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BACKWARD 3-STEP EXTENSIONS OF RECURSIVELY GENERATED WEIGHTED SHIFTS: A RANGE OF QUADRATIC HYPONORMALITY

GEORGE R. EXNER, IL BONG JUNG ✉, MI RYEONG LEE and SUN HYUN PARK

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Abstract

Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ be a backward 3-step extension of a recursively generated weighted sequence of positive real numbers with $1 \le x \le u \le v \le w$ and let W_α be the associated weighted shift with weight sequence α. The set of positive real numbers x such that W_α is quadratically hyponormal for some u, v and w is described, solving an open problem due to Curto and Jung ['Quadratically hyponormal weighted shifts with two equal weights', *Integr. Equ. Oper. Theory* **37** (2000), 208–231].

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1. Introduction

Let $\mathcal{H}$ be a separable infinite-dimensional complex Hilbert space and let $L(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For $X, Y \in L(\mathcal{H})$, one sets $[X, Y] = XY - YX$. An operator T in $L(\mathcal{H})$ is *normal* if $[T^*, T] = 0$, *subnormal* if it has a normal extension, and *hyponormal* if $[T^*, T] \ge 0$. An operator T in $L(\mathcal{H})$ is *weakly k-hyponormal* if $p(T)$ is hyponormal for every complex polynomial p of degree k or less [2]. An operator T is *polynomially hyponormal* if $p(T)$ is hyponormal for every complex polynomial p. Obviously 'subnormal $\Rightarrow$ polynomially hyponormal $\Rightarrow \cdots \Rightarrow$ weakly 2-hyponormal $\Rightarrow$ hyponormal'. In particular, weak 2-hyponormality is referred to as quadratic hyponormality. Note that the structure of quadratically hyponormal weighted shifts has been studied at some length since 1990 (see [3, 6–10]).

Some of these and related studies have considered backward extensions, of length 1, 2 or 3, of the subnormal shift with weight sequence $(\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$, where this

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is Stampfli's subnormal completion of three positive real values $\sqrt{u}$, $\sqrt{v}$, $\sqrt{w}$ ([12]; definitions are reviewed below). Many of these [3, 6, 9, 11] have considered *flatness* (in which, under some condition, equality of two or three weights forces equality of all or almost all weights), and it is understood that to avoid trivialities there is no loss of generality in assuming that the weight sequence is strictly increasing except perhaps for the zeroth and first weights. An early question posed in [4] was to *describe all quadratically hyponormal weighted shifts with the first two weights equal*. Since all properties of interest are retained under scaling, we may as well assume that the first two weights are 1.

Recall that, for a weight sequence $\alpha : 1, 1, \alpha_2, \alpha_3, \ldots$, if its associated weighted shift W_α is quadratically hyponormal, a symbolic manipulation shows that there are no such shifts with $1 < \alpha_2 = \alpha_3$ [2, Proposition 11], that α_2 is always less than $\sqrt{2}$, and that $\alpha_3 \geq (2 - \alpha_2^2)^{-1/2}$. Thus it is worthwhile to find the range of α_2 and α_3 in this case. As a related question, the following problem, which remains open, was posed in [6].

PROBLEM 1.1 [6, Problem 5.3]. Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with $1 \leq x \leq u \leq v \leq w$. Describe the set of positive real numbers x such that the weighted shift W_α is quadratically hyponormal for some positive real numbers u, v and w.

In this note, we solve Problem 1.1 using tools from [8] for positive quadratic hyponormality.

2. Preliminaries and notation

We denote by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ the positive integers, real numbers, and complex numbers, respectively. Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and denote by $\mathbb{R}_+$ the set of positive real numbers. For a subset $\mathcal{M}$ of $\mathcal{H}$, $\vee\mathcal{M}$ denotes the closed linear span of $\mathcal{M}$.

Let $\alpha := \{\alpha_i\}_{i=0}^\infty$ denote a positive term weight sequence. The weighted shift W_α acting on $\ell^2(\mathbb{N}_0)$, with standard basis $\{e_i\}_{i=0}^\infty$, is defined by $W_\alpha(e_i) = \alpha_i e_{i+1}$ for all $i \in \mathbb{N}_0$, and extended by linearity. It is well known that W_α is quadratically hyponormal if and only if $W_\alpha + sW_\alpha^2$ is hyponormal for any $s \in \mathbb{C}$.

We give here a much abbreviated discussion of certain standard notation and computational results in this context; for a fuller discussion, see [10] or [8] amongst others. Let P_n be the orthogonal projection onto $\vee\{e_i\}_{i=0}^n$. For $s \in \mathbb{C}$ and $n \in \mathbb{N}_0$, define D_n by

$$D_n := D_n(s) = P_n[(W_\alpha + sW_\alpha^2)^*, \quad W_\alpha + sW_\alpha^2]P_n.$$

It turns out that D_n is tridiagonal, with diagonal entries q_k $(0 \leq k \leq n)$ and super-diagonal (respectively, sub-diagonal) entries $\overline{r_j}$ (respectively, r_j) $(1 \leq j \leq n-1)$. Computations show, setting $\alpha_{-1} = \alpha_{-2} := 0$ for convenience, that

$$q_k = u_k + |s|^2 v_k, \quad r_k = s\sqrt{w_k}, \quad u_k = \alpha_k^2 - \alpha_{k-1}^2,$$
$$v_k = \alpha_k^2\alpha_{k+1}^2 - \alpha_{k-1}^2\alpha_{k-2}^2, \quad w_k = \alpha_k^2(\alpha_{k+1}^2 - \alpha_{k-1}^2)^2, \quad k \geq 0.$$

Since W_α is quadratically hyponormal if and only if $D_n(s) \geq 0$ for every $s \in \mathbb{C}$ and every $n \geq 0$, we consider $d_n(\cdot) := \det(D_n(\cdot))$ in anticipation of the usual use of Sylvester's

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criterion (also known as the nested determinant test) for positivity of a matrix; d_n turns out to be a polynomial in $t := |s|^2$ of degree $n+1$, having Maclaurin's expansion

$$d_n(t) := \sum_{i=0}^{n+1} c(n,i)t^i.$$

The coefficients $c(n,i)$ have well-known expressions, including some recursive ones, in terms of the u_k, v_k and w_k (see [10]). We record for future use that u_n, v_n and w_n are nonnegative for all $n \geq 0$, under the assumption that the weights are strictly increasing.

A weighted shift W_α is defined to be *positively quadratically hyponormal* if $c(n, n+1) > 0$ and $c(n,i) \geq 0$, $0 \leq i \leq n$, $n \in \mathbb{N}_0$ (see [1, 5]); obviously this is an assumption (strong but) sufficient to have all the d_n nonnegative to ensure quadratic hyponormality.

Let $\alpha_1, \alpha_2, \alpha_3$ be the first three weights in $\mathbb{R}_+$ such that $\alpha_0 < \alpha_1 < \alpha_2$ (to avoid flatness). Then there is a canonical way due to Stampfli [12] to generate a satisfactory completion recursively: define

$$\widehat{\alpha}_n = \left(\Psi_1 + \frac{\Psi_0}{\alpha_{n-1}^2}\right)^{1/2}, \quad n \geq 3,$$

where

$$\Psi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2 - \alpha_1^2)}{\alpha_1^2 - \alpha_0^2}, \quad \Psi_1 = \frac{\alpha_1^2(\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2}.$$

This produces a bounded sequence $\widehat{\alpha} := \{\widehat{\alpha}_i\}_{i=0}^{\infty}$, for which $\widehat{\alpha}_i = \alpha_i$ $(0 \leq i \leq 2)$ and such that the resulting weighted shift is subnormal. We usually write the weight sequence $(\alpha_0, \alpha_1, \alpha_2)^\wedge$ and the resulting weighted shift as $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$. Associated with this process is a value K (to appear in the next theorem) in terms of Ψ_0 and Ψ_1 but whose value is not important here.

We recall the criterion for positive quadratic hyponormality of the weighted shift W_α with $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ as follows.

Theorem 2.1 [8, Theorem 3.5]. *Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with $1 < x < u < v < w$. Then W_α is positively quadratically hyponormal if and only if:*

(i) $c(3,2) \geq 0$ *(that is, $x \leq 2 - 1/u$);*
(ii) $c(4,4) \geq 0$, $c(4,3) \geq 0$*;*
(iii) $c(5,5) \geq 0$, $c(5,4) \geq 0$*;*
(iv) *it holds that, with $\eta_1 := v_4 c(3,4)$ and $\eta_2 := v_4 c(3,3) - w_3 c(2,3)$,*
  (a) *$\eta_2 \geq 0$, or*
  (b) *$\eta_2 < 0$ and $K \leq \eta_1/|\eta_2|$;*
(v) *it holds that, with $\eta_3 := v_4 c(3,2) - w_3 c(2,2)$,*
  (a) *if $\eta_2 + ((v^2(u-w)^3)/(u(u-v)^2(v-w)))\eta_3 \geq 0$, then $\eta_1 + (v_5/u_5)\eta_2 + (v_5/u_5)(v_6/u_6)\eta_3 \geq 0$ or equivalently, $c(6,5) \geq 0$,*
  (b) *if $\eta_2 + ((v^2(u-w)^3)/(u(u-v)^2(v-w)))\eta_3 < 0$, then $\eta_1 + \eta_2 K + \eta_3 K^2 \geq 0$.*

3. A solution of Problem 1.1

We now solve Problem 1.1, and, as noted, need only consider the case in which the first two weights are one.

THEOREM 3.1. *Let $\alpha(x): 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ be a weight sequence with positive real variables x, u, v, w such that $1 < x < u < v < w$. Let*

$$(\mathrm{PQH}) := \{x \mid W_{\alpha(x)} \text{ is positively quadratically hyponormal for some } u, v, w\},$$
$$(\mathrm{QH}) := \{x \mid W_{\alpha(x)} \text{ is quadratically hyponormal for some } u, v, w\}.$$

Then $(\mathrm{PQH}) = (\mathrm{QH}) = (1, 2)$.

PROOF. We begin by presenting the results for some of the $c(n, i)$ appearing in Theorem 2.1:

$$c(3,2) = x(x-1)(2u-1-ux), \quad c(4,3) = x(x-1)f_1(x), \quad c(4,4) = xf_2(x),$$

where

$$f_1(x) = (2u^2v - 2uv - uvw + u)x - (u^3v + u^2 + vw - uv - 2uvw),$$
$$f_2(x) = u^2x^3 + (-uv - 2uvw - u^3v - 2u^2 + 2u^2v + u)x^2$$
$$+ (v - u + u^2v^2w + 3uvw + u^2 - 2uv)x + (u^2v - uvw - u^2v^2w).$$

We now claim that $(\mathrm{QH}) \subset (1, 2)$. Let $x \in (\mathrm{QH})$, that is, $\alpha(x)$ induces a quadratically hyponormal weighted shift $W_{\alpha(x)}$ for some u, v, w with $1 < x < u < v < w$. Obviously, since $D_3(s) \geq 0$ for all $s \in \mathbb{C}$, we have $d_3(t) \geq 0$ for all $t \geq 0$. Hence, necessarily,

$$c(3,2) = x(x-1)(2u-1-ux) \geq 0.$$

That is, $2u - 1 - ux \geq 0$, so $1 < x \leq 2 - 1/u < 2$.

Since $(\mathrm{PQH}) \subset (\mathrm{QH})$, it is sufficient to show that $(\mathrm{PQH}) = (1, 2)$. To do so, we will show that conditions (i)–(v) of Theorem 2.1 are met with u, v, w sufficiently large. It aids the computations to set $x = 1 + \varepsilon$ with $0 < \varepsilon < 1$. Fix $x < 2$, so $\varepsilon < 1$. Clearly for any $x < 2$, if we choose u sufficiently large we will satisfy condition (i) of Theorem 2.1. One computes that $f_1(1 + \varepsilon, u, v, w)$ is linear in w, of the form

$$\widehat{f_1}(\varepsilon, u, v) + (v(-1 + u(1 - \varepsilon)))w.$$

Clearly if u is large, the coefficient of the highest power of w (in this case, w^1) is positive. So for u, w sufficiently large we can ensure that $f_1(x, u, v, w) > 0$. The same argument, *mutatis mutandis*, handles f_2, and together these yield condition (ii) of Theorem 2.1.

(†) Note that in taking u large we must keep $v > u$, but this gives no difficulty in keeping some term positive.

The coefficient $c(5, 5)$ is of the form

$$\frac{1}{v-u}(1+\varepsilon) \cdot \sum_{i=0}^{3} a_{5,5,i} \cdot w^i,$$

where $a_{5,5,i}$ is a function of ε, u, and v. The coefficient $a_{5,5,3} = uv^2(-1 + uv - 2\varepsilon)\varepsilon$ and this is surely positive for u large enough (and see (†) above). So $c(5, 5) > 0$ for u and w sufficiently large. Entirely similar arguments for the coefficient $c(5, 4)$ with highest power w^3 having coefficient $v^2\varepsilon(-1 + u(1 - \varepsilon))$ show that this is positive for u, v and w sufficiently large. This completes condition (iii).

Now $\eta_2(1 + \varepsilon, u, v, w)$ is linear in w of the form

$$\widehat{\eta_2}(\varepsilon, u, v) + w(uv\varepsilon(1 + \varepsilon)(-1 + uv - 2\varepsilon)).$$

Again the coefficient is positive for u sufficiently large and so with u, w sufficiently large we are in the case $\eta_2 > 0$. This means we need not consider condition (iv)(b).

We next show that if w is sufficiently large we are in case (v)(a) of Theorem 2.1, that is, $g := \eta_2 + ((v^2(u - w)^3)/(u(u - v)^2(v - w)))\eta_3 \geq 0$. This is of the form

$$g = \frac{1}{u(u - v)^2(w - v)} \cdot \sum_{i=0}^{4} g_i \cdot w^i,$$

where g_i is a function of ε, u, and v. Now (excluding multipliers obviously positive) $g_4 = v^3\varepsilon(-1 + u(1 - \varepsilon))$, which is positive for u large. Thus if u and w are sufficiently large we are in the 'increasing' case of (v)(a) of Theorem 2.1.

Finally, consider $h := \eta_1 + z_5\eta_2 + z_5z_6\eta_3$. This is of the form

$$h = \frac{1}{u^2} \frac{1}{(u - v)^2} \frac{1}{(w - v)^2} \sum_{i=0}^{6} h_i \cdot w^i,$$

where each h_i is a function of ε, u, and v. The function $h_6 = v^4\varepsilon(-1 + u(1 - \varepsilon))$ is positive for u large, so $\eta_1 + z_5\eta_2 + z_5z_6\eta_3 \geq 0$ for u, w large. This last argument completes condition (v)(a) of Theorem 2.1 and we are done. □

In fact, we have the following *a priori* stronger result.

THEOREM 3.2. *Let δ be an arbitrary real number in the open interval $(1, 2)$. Then there exist u_δ, v_δ and w_δ with $u_\delta < v_\delta < w_\delta$ such that for all $x \in (1, \delta]$, $W_{\alpha(x)}$ is a positively quadratically hyponormal weighted shift with the weight sequence $\alpha(x) : 1, 1, \sqrt{x}, (\sqrt{u_\delta}, \sqrt{v_\delta}, \sqrt{w_\delta})^\wedge$.*

PROOF. Consider the 'highest power of w' coefficients of the expressions arising as in the proof of Theorem 3.1, and with its ε, from the conditions (i)–(v) of Theorem 2.1, including $\eta_2 > 0$ and h in case (v)(a):

$$\begin{aligned}
f_1 &: v(-1 + u(1 - \varepsilon)),\\
f_2 &: uv(-1 - 2\varepsilon + uv),\\
c(5, 5) &: uv^2(-1 - 2\varepsilon + uv),\\
c(5, 4) &: v^2\varepsilon(-1 + u(1 - \varepsilon)),\\
\eta_2 &: uv\varepsilon(1 + \varepsilon)(-1 - 2\varepsilon + uv),
\end{aligned}$$

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$$g : v^3\varepsilon(-1 + u(1 - \varepsilon)),$$
$$h : v^4\varepsilon(-1 + u(1 - \varepsilon)).$$

The collection is finite and, as long as ε is bounded away from 1 (since $x \leq \delta < 2$), we may clearly choose u (and $v > u$) sufficiently large so that all these coefficients are positive for all $0 < \varepsilon \leq \delta - 1$ corresponding to $1 < x \leq \delta$. We may then choose w so large that all the expressions $f_1, f_2, \ldots, h$ are positive for all $0 < \varepsilon \leq \delta - 1$, that is, all $1 < x \leq \delta$. □

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GEORGE R. EXNER, Department of Mathematics, Bucknell University, Lewisburg, Pennsylvania 17837, USA
e-mail: exner@bucknell.edu

IL BONG JUNG, Department of Mathematics, Kyungpook National University, Daegu 702-701, Korea
e-mail: ibjung@knu.ac.kr

MI RYEONG LEE, Institute of Liberal Education, Catholic University of Daegu, Gyeongsan 712-702, Korea
e-mail: leemr@cu.ac.kr

SUN HYUN PARK, Department of Mathematics, Kyungpook National University, Daegu 702-701, Korea
e-mail: sm1907s4@knu.ac.kr

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