)$ *and all those of the complementary series of* G, *remain irreducible when restricted to* Γ.

REMARK 4.3. An analogous result holds for the representations of the analytic continuation of the principal series, associated to the characters $\chi(x) = |x|^z \operatorname{sgn}_\tau(x)$, $z \in S$.

Relative to the special series of G, it is known that it consists of the representations $S_{\chi^*,t}$, $t = h\pi/\log p$, $h \in \mathbf{Z}$, acting on the space of functions on $\mathbf{Q}_p$ for which $\int f(x)\,dx = 0$, with the inner product

$$(f_1, f_2)_1 = \iint f_1(x) f_2(y) \log|x - y|\,dx\,dy.$$

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The action is defined by

$$S_{\chi^*,t}(g)f(x) = \operatorname{sgn}_\tau(\det g)\frac{|\det g|^{1+it}}{|\beta x+\delta|^{2(1+it)}}f\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right).$$

Because the matrix coefficients for these representations belong to $L^2(G)$, they are subrepresentations of the regular representation of G and therefore their restrictions of the discrete subgroup Γ are reducible (see [19]).

The same argument proves that the restrictions to Γ of the discrete series of G are reducible.

Finally we observe that the degenerate series consists of the representations acting on the one-dimensional space of constant functions, in the following way

$$W_{\chi^*,t}(g) = \chi^*(\det g)|\det g|^{it}, \qquad t = h\pi/\log p,\ h \in \mathbb{Z},$$

and their restrictions to Γ are the one-dimensional representations π_{it} or $(-1)^{|g|}\pi_{it}$.

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Recently we have been informed (oral communication) that T. Pytlik and R. Szwarc also discuss, in forthcoming papers, the case of finitely generated free groups.

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