The Condensation of Continuants.

By Thomas Muir, LL.D.

1. Rather more than twenty years ago, in a note on this subject, it was shown to the Edinburgh Mathematical Society (*Proceedings*, II., pp. 16–18) that a special form of continuant, viz., one with univarial diagonals, could be expressed by means of a similar continuant of much lower order. A new mode of proving this theorem, which has lately been hit upon, has unexpectedly led to the discovery that the peculiarity in question is not confined to this special form, but characterises continuants of any form whatever.

2. Taking first a continuant of *odd* order, viz.,

$$\begin{vmatrix} a_1 & b_1 & . & . & . & . & . \\ -1 & a_2 & b_2 & . & . & . & . \\ . & -1 & a_3 & b_3 & . & . & . \\ . & . & -1 & a_4 & b_4 & . & . \\ . & . & . & -1 & a_5 & b_5 & . \\ . & . & . & . & -1 & a_6 & b_6 \\ . & . & . & . & . & -1 & a_7 \end{vmatrix}, \text{ or } K_{1,7} \text{ say,}$$

and multiplying it by $a_1a_3^2a_5^2a_7$ in the form

$$\begin{vmatrix} 1 & . & . & . & . & . & . \\ -b_1a_3 & a_1a_3 & a_1 & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & -b_3a_5 & a_3a_5 & a_3 & . & . \\ . & . & . & . & 1 & . & . \\ . & . & . & . & -b_5a_7 & a_5a_7 & a_5 \\ . & . & . & . & . & . & 1 \end{vmatrix}$$

we obtain

$$\begin{vmatrix} a_1 & . & . & . & . & . & . \\ -1 & |a_1a_2a_3| & b_2 & -b_2b_3a_5 & . & . & . \\ . & . & a_3 & . & . & . & . \\ . & -a_1 & -1 & |a_3a_4a_5| & b_4 & -b_4b_5a_7 & . \\ . & . & . & . & a_5 & . & . \\ . & . & . & -a_3 & -1 & |a_5a_6a_7| & b_6 \\ . & . & . & . & . & . & a_7 \end{vmatrix}$$

where $|a_1a_2a_3|$, $|a_3a_4a_5|$, $|a_5a_6a_7|$ are coaxial minors of $K_{1,7}$, viz.,

$$\begin{vmatrix} a_1 & b_1 & . \\ -1 & a_2 & b_2 \\ . & -1 & a_3 \end{vmatrix}, \quad \begin{vmatrix} a_3 & b_3 & . \\ -1 & a_4 & b_4 \\ . & -1 & a_5 \end{vmatrix}, \quad \begin{vmatrix} a_5 & b_5 & . \\ -1 & a_6 & b_6 \\ . & -1 & a_7 \end{vmatrix}.$$

This seven-line determinant, however, is evidently resolvable into

$$a_1a_3a_5a_7 . \begin{vmatrix} |a_1a_2a_3| & -b_2b_3a_5 & . \\ -a_1 & |a_3a_4a_5| & -b_4b_5a_7 \\ . & -a_3 & |a_5a_6a_7| \end{vmatrix};$$

consequently there results

$$K_{1,7} \cdot a_3a_5 = \begin{vmatrix} |a_1a_2a_3| & b_2b_3 & . \\ a_1a_5 & |a_3a_4a_5| & b_4b_5 \\ . & a_3a_7 & |a_5a_6a_7| \end{vmatrix} \qquad \text{(I)}.$$

When the given continuant is $K_{1,2n+1}$ its co-factor on the left is $a_3a_5 \dots a_{2n-1}$, and the continuant on the right is of the nth order.

3. For the case where the given continuant is of even order, no separate investigation is necessary, for, putting $a_7 = 1$, $b_6 = 0$ in the preceding result we have

$$K_{1,6} \cdot a_3a_5 = \begin{vmatrix} |a_1a_2a_3| & b_2b_3 & . \\ a_1a_5 & |a_3a_4a_5| & b_4b_5 \\ . & a_3 & |a_5a_6| \end{vmatrix} \qquad \text{(II)}$$

the want of symmetry in which at once suggests the alternative result

$$K_{1,6} \cdot a_2a_4 = \begin{vmatrix} |a_1a_2| & b_1b_2 & . \\ a_4 & |a_2a_3a_4| & b_3b_4 \\ . & a_6a_2 & |a_4a_5a_6| \end{vmatrix}. \qquad \text{(II}'\text{)}$$

Putting in this last each of the a's equal to x and each of the b's equal to $-bc$ we obtain

$$K_{1,6}\cdot x^2 = \begin{vmatrix} x^2-bc & b^2c^2 & . \\ x & x(x^2-2bc) & b^2c^2 \\ . & x^2 & x(x^2-2bc) \end{vmatrix}$$

and therefore in the notation of the paper of 1884

$$\begin{aligned} F(b,\, x,\, c,\, 6) &= \begin{vmatrix} x^2-bc & b^2 & . \\ c^2 & x^2-2bc & b^2 \\ . & c^2 & x^2-2bc \end{vmatrix} \\ &= F(b^2,\, x^2-2bc,\, c^2,\, 3) + bcF(b^2,\, x^2-2bc,\, c^2,\, 2). \end{aligned}$$

4. Since from (II′) we have

$$K_{1,8}\cdot a_2a_4a_6 = \begin{vmatrix} |a_1a_2| & b_1b_2 & . & . \\ a_4 & |a_2a_3a_4| & b_3b_4 & . \\ . & a_6a_2 & |a_4a_5a_6| & b_5b_6 \\ . & . & a_8a_4 & |a_6a_7a_8| \end{vmatrix}$$

it follows on putting $a_8=1$, $b_7=0$ that

$$K_{1,7}\cdot a_2a_4a_6 = \begin{vmatrix} |a_1a_2| & b_1b_2 & . & . \\ a_4 & |a_2a_3a_4| & b_3b_4 & . \\ . & a_6a_2 & |a_4a_5a_6| & b_5b_6 \\ . & . & a_4 & |a_6a_7| \end{vmatrix} \qquad \text{(III)}$$

which is an alternative to (I).

5. If now we take identities which give the equivalent of an even-ordered K, say $K_{1,8}$, and the equivalent of the differential quotient of this with respect to a, viz., the first identity of the preceding paragraph and the identity

$$K_{2,8}\cdot a_4a_6 = \begin{vmatrix} |a_2a_3a_4| & b_3b_4 & . \\ a_6a_2 & |a_4a_5a_6| & b_5b_6 \\ . & a_4 & |a_6a_7a_8| \end{vmatrix},$$

we obtain at once by division

$$a_2\cdot\left\{a_1+\frac{b_1}{a_2}+\frac{b_2}{a_3}+\dots+\frac{b_7}{a_8}\right\} = |a_1a_2| - \frac{b_1b_2a_4}{|a_2a_3a_4|} - \frac{b_3b_4a_6a_2}{|a_4a_5a_6|} - \frac{b_5b_6a_8a_4}{|a_6a_7a_8|} \qquad \text{(IV).}$$

The corresponding identity in which the number of a's is odd is most readily got as before by putting in this $a_8 = 1$, $b_7 = 0$.

Substituting x for each of the odd-numbered a's and 1 for each of the even-numbered we find

$$x + \frac{b_1}{1} + \frac{b_2}{x} + \frac{b_3}{1} + \cdots + \frac{b_7}{1} = x - \frac{b_1 b_2}{x + b_2 + b_3} - \frac{b_3 b_4}{x + b_4 + b_5} - \frac{b_5 b_6}{x + b_6 + b_7}$$

—a result said to have been first published in 1860 by Heilermann (*Zeitschr. f. Math. u. Phys.* V. pp. 262–263).

6. Again since we also have

$$\mathrm{K}_{1,8} \cdot a_3 a_5 a_7 = \begin{vmatrix} |a_1 a_2 a_3| & b_2 b_3 & \cdot & \cdot \\ a_1 a_5 & |a_3 a_4 a_5| & b_4 b_5 & \cdot \\ \cdot & a_3 a_7 & |a_5 a_6 a_7| & b_6 b_7 \\ \cdot & \cdot & a_5 & |a_7 a_8| \end{vmatrix}$$

and

$$\mathrm{K}_{3,8} \cdot a_5 a_7 = \begin{vmatrix} |a_3 a_4 a_5| & b_4 b_5 & \cdot \\ a_3 a_7 & |a_5 a_6 a_7| & b_6 b_7 \\ \cdot & a_5 & |a_7 a_8| \end{vmatrix}$$

it follows by division that

$$a_3 \frac{\mathrm{K}_{1,8}}{\mathrm{K}_{3,8}} = |a_1 a_2 a_3| - \frac{b_2 b_3 a_1 a_5}{|a_3 a_4 a_5|} - \frac{b_4 b_5 a_3 a_7}{|a_5 a_6 a_7|} - \frac{b_6 b_7 a_5}{|a_7 a_8|}, \qquad \text{(V)}$$

—that is to say, we can now express as a continued fraction not only the ratio of K to $\partial \mathrm{K}/\partial a_1$, but also the ratio of K to $\partial^2 \mathrm{K}/\partial a_1 \, \partial a_2$.

Further, since

$$\frac{\mathrm{K}_{1,8}}{\mathrm{K}_{3,8}} = \frac{\mathrm{K}_{1,8}}{\mathrm{K}_{2,8}} \cdot \frac{\mathrm{K}_{2,8}}{\mathrm{K}_{3,8}},$$

we deduce

$$a_3 \left\{ a_1 + \frac{b_1}{a_2} + \frac{b_2}{a_3} + \cdots + \frac{b_7}{a_8} \right\} \cdot \left\{ a_2 + \frac{b_2}{a_3} + \cdots + \frac{b_7}{a_8} \right\}$$

$$= |a_1 a_2 a_3| - \frac{b_2 b_3 a_1 a_5}{|a_3 a_4 a_5|} - \cdots - \frac{b_6 b_7 a_5}{|a_7 a_8|}. \qquad \text{(V')}$$

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7. From (IV) by putting each of the b's equal to 1 we have

$$a_2\left\{a_1+\frac{1}{a_2}+\frac{1}{a_3}+\cdots\right\}=(a_1a_2)-\frac{a_4}{(a_2a_3a_4)}-\frac{a_6a_2}{(a_4a_5a_6)}-\frac{a_8a_4}{(a_6a_7a_8)}-\cdots$$

where (a_1a_2), $(a_2a_3a_4)$, ... are now "simple" continuants. This has a special interest when the continued fraction on the left is the representative of a quadratic surd. Knowing, for example, that

$$\sqrt{13}=3+\frac{1}{\dot{1}}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{\dot{6}}+\ldots$$

we conclude from it that

$$\sqrt{13}=4-\frac{1}{\dot{3}}-\frac{6}{13}-\frac{1}{13}-\frac{6}{3}-\frac{1}{\dot{8}}-\ldots.$$

The convergents to the former continued fraction are

$$3,\ 4,\ 3\tfrac{1}{2},\ 3\tfrac{2}{3},\ 3\tfrac{3}{5},\ 3\tfrac{20}{33},\ 3\tfrac{23}{38},\ 3\tfrac{43}{71},\ldots$$

and to the latter

$$4,\qquad 3\tfrac{2}{3},\qquad 3\tfrac{20}{33},\qquad 3\tfrac{43}{71},\ldots$$

the rth convergent in the second case being the same as the $2r$th convergent in the first.

The process of condensation may, of course, be continued indefinitely. In the case of $\sqrt{13}$ the next result in order is

$$\sqrt{13}=\frac{11}{3}-\frac{26}{\dot{426}}-\frac{144}{251}-\frac{1014}{226}-\frac{144}{476}-\frac{234}{\dot{51}}-\ldots$$

where the 1st, 2nd, 3rd, .. convergents are the 4th, 8th, 12th,... of the original continued fraction.

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