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EXPLICIT SOLUTIONS FOR A SYSTEM OF COUPLED LYAPUNOV DIFFERENTIAL MATRIX EQUATIONS

by L. JODAR and M. MARITON

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1. Introduction

This paper is concerned with the problem of obtaining explicit expressions of solutions of a system of coupled Lyapunov matrix differential equations of the type

$$\begin{aligned} \dot{X}_i(t) &= A_i(t) + B_i(t)X_i(t) + X_i(t)C_i(t) + \sum_{j=1}^{N} D_{ij}(t)X_j(t) \\ X_i(b) &= F_i, \qquad 1 \leqq i, j \leqq N \end{aligned} \tag{1.1}$$

where F_i, $A_i(t)$, $B_i(t)$, $C_i(t)$ and $D_{ij}(t)$ are $m \times m$ complex matrices (members of $\mathbb{C}_{m\times m}$), for $1 \leqq i$, $j \leqq N$, and t in the interval $[a, b]$. When the coefficient matrices of (1.1) are time-invariant, D_{ij} are scalar multiples of the identity matrix of the type $D_{ij} = d_{ij}I$, where d_{ij} are real positive numbers, for $1 \leqq i, j \leqq N$ C_i is the transposed matrix of B_i, and $F_i = 0$, for $1 \leqq i \leqq N$, the Cauchy problem (1.1) arises in control theory of continuous-time jump linear quadratic systems [**9–11**]. Algorithms for solving the above particular case can be found in [**12**]. These methods yield approximations to the solution. Without knowing the explicit expression of the solutions and in order to avoid the error accumulation it is interesting to know an explicit expression for the exact solution. In Section 2, we obtain an explicit expression of the solution of the Cauchy problem (1.1) and of two-point boundary value problems related to the system arising in (1.1). Stability conditions for the solutions of the system of (1.1) are given. Because of developed techniques this paper can be regarded as a continuation of the sequence [**3**, **4**, **5**, **6**].

2. Explicit solutions

Let A be a $m \times n$ complex matrix, $A \in \mathbb{C}_{m\times n}$ and let B be a $k \times s$ complex matrix, $B \in \mathbb{C}_{k\times s}$, then the tensor product of A and B written $A \otimes B$, is defined to be the partitioned matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}.$$

It must be assumed that the reader has some familiarity with this concept. An account of the uses and applications of this operation can be found in [7, Chapter 8] and in [8]. If $A \in \mathbb{C}_{m \times n}$, we denote

$$\hat{A} = \operatorname{vec} A = \begin{bmatrix} A_{.1} \\ \vdots \\ A_{.n} \end{bmatrix}, \quad A_{.j} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}, \qquad 1 \leqq j \leqq n.$$

If M, N and P are matrices with suitable dimensions, and P^T denotes the transposed matrix of P, then from the column lemma [1], one gets $\operatorname{vec}(MNP) = (P^T \otimes M) \operatorname{vec} N$, and if $F_i, 1 \leqq i \leqq N$, are matrices we denote

$$F = \begin{bmatrix} F_1 \\ \vdots \\ F_N \end{bmatrix}, \operatorname{vec} F = \hat{F} = \begin{bmatrix} \hat{F}_1 \\ \vdots \\ \hat{F}_N \end{bmatrix}.$$

Given the problem (1.1) let $M(t) = M_{ij}(t)$, where $t \in [a, b]$, $1 \leqq i, j \leqq N$, defined by

$$\begin{aligned} M_{ii}(t) &= (I \otimes B_i(t)) + (C_i(t)^T \otimes I) + (I \otimes D_{ii}(t)) \\ M_{ij}(t) &= I \otimes D_{ij}(t); \qquad 1 \leqq i, j \leqq N, \quad i \neq j \end{aligned} \tag{2.1}$$

and let $\Phi(t, s)$ the transition state matrix of the linear system

$$\dot{Y} = M\,Y \tag{2.2}$$

such that $\Phi(b, b) = I$ and $\Phi(t, s) = \exp(\int_s^t M(u)\,du)$, [2].

With the previous notation, the next result contains an explicit solution of the Cauchy problem (1.1) in the general case.

Theorem 1. *Let us consider the Cauchy problem* (1.1) *where the coefficients are continuous matrix valued functions defined on the interval* $[a, b]$. *Then the only solution of this problem is given by the expressions*

$$\hat{X}(t) = \Phi(t, b)\left\{\hat{F} + \int_b^t \Phi(b, s)\hat{A}(s)\,ds\right\} \tag{2.3}$$

$$X(t) = \begin{bmatrix} X_1(t) \\ \vdots \\ X_N(t) \end{bmatrix}, \quad A(s) = \begin{bmatrix} \hat{A}_1(s) \\ \vdots \\ A_N(s) \end{bmatrix} \tag{2.4}$$

where s, t *belong to* $[a, b]$.

Proof. By application of the column lemma, [1], the Cauchy problem (1.1) is equivalent to the following one

$$\dot{\hat{X}}_i(t) = \hat{A}_i(t) + (I \otimes B_i(t))\hat{X}_i(t) + (C_i(t)^T \otimes I)\hat{X}_i(t) + \sum_{j=1}^{N} (I \otimes D_{ij}(t))\hat{X}_j(t)$$
$$\hat{X}_i(b) = \hat{F}_i, \qquad 1 \leqq i, j \leqq N. \tag{2.5}$$

If J_i denotes the set $\{1, \ldots, i-1, i+1, \ldots, N\}$, for $1 \leqq i \leqq N$, Problem (2.5) can be expressed by the form

$$\dot{\hat{X}}_i(t) = M_{ii}(t)\hat{X}_i(t) + \sum_{j \in J_i} M_{ij}(t)\hat{X}_j(t) + \hat{A}_i(t)$$
$$\hat{X}_i(b) = \hat{F}_i, \qquad 1 \leqq i, j \leqq N \tag{2.6}$$

where $M_{ij}(t)$ are given by (2.1) From (2.6) Problem (1.1) is equivalent to the extended linear system

$$\dot{\hat{X}}(t) = M(t)\hat{X}(t) + \hat{A}(t)$$
$$\hat{X}(b) = \hat{F} \tag{2.7}$$

where X and A are given by (2.4). From (2.7) and [**2**, p. 40], the only solution of Problem (1.1) is given by (2.3).

For the time-invariant case, $M = M(t)$ is a constant matrix and the transition state matrix is $\Phi(t,s) = \exp((t-s)M)$. Thus for the time invariant case, the solution of Problem (1.1) is obtained from (2.3), substituting $\Phi(t,s)$ by the exponential matrix $\exp((t-s)M)$. Thinking of the application to the control theory of continuous-time jump linear quadratic systems, we are interested in the explicit expression of the following particular case.

Corollary 1. *Let us consider Problem* (1.1) *where* $F_i = 0$, *and* $C_i = B_i^T$, $D_{ij} = d_{ij}I$, $A_i = Q$, *are time-invariant matrices for* $1 \leqq i, j \leqq N$, *and* d_{ij} *is a positive number for each* i, j. *In this case the solution of Problem* (1.1) *is given by*

$$\hat{X}(t) = \exp(Mt)\left(\int_b^t \exp(-sM)\, ds\right)\hat{A}, \qquad A = \begin{bmatrix} Q \\ \vdots \\ Q \end{bmatrix} \tag{2.8}$$

where $M = (M_{ij})$,

$$M_{ii} = (1 \otimes B_i) + (B_i \otimes I) + I \otimes (d_{ii}I)$$
$$M_{ij} = I \otimes (d_{ij}I); \qquad 1 \leqq i, j \leqq N, \quad i \neq j. \tag{2.9}$$

Proof. It is a consequence of Theorem 1.

Corollary 2. *With the hypothesis and the notation of Corollary* 1, *if* M *is an invertible matrix, the solution of the Cauchy problem* (1.1) *is given by*

$$\hat{X}(t)=M^{-1}(I-\exp((b-t)M))\hat{A}. \tag{2.10}$$

Proof. By integration in the expression (2.8) of Corollary 1, the result is established.

In order to obtain an effective computation of the solution of the Cauchy problem (1.1) for the time-invariant case studied in Corollaries 1 and 2, we recall that in [**14**], useful methods for computing the integral (2.8) are given. These procedures are extremely easy to implement and yield an estimation of the approximation error. For computing the expression (2.10), a numerically useful method is given in [**13**], where one can compute the exponential $\exp(uM)$ avoiding the computation of the eigenvalues of M.

For the time varying case, in general a computable expression of the transition state matrix $\Phi(t,s)$ is not available, though there exist several classes of systems such that this matrix is computable in the same manner as the linear invariant systems. In accordance with Definition 2 given by Wu in [**15**], a set of $m\times m$ constant matrices $\{M_i\}p_{i=1}$ is said to be mutually commutative if and only if, $M_iM_j=M_jM_i$, for all i,j, such that $1\leqq i,j\leqq p$.

Let us consider the time-varying Problem (1.1) and let us suppose that the matrix $M(t)$ given by (2.1) can be written as

$$M(t)=\sum_{i=1}^{p} c_i(t)M_i \tag{2.11}$$

where M_i's are constant matrices and $c_i(t)$'s are linearly independent sets of scalar time functions extracted from elements of $M(t)$, such that $\{M_i\}_{i=1}^{p}$ is mutually commutative, then the transition state matrix $\Phi(t,s)$ of the linear system (2.7) can be computed by

$$\Phi(t,b)=\prod_{i=1}^{p}\exp(M_ib_i(t,b)) \tag{2.12}$$

where

$$b_i(t,b)=\int_b^t c_i(s)\,ds \tag{2.13}$$

and M_i's and $c_i(s)$'s are defined by (2.11).

The following corollary is a consequence of Theorem 1 and Theorems 1 and 2 of [**15**].

Corollary 3. *Let us consider Problem* (1.1) *defined on the real line, and let us assume that the matrix* $M(t)$ *given by* (2.7) *can be expressed by* (2.11) *and* $\{M_i\}_{i=1}^{p}$ *is mutually* *commutative. Then it follows that.*

(i) *The Cauchy problem* (1.1) *has only one solution given by* (2.3) *where* $\Phi(t,s)$ *is given by* (2.12)–(2.13).

(ii) *The system* (1.1) *is stable if* $|b_i(t,b)|<\infty$, *for all* t *and* b *and for* $1\leqq i\leqq p$.

(iii) *The system* (1.1) *is asymptotically stable if for all* $i=1,2,\dots,p$, $b_i(t,b)>0$ *for all* t *and* b *and* $b_i(t,b)\to\infty$ *as* $t\to\infty$ *and*

$$\operatorname{Re}\lambda_k(M_i)<0, \qquad 1\leqq i\leqq p$$

where $\lambda_k(M_i)$ *is the kth eigenvalue of the matrix* M_i.

Example. Let us consider the coupled Lyapunov system

$$\dot{X}_i(t)=A_i(t)+B_i(t)X_i(t)+X_i(t)C_i(t)+\sum_{j=1}^{2}D_{ij}(t)X_j(t), \qquad 1\leqq i,j\leqq 2 \tag{2.14}$$

where A_1 and A_2 are continuous matrix functions defined on the real line, and

$$C_1(t)=\begin{bmatrix}\cos t & 0\\ 0 & 0\end{bmatrix};\ C_2(t)=\begin{bmatrix}1 & 0\\ 0 & \cos t\end{bmatrix};\ B_1(t)=\begin{bmatrix}-1 & 0\\ -\sin t & \cos t-1\end{bmatrix}$$

$$B_2(t)=\begin{bmatrix}\cos t & 0\\ \sin t & 1\end{bmatrix};\ D_{12}(t)=-D_{21}(t)=\begin{bmatrix}\sin t & 0\\ 0 & \sin t\end{bmatrix};\ D_{11}(t)=\begin{bmatrix}1 & 0\\ \sin t & 1-\cos t\end{bmatrix}$$

and

$$D_{22}(t)=\begin{bmatrix}-\cos t & 0\\ -\sin t & -1\end{bmatrix}.$$

It is easy to show that in this case the matrix $M(t)$ of the corresponding system (2.2) associated to (2.14) is given by

$$M(t)=\cos t\,M_1+\sin t\,M_2;\ M_1=I_{8\times 8},$$

$$M_2=\left[\begin{array}{c|c} 0_{4\times 4} & \begin{array}{cc|c} 1 & 0 & \multirow{2}{*}{$0_{2\times 2}$}\\ 0 & 1 & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \begin{array}{cc}1 & 0\\ 0 & 1\end{array} \end{array}\\ \hline \begin{array}{cc|c} -1 & 0 & \multirow{2}{*}{$0_{2\times 2}$}\\ 0 & -1 & \\ \hline \multicolumn{2}{c|}{0_{2\times 2}} & \begin{array}{cc}-1 & 0\\ 0 & -1\end{array} \end{array} & 0_{4\times 4} \end{array}\right] \tag{2.15}$$

It is clear that $M_1M_2=M_2M_1$, and following the notation of (2.11)–(2.13), one gets

$$b_1(t,b)=\sin t-\sin b;\ b_2(t,b)=\cos b-\cos t,$$

and

$$\Phi(t,b)=\exp((\sin t-\sin b)I)\exp((\cos b-\cos t)M_2),$$

where M_2 is given by (2.15). From Corollary 3, the system (2.14) is stable. For a concrete initial condition and for given matrices $A_i(t)$, $i=1,2$, the explicit expression of the solution of the correspondient Cauchy problem (1.1), is given by Expression (2.3).

The following result is concerned with a two-point boundary value problem of the type

$$\begin{aligned}&\dot{X}_i(t)=A_i(t)+B_i(t)X_i(t)+X_i(t)C_i(t)+\sum_{j=1}^{N}D_{ij}(t)X_j(t)\\&X_i(b)-X_i(a)=E_i;\qquad 1\leqq i,j\leqq N,\quad a\leqq t\leqq b\end{aligned}\tag{2.16}$$

Theorem 2. *Let us consider Problem* (2.16) *and let us suppose that the transition state matrix* $\Phi(t,s)$ *of the linear system* (2.7) *satisfies the property.*

$$\text{For all eigenvalue } \lambda \text{ of } \Phi(a,b),\qquad \lambda\neq 1.\tag{2.17}$$

Then there is only one solution of (2.1) *given by the Expression* (2.3) *taking*

$$F=(I-\Phi(a,b))^{-1}\left\{\hat{E}+\int_b^a\Phi(b,s)\hat{A}(s)\,ds\right\}\tag{2.18}$$

where

$$E=\begin{bmatrix}E_1\\ \vdots\\ E_N\end{bmatrix},$$

and $t\in[a,b]$.

Proof. From Theorem 1, the only solution of a Cauchy problem for the differential equation of (1.1) is given by (2.3), where $F=X(b)$. Imposing on the solution the boundary value condition (2.16), it follows that $X(b)$ satisfies the equation

$$(I-\Phi(a,b))\hat{X}(b)=\hat{E}+\int_b^a\Phi(b,s)\hat{A}(s)\,ds.\tag{2.19}$$

From the hypothesis (2.17), the matrix $I-\Phi(a,b)$ is invertible and from (2.19), $X(b)$ is given by (2.18). Thus the only solution of (2.14) is given by (2.3) taking F from (2.18).

An analogous result to Theorem 2 can be obtained for the time-invariant case, taking into account that in this case $\Phi(t,s)=\exp((t-s)M)$. In fact, for the time-invariant case, Condition (2.17) is equivalent to the following one.

$$\text{For all eigenvalue } \lambda \text{ of } M, \qquad \lambda \neq 2k\pi i/(b-a),\ k \text{ integer.} \tag{2.20}$$

Under this hypothesis, the only solution of the correspondent problem (2.16) is given by (2.3), with F given by (2.18) and $\Phi(t,s) = \exp((t-s)M)$.

Corollary 4. *Let us consider the coupled system* (1.1) *and let us assume that the coefficient matrix valued functions are continuous T-periodic in the real line, where T is a positive real number. Under the hypothesis*

$$\text{For all eigenvalue } \lambda \text{ of } \Phi(T,0), \qquad \lambda \neq 1 \tag{2.21}$$

the only T-periodic solution of the coupled system (1.1) *is given by* (2.3), *where* $a = T$, $b = 0$, *and* $E = 0$.

Proof. Let us consider the boundary value problem (2.1), with $a = T$, $b = 0$ and $E_i = 0$ for $1 \leqq i \leqq N$. The necessary and sufficient condition in order for the solution of a Cauchy problem for the system (1.1) to be T-periodic, is $x(T) = x(0)$. This condition is the boundary value condition of (2.16), with $E_i = 0$, $a = T$, and $b = 0$. The result is a consequence of Theorem 2, because given the solution of (1.1) with $X(T) = X(0)$, extending T-periodically to the real line, one gets a T-periodic solution on the real line.

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Department of Applied Mathematics
Polytechnical University of Valencia
P.O. Box 22.012
Valencia
Spain

Laboratoire des Signaux et Systemes
C.N.R.S., E.S.E.
Plateau du Moulon
91.190 GIF
France