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Automatic Continuity of Homomorphisms in Non-associative Banach Algebras

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Abstract. We introduce the concept of a rare element in a non-associative normed algebra and show that the existence of such an element is the only obstruction to continuity of a surjective homomorphism from a non-associative Banach algebra to a unital normed algebra with simple completion. Unital associative algebras do not admit any rare elements, and hence automatic continuity holds.

1 Introduction

All vector spaces in this paper are over the complex field, and all algebras are not assumed to be associative. To highlight non-associativity and by a slight abuse of language, the term *non-associative* (*Banach*) *algebra* is used to mean a (Banach) algebra that may or may not be associative.

In what follows, an *algebra* $\mathcal{A}$ is a complex vector space equipped with a bilinear product, which is usually written by juxtaposition. It is called *associative* if the product is associative. An *algebra norm* on $\mathcal{A}$ is a norm $\|\cdot\|$ satisfying $\|ab\| \leq \|a\|\|b\|$ for $a, b \in \mathcal{A}$. A *normed algebra* $\mathcal{A}$ is an algebra equipped with an algebra norm, and if the norm is complete, we call $\mathcal{A}$ a *Banach algebra*.

Important natural examples of non-associative Banach algebras include the evolution algebras in genetics [17] and Banach Lie algebras in geometry [1, 18], where the complete holomorphic vector fields on a complex Banach manifold form a Banach Lie algebra.

In the study of associative Banach algebras, the theory of automatic continuity of homomorphisms and uniqueness of complete algebra norm is fundamental and has been well-established since the seminal works of Rickart [12] and Johnson [5]. We refer to [2, 11] for details.

For non-associative algebras, the question of automatic continuity has been investigated by several authors; see, for example, [8, 9, 13–15, 19, 20], where the background and connections to spectral theory have been explained succinctly. In this paper, we examine the issue of spectral theory in the context of automatic continuity for non-associative algebras. Indeed, we introduce in Definition 3.1 the concept of a *rare* linear operator on a normed vector space in terms of a special spectral property of the operator. A *rare element* in a normed algebra is an element x for which the left multiplication L_x and the right multiplication R_x are rare operators.

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We show that the existence of a rare element is the only obstruction to automatic continuity. More precisely, we show in Theorem 4.2 that a surjective homomorphism $\theta\colon \mathcal{A} \to \mathcal{B}$ from a non-associative Banach algebra $\mathcal{A}$ to a unital normed algebra $\mathcal{B}$ with a simple completion is continuous if and only if $\mathcal{B}$ does not have any rare elements. Unital associative algebras do not have rare elements and hence automatic continuity follows. This reveals a distinguishing feature between associativity and non-associativity, namely, the existence of rare elements, and also suggests that the study of rare operators should lead to interesting and fruitful consequences. One consequence is the dichotomy in Theorem 4.5 that, given a unital normed algebra $\mathcal{B}$ with simple completion, either $\mathcal{B}$ admits a dominating complete algebra norm, in which case all homomorphisms from a non-associative Banach algebra $\mathcal{A}$ onto $\mathcal{B}$ are continuous, or $\mathcal{B}$ does not admit a dominating complete algebra norm, in which case all homomorphisms from $\mathcal{A}$ onto $\mathcal{B}$ are discontinuous. If $\mathcal{B}$ is unital, power associative, and has a simple completion, then it is shown in Corollary 4.12 that any complete algebra norm on $\mathcal{B}$ is dominating. Another notable result is that, given a homomorphism θ from a non-associative Banach algebra $\mathcal{A}$ onto a simple normed algebra $\mathcal{B}$, either the kernel $\ker\theta$ is closed or the multiplication operators on $\mathcal{B}$ form a nowhere dense set in the space $L(\mathcal{B})$ of bounded operators on $\mathcal{B}$. It follows that every surjective homomorphism θ from $\mathcal{A}$ onto a power-associative normed algebra $\mathcal{B}$ with simple completion is continuous whenever the multiplication algebra $M(\mathcal{B})$ is of second category in $L(\mathcal{B})$.

2 Spectra for Incomplete Normed Spaces

We begin with some notations and basic properties of various spectra of a linear operator on a normed space that need not be complete. Let X be a normed vector space with dual X^*. For each $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball centred at x of radius r. The norm closure of a set $E \subset X$ is denoted by $\overline{E}$. Let $L(X)$ denote the normed algebra of all bounded linear operators from X to itself. For each operator $T \in L(X)$, we denote the spectrum of T by

$$\sigma(T) = \left\{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in } L(X) \right\}.$$

We have $\sigma(T) = \sigma_s(T) \cup \sigma_p(T) \cup \sigma_a(T) = \sigma_s(T) \cup \sigma_a(T)$, where $\sigma_s(T)$ is the *surjective spectrum* of T defined by

$$\sigma_s(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not surjective}\}.$$

The *point spectrum*

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not injective}\}$$

is the set of eigenvalues of T, contained in the *approximate spectrum* of T:

$$\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not bounded below}\}.$$

In the sequel, we will write $\sigma^X(T)$ for $\sigma(T)$ if it is necessary to highlight the underlying space X. The same convention applies to all other spectra.

If X is a Banach space, then the open mapping theorem implies that $\sigma(T) = \sigma_s(T) \cup \sigma_p(T)$, which is a nonempty compact set in $\mathbb{C}$. We also have (*cf.* [7])

$$\sigma_s(T) = \sigma_a(T^*) \quad \text{and} \quad \sigma_a(T) = \sigma_s(T^*). \tag{2.1}$$

However, if X is incomplete, the spectrum $\sigma^X(T)$ need not be compact, as shown in Example 3.8.

Given a normed space X and $T \in L(X)$, we will always denote by $\widehat{X}$ the completion of X and by $\widehat{T} \in L(\widehat{X})$ the unique continuous extension of T to $\widehat{X}$. We note that the completion $\widehat{X}$ of an incomplete normed space X can be reflexive. For instance, $L_p(\mathbb{R}^n)$ is the completion of the space $C_c(\mathbb{R}^n)$ of continuous functions with compact support, equipped with the L_p-norm, for $1 < p < \infty$.

If X is an incomplete normed space, then we have $\sigma^{\widehat{X}}(\widehat{T}) \subset \sigma^X(T)$ and $\sigma_p^X(T) \subset \sigma_p^{\widehat{X}}(\widehat{T})$.

Lemma 2.1 *Let X be a normed space and $T \in L(X)$. Then we have $\sigma_a^X(T) = \sigma_a^{\widehat{X}}(\widehat{T})$.*

Proof If $F \in L(X)$ and the completion $\widehat{F}$ is bounded below, then evidently F is also bounded below. Conversely, let $F \in L(X)$ satisfy $\|F(x)\| \geq k\|x\|$ for all $x \in X$. If $\widehat{x} = \lim_n x_n$, then

$$\|\widehat{F}(\widehat{x})\| = \lim_n \|F(x_n)\| \geq \lim_n k\|x_n\| = k\|\widehat{x}\|.$$

Hence $\widehat{F}$ is bounded below. ∎

Lemma 2.2 *Let X be a normed space and $T \in L(X)$. Then we have*

$$\sigma_s^{\widehat{X}}(\widehat{T}) \backslash \sigma_a^{\widehat{X}}(\widehat{T}) \subset \sigma_s^X(T).$$

Proof This follows from $\sigma^{\widehat{X}}(\widehat{T}) \subset \sigma^X(T) = \sigma_s^X(T) \cup \sigma_a^X(T)$ and Lemma 2.1. ∎

Example 2.3 Let $B(H)$ be the von Neumann algebra of bounded linear operators on an infinite dimensional Hilbert space H. Pick a projection $p \in B(H)$ that is equivalent, but not equal, to the identity $\mathbf{1} \in B(H)$. We have $\mathbf{1} = u^*u$ and $p = uu^*$ for some partial isometry $u \in B(H)$. Let $T\colon B(H) \to B(H)$ be the linear map $T(x) = ux$. Then $\mathbf{1} \notin T(B(H))$, and T is a non-surjective isometry. Hence $0 \in \sigma_s(T) \backslash \sigma_a(T)$.

For completeness, we include a short proof of the following fact.

Lemma 2.4 *Let X be a Banach space and $T \in L(X)$. Then the surjective spectrum $\sigma_s(T)$ is nonempty and compact.*

Proof First, $\sigma_s(T)$ is nonempty because it contains the boundary of the spectrum $\sigma(T)$ (cf. [7]).

For compactness, it suffices to show that $\sigma_s(T)$ is closed in $\mathbb{C}$. Indeed, given $\lambda \in \mathbb{C} \backslash \sigma_s(T)$, the operator $T - \lambda\colon X \to X$ is surjective. Since the surjective operators in $L(X)$ form an open set [3], there exists $\varepsilon > 0$ such that each $S \in L(X)$ satisfying $\|S - (T - \lambda)\| < \varepsilon$ is surjective. It follows that each $\mu \in \mathbb{C}$ satisfying $|\mu - \lambda| < \varepsilon$ is outside the surjective spectrum $\sigma_s(T)$. ∎

In contrast to Lemma 2.4, we shall see that the surjective spectrum $\sigma_s^X(T)$ could be a large open set if X is not complete.

3 Rare Operators

We introduce rare linear operators in this section and discuss some of their properties, which will be used to derive several results on automatic continuity. The subject of rare operators itself may also be of independent interest.

Definition 3.1 Let X be a normed vector space with completion $\widehat{X}$. A linear operator $T \in L(X)$ is called *rare* if

$$\sigma_s^X(T) \cap \sigma_s^{\widehat{X}}(\widehat{T}) = \varnothing.$$

It is readily seen that if T is a rare operator, then so are λT and $\lambda \mathbf{1} \pm T$ for $\lambda \neq 0$, where $\mathbf{1}$ denotes henceforth the identity operator on a vector space. Evidently, if X is complete, then there is no rare operator on X by Lemma 2.4.

Given a normed space X and $T \in L(X)$, we recall that the *compression spectrum* of T is defined by

$$\sigma_c(T) := \{\lambda \in \mathbb{C} : (T - \lambda)(X) \text{ is not dense in } X\}.$$

If X is a Banach space, then $\sigma^X(T) = \sigma_c^X(T) \cup \sigma_a^X(T)$. For any normed space X, we have $\sigma_c^{\widehat{X}}(\widehat{T}) = \sigma_c^X(T)$.

Lemma 3.2 *Let X be a normed space and let $T \in L(X)$ be a rare operator. Then we have $\sigma_c^{\widehat{X}}(\widehat{T}) = \varnothing$ and $\sigma^X(T) = \sigma_s^X(T) \cup \sigma^{\widehat{X}}(\widehat{T})$.*

Proof Indeed, $\sigma_c^{\widehat{X}}(\widehat{T}) = \sigma_c^X(T) \subset \sigma_s^X(T) \cap \sigma_s^{\widehat{X}}(\widehat{T}) = \varnothing$, and the second assertion follows from Lemma 2.1. ■

We note that an operator $T \in L(X)$ has dense range if and only if its dual $T^* \in L(X^*)$ is injective. It follows that $\sigma_p^{X^*}(T^*) = \sigma_c^X(T)$. Therefore, $\sigma_p^{X^*}(T^*) = \varnothing$ if T is rare. While Lemma 3.2 shows how the spectra of T and $\widehat{T}$ differ for a rare operator T, the spectra of the dual operators T^* and $\widehat{T}^*$ are identical: $\sigma^{\widehat{X}}(\widehat{T}) = \sigma^{\widehat{X}^*}(\widehat{T}^*) = \sigma^{X^*}(T^*)$, since X^* is isometric to $\widehat{X}^*$.

Given a compact operator T on an infinite dimensional normed space X, it has been shown in [16] that T cannot be surjective, that is, $0 \in \sigma_s^X(T)$. Since the dual T^* is also a compact operator on the Banach space X^*, we have $\sigma_p^{X^*}(T^*) \neq \varnothing$ or $\sigma^{X^*}(T^*) = \{0\}$. In the latter case, we have $\sigma^{\widehat{X}}(\widehat{T}) = \sigma^{X^*}(T^*) = \{0\}$ and hence $0 \in \sigma_s^X(T) \cap \sigma_s^{\widehat{X}}(\widehat{T})$. Therefore compact operators are not rare. In fact, neither are the weakly compact operators, which will be shown later.

We recall that a *rare set* or a set of *first category* in a metric space is a countable union of nowhere dense sets. We will show that the rare operators in $L(X)$ form a rare set, which explains the terminology for a rare operator. We first give some criteria for rare operators and an appropriate converse for Lemma 3.2.

Proposition 3.3 *Let X be a normed space and let $T \in L(X)$. The following conditions are equivalent:*

(i) *T is a rare operator;*

(ii) $\sigma_c^X(T) = \varnothing$ *and* $\sigma_s^X(T) \cap \sigma_s^{\widehat{X}}(\widehat{T}) \cap \sigma_a^X(T) = \varnothing$;
(iii) $\sigma^X(T) \backslash \sigma_s^X(T) = \sigma_s^{\widehat{X}}(\widehat{T}) \cup (\sigma_a^X(T) \backslash \sigma_s^X(T))$.

Proof (i) $\Rightarrow$ (ii). This is immediate from Lemma 3.2.

(ii) $\Rightarrow$ (iii). Since $\sigma^{\widehat{X}}(\widehat{T}) = \sigma_c^{\widehat{X}}(\widehat{T}) \cup \sigma_a^{\widehat{X}}(\widehat{T})$, condition (ii) implies $\sigma_s^{\widehat{X}}(\widehat{T}) \subset \sigma_a^{\widehat{X}}(\widehat{T})$ and $\sigma_s^{\widehat{X}}(\widehat{T}) \subset \sigma^X(T) \backslash \sigma_s^X(T)$, which gives (iii) readily.

Finally, it is evident that (iii) implies $\sigma_s^X(T) \cap \sigma_s^{\widehat{X}}(\widehat{T}) = \varnothing$, which completes the proof. ■

If T is a rare operator and $\sigma_s(T) \nsubseteq \sigma_a(T)$, then Lemma 3.2 shows that $\sigma(T) \neq \sigma(T^*)$. However, for a weakly compact operator T on a normed space X, we always have $\sigma(T) = \sigma(T^*)$. Indeed, by [10, Corollary 3.3], we have $\sigma_s(T) \cup \sigma_p(T) = \sigma(T^*)$. It follows that $\sigma^{\widehat{X}}(\widehat{T}) = \sigma_s(T) \cup \sigma_p(T)$ and $\sigma(T) = \sigma_s(T) \cup \sigma_p(T)$ by Lemma 3.2. In fact, T is not a rare operator as shown below.

Proposition 3.4 *A weakly compact operator T on a normed space X is not rare.*

Proof We may assume that X is incomplete. Let $T \in L(X)$ be weakly compact. We first observe, by (2.1), that

$$\sigma_s^{\widehat{X}}(\widehat{T}) = \sigma_a^{X^*}(T^*) = \sigma_s^{X^{**}}(T^{**}).$$

By [10, Proposition 2.5], we have

$$\sigma_s^X(T) \cup \{0\} = \sigma_s^{X^{**}}(T^{**}) \cup \{0\} = \sigma_s^{\widehat{X}}(\widehat{T}) \cup \{0\}.$$

If $0 \in \sigma_s^X(T)$, then $\sigma_s^X(T) \cap \sigma_s^{\widehat{X}}(\widehat{T}) \neq \varnothing$ as the surjective spectrum $\sigma_s^{\widehat{X}}(\widehat{T})$ is never empty. If $0 \notin \sigma_s^X(T)$, then T is surjective, and by [10, Theorem 4.3], $\sigma^X(T)$ contains a disc of eigenvalues, centred at 0. Since $\sigma_s^{\widehat{X}}(\widehat{T})$ contains the boundary of $\sigma^{\widehat{X}}(\widehat{T}) = \sigma^X(T)$, there must be a nonzero element in $\sigma_s^{\widehat{X}}(\widehat{T})$, and we also have $\sigma_s^X(T) \cap \sigma_s^{\widehat{X}}(\widehat{T}) \neq \varnothing$. ■

It is well known that finite dimensional subspaces are complemented in any normed space X. Although a finite rank projection on a dual space X^* need not be weak* continuous, we have nevertheless the following fact.

Lemma 3.5 *Let X be a normed space and let E be a one dimensional subspace of X^*. Then there is a weak* continuous contractive projection $P\colon X^* \to E$.*

Proof Let $E = \mathbb{C}g$ for some $g \in X^*$. Pick $a \in X$ such that $g(a) = 1$. Then the contractive projection $P\colon X^* \to E$ defined by

$$P(f) = f(a)\,g \qquad (f \in X^*)$$

is weak* continuous. ■

Given a normed space X, the set

$$\mathcal{U}(X) = \{F \in L(X^*) : F \text{ has an isolated eigenvalue with finite dimensional eigenspace}\}$$

is an open set in $L(X^*)$ (*cf.* [6]), and it has been shown in [4, Theorem 1.5] that $\mathcal{U}(X)$ is actually dense in $L(X^*)$.

Theorem 3.6 *For any normed space X, the set*

$$\mathcal{S} = \{T \in L(X) : T \text{ is rare}\}$$

of rare operators is nowhere dense in $L(X)$.

Proof We need to show that the normed closure $\overline{\mathcal{S}}$ of $\mathcal{S}$ has empty interior. Suppose otherwise. Then there is an operator $T \in \overline{\mathcal{S}}$ and $\varepsilon > 0$ such that the ε-open ball $B(T, \varepsilon)$ centred at T is contained in $\overline{\mathcal{S}}$.

Let $\mathcal{L} = \{F \in L(X^*) : \sigma_p(F) \neq \varnothing\}$. Then $\mathcal{L}$ contains the open dense set $\mathcal{U}(X)$ defined above. By [4], there exists a finite rank operator $S \in L(X^*)$ with $\|S\| < \varepsilon$ such that $T^* + S \in \mathcal{U}(X)$, where S is constructed as follows. As in [4], there is a boundary point α of the spectrum $\sigma(T^*)$ and a vector $g \in X^*$ such that $\|(T^* - \alpha)(g)\| < \varepsilon/2$. There is a weak* continuous contractive projection $P\colon X^* \to \mathbb{C}g$ by Lemma 3.5. Let $S = UP$, where the operator $U\colon \mathbb{C}g \to \mathbb{C}v$ satisfies $U(g) = v$ and $v = \mu g - T^*g$ for some μ in the resolvent set $\rho(T^*)$. Then S is as required.

Since $S\colon X^* \to X^*$ is weak* continuous, it has a predual $S_1 \colon X \to X$ with $\|S_1\| = \|S\| < \varepsilon$. Hence, $T + S_1 \in B(T, \varepsilon) \subset \overline{\mathcal{S}}$ implies $T^* + S = (T + S_1)^* \in L(X^*)\backslash\mathcal{U}(X)$, since $F \in \mathcal{S}$ implies $F^* \in L(X^*)\backslash\mathcal{L}$ by the remark after Lemma 3.2. This contradicts $T^* + S \in \mathcal{U}(X)$, which completes the proof. ∎

Although the set of rare operators in $L(X)$ is nowhere dense, it is nevertheless nonempty, as shown below.

Example 3.7 [1] Let $X = \mathbb{C}[x]$ be the space of polynomial functions $p(x)$ on $[0, 1]$, equipped with the norm

$$|p| = \sum_{k\geq 0} \|p^{(k)}\| \qquad (p \in X)$$

where $p^{(0)} = p$ and $\|p^{(k)}\|$ denotes the supremum norm of the k-th derivative of p. Let $T\colon X \to X$ be the differential operator $T = \frac{d}{dx}$. Then T is rare since $\sigma_s(T) = \varnothing$.

We now give an example of a rare operator with nonempty surjective spectrum.

Example 3.8 [2] Let $1 \leq p < \infty$ and let $X = c_c(\mathbb{Z})$ be the space of complex sequences $x = (x_n)_{n\in\mathbb{Z}}$, with finite support, equipped with the ℓ_p-norm $\|x\|_p^p = \sum_n |x_n|^p$. The completion of X is the Banach space $\widehat{X} = \ell_p(\mathbb{Z})$.

[1]We thank M. Mbekhta for this example.
[2]This example originates from a discussion with L. L. Stacho, for which we are thankful.

Let $\{e_n : n \in \mathbb{Z}\}$ be the canonical basis in $\ell_p(\mathbb{Z})$, where

$$e_n(m) = \delta_{mn} \qquad (m \in \mathbb{Z}).$$

Let $(w_n)_{n\in\mathbb{Z}}$ be a weight sequence given by $w_n = \frac{1}{2^{|n|}}$. Let $T\colon X \to X$ be the weighted shift

$$Te_n = w_n e_{n+1} \quad (n \in \mathbb{Z}).$$

Although T is invertible, its inverse is unbounded. Hence $0 \in \sigma_a^X(T)\backslash\sigma_s^X(T)$. Since

$$\widehat{T}^N(x_n) = \widehat{T}^N\Big(\sum_n x_n e_n\Big) = \sum_n x_n(\widehat{T}^N e_n) = \sum_n x_n w_n w_{n+1} \cdots w_{n+N-1} e_{n+N},$$

we have

$$\begin{aligned}\|\widehat{T}^N(x_n)\|_p^p &= \sum_n |x_n w_n w_{n+1} \cdots w_{n+N-1}|^p \le (w_{-1} w_0 w_1 \cdots w_{N-2})^p \sum_n |x_n|^p, \\ &= \Big(\frac{1}{2}\frac{1}{2}\frac{1}{2^2}\cdots\frac{1}{2^{N-2}}\Big)^p \sum_n |x_n|^p < \Big(\frac{1}{2^{(N-1)(N-2)/2}}\Big)^p \sum_n |x_n|^p\end{aligned}$$

and hence

$$\|\widehat{T}^N\|_p^{1/N} < \frac{1}{2^{(N-1)(N-2)/2N}} \longrightarrow 0$$

as $N \to \infty$. Therefore we have $\sigma^{\widehat{X}}(\widehat{T}) = \{0\}$.

On the other hand, $(T - \lambda I)X \neq X$, for every $\lambda \neq 0$, because the support of $(T - \lambda I)x$ contains at least two points, for any sequence $x \in X\backslash\{0\}$, and hence the equations $(T - \lambda I)x = e_n$ $(n \in Z)$ cannot be solved. Indeed, if $x_n \neq 0$, then the sequence $(T - \lambda I)x$ is nonzero at n and $n + 1$. It follows that

$$\sigma^X(T) = \mathbb{C}, \quad \sigma_s^X(T) = \mathbb{C} \setminus \{0\}, \quad \text{and} \quad \sigma_s^X(T) \cap \sigma^{\widehat{X}}(\widehat{T}) = \varnothing,$$

and T is a rare operator.

Given a normed algebra $\mathcal{A}$ and an element $a \in \mathcal{A}$, the question of rareness of a left multiplication operator $L_a\colon x \in \mathcal{A} \mapsto ax \in \mathcal{A}$ or right multiplication $R_a\colon x \in \mathcal{A} \mapsto xa \in \mathcal{A}$ is closely related to our problem of automatic continuity of homomorphisms in the setting of non-associative normed algebras.

Let $M(\mathcal{A})$ be the *multiplication algebra* $M(\mathcal{A})$ of $\mathcal{A}$, that is, $M(\mathcal{A})$ is the subalgebra of $L(\mathcal{A})$ generated by the left and right multiplication operators. We prove below that the left and right multiplication operators on a unital associative algebra $\mathcal{A}$ are not rare.

Proposition 3.9 *Let $\mathcal{A}$ be an associative normed algebra with identity* $\mathbf{1}$*. Then for each $a \in \mathcal{A}$, the left multiplication L_a and the right multiplication R_a are not rare operators on $\mathcal{A}$.*

If, moreover, $\mathcal{A}$ is commutative, then all operators T in $M(\mathcal{A})$ are not rare.

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Proof Let $T = L_a$ or $T = R_a$. By associativity of $\mathcal{A}$, it is readily seen that T is surjective if and only if $\mathbf{1} \in T(\mathcal{A})$. Therefore, if $\widehat{\mathcal{A}}$ is the completion of $\mathcal{A}$, then $\widehat{T}$ is not surjective if and only if $\widehat{T}(\widehat{\mathcal{A}}) \cap \operatorname{inv}(\widehat{\mathcal{A}}) = \varnothing$ where $\operatorname{inv}(\widehat{\mathcal{A}})$ denotes the open set of invertible elements in $\widehat{\mathcal{A}}$. In particular, $\widehat{T}(\widehat{\mathcal{A}})$ is not dense in $\widehat{\mathcal{A}}$ if $\widehat{T}$ is not surjective. It follows that $\sigma_c^{\widehat{\mathcal{A}}}(\widehat{T}) = \sigma_s^{\widehat{\mathcal{A}}}(\widehat{T}) \neq \varnothing$ and hence T is not rare by Lemma 3.2.

If $\mathcal{A}$ is commutative, then the above arguments apply to all multiplication operators $T \in M(\mathcal{A})$. ∎

Example 3.10 Let T be the operator in Example 3.8. Although T itself is a rare operator on X, the left multiplication $L_T \colon L(X) \to L(X)$ is not a rare operator on the associative algebra $L(X)$.

In contrast to the associative case, multiplication operators can be rare. We provide in the following example a device to construct multiplication operators on non-associative algebras with prescribed properties.

Example 3.11 Let X be a normed space and $T \in L(X)$ with $\|T\| = 1$. Pick $x_0 \in X$ and $f \in X^*$ such that $f(x_0) = 1 = \|f\|$. Define a non-associative product $\circ$ on X by

$$x \circ y = f(x)T(y) \qquad (x, y \in X).$$

Then on the normed algebra $(X, \circ)$, the left multiplication $L_{x_0} \colon X \to X$ is just the operator T. Hence L_{x_0} is rare if T is rare. Also, L_{x_0} is compact if T is compact.

By a *character* of a normed algebra $\mathcal{A}$, we mean a nonzero homomorphism $\psi \colon \mathcal{A} \to \mathbb{C}$.

Example 3.12 If a normed algebra $\mathcal{A}$ admits a character, then the left and right multiplication operators on $\mathcal{A}$ are not rare. Indeed, let ψ be a character of $\mathcal{A}$ and let $a \in \mathcal{A}$. Then we have

$$(L_a^*)(\psi)(x) = \psi(ax) = \psi(a)\psi(x) \qquad (x \in \mathcal{A})$$

and hence $\psi(a) \in \sigma_p(L_a^*)$, which is nonempty. Therefore L_a cannot be rare.

Definition 3.13 Let $\mathcal{A}$ be a normed algebra. An element $a \in \mathcal{A}$ is called *rare* if the left and right multiplication operators L_a and R_a are rare operators on $\mathcal{A}$. We denote by $R(\mathcal{A})$ the set of rare elements in $\mathcal{A}$ and call it the *rarity* of $\mathcal{A}$.

Let $\mathcal{A}$ be a normed algebra. We recall that the set

$$\mathcal{U}(\mathcal{A}) = \{F \in L(\mathcal{A}^*) : F \text{ has an isolated eigenvalue with finite dimensional eigenspace}\}$$

is open and dense in $L(\mathcal{A}^*)$.

Lemma 3.14 *Let $\mathcal{A}$ be a normed algebra. If there exists $a \in \mathcal{A}$ such that $L_a^* \in \mathcal{U}(\mathcal{A})$ or $R_a^* \in \mathcal{U}(\mathcal{A})$, then the rarity $R(\mathcal{A})$ is not dense in $\mathcal{A}$.*

Proof We need only consider the case $L_a^* \in \mathcal{U}(\mathcal{A})$, the other case can be proved analogously. Since $\sigma_p(L_a^*) \neq \varnothing$, we have $a \notin R(\mathcal{A})$. Since $\mathcal{U}(\mathcal{A})$ is open, we have $B(L_a^*, r) \subset \mathcal{U}(\mathcal{A})$ for some $r > 0$. Hence for each $b \in B(a, r)$, we have $L_b^* \in B(L_a^*, r)$ and $b \notin R(\mathcal{A})$. Therefore $R(\mathcal{A})$ is not dense in $\mathcal{A}$. ∎

By Example 3.11, a multiplication operator on a normed algebra can be compact.

Proposition 3.15 *Let $\mathcal{A}$ be a normed algebra. If there exists $a \in \mathcal{A}$ such that L_a or R_a is of finite rank, or a compact operator with nonzero spectrum, then $R(\mathcal{A})$ is not dense in $\mathcal{A}$.*

Proof Let $T = L_a$ or R_a. Then $T^* \in \mathcal{U}(\mathcal{A})$ if it is of finite rank. Let T be compact with nonzero spectrum. As noted before, we have $\sigma(T) = \sigma(T^*)$, which, by assumption, contains some $\lambda \neq 0$. Hence the compact operator T^* has a nonzero eigenvalue and therefore $T^* \in \mathcal{U}(\mathcal{A})$. By Lemma 3.14, $R(\mathcal{A})$ is not dense in $\mathcal{A}$. ∎

We have seen that the rarity $R(\mathcal{A})$ is empty for a unital associative algebra $\mathcal{A}$, but it will be seen in Example 4.11 that a non-associative unital normed algebra can contain many rare elements. In fact, obstruction to automatic continuity in non-associative algebras is the existence of rare elements, which will be shown in the next section.

4 Continuity of Homomorphisms

We shall now apply the notion of a rare element to the study of automatic continuity of homomorphisms in non-associative algebras. We begin with a lemma and reiterate that a normed algebra need not be associative.

Lemma 4.1 *Let $\theta\colon \mathcal{A} \to \mathcal{B}$ be a continuous homomorphism from a non-associative Banach algebra $\mathcal{A}$ onto a normed algebra $\mathcal{B}$. Then $R(\mathcal{B}) = \varnothing$.*

Proof Since θ is continuous, its kernel $\ker\theta$ is closed in $\mathcal{A}$ and the quotient $\mathcal{A}/\ker\theta$ is a (non-associative) Banach algebra. The induced map

$$\tau\colon \mathcal{A}/\ker\theta \to \mathcal{B}$$

given by

$$\tau(x + \ker\theta) = \theta(x) \qquad (x \in \mathcal{A})$$

is a continuous bijection.

We show that $R(\mathcal{B}) = \varnothing$. Let $x \in \mathcal{A}$. The left multiplication $L_{\theta(x)}$ on $\mathcal{B}$ is related to the left multiplication $L_{x+\ker\theta}$ on the quotient $\mathcal{A}/\ker\theta$ by

$$L_{\theta(x)} - \lambda\mathbf{1} = \tau^{-1}(L_{x+\ker\theta} - \lambda\mathbf{1})\tau \qquad (\lambda \in \mathbb{C}),$$

where $\mathbf{1}$ denotes the identity operator. Hence the two left multiplication operators have the same surjective spectra,

$$\sigma_s^{\mathcal{B}}(L_{\theta(x)}) = \sigma_s^{\mathcal{A}/\ker\theta}(L_{x+\ker\theta}),$$

which is nonempty since $\mathcal{A}/\ker\theta$ is complete. The same can be said about the right multiplication operators:

$$\sigma_s^{\mathcal{B}}(R_{\theta(x)}) = \sigma_s^{\mathcal{A}/\ker\theta}(R_{x+\ker\theta}) \neq \varnothing.$$

For the dual operators, we have, for $\lambda \in \mathbb{C}$,

$$\begin{aligned} L^*_{\theta(x)} - \lambda\mathbf{1} &= \tau^{*^{-1}}(L^*_{x+\ker\theta} - \lambda\mathbf{1})\tau^*, \\ R^*_{\theta(x)} - \lambda\mathbf{1} &= \tau^{*^{-1}}(R^*_{x+\ker\theta} - \lambda\mathbf{1})\tau^*, \end{aligned} \tag{4.1}$$

where $\tau^* \colon \mathcal{B}^* \to (\mathcal{A}/\ker\theta)^*$ is a linear homemomorphism by the open mapping theorem. By (4.1), we have

$$\sigma_s(L_{\theta(x)}) = \sigma_s(L_{x+\ker\theta}) = \sigma_a(L^*_{x+\ker\theta}) = \sigma_a(L^*_{\theta(x)}) = \sigma_s(\widehat{L}_{\theta(x)}).$$

Hence $L_{\theta(x)}$ cannot be a rare operator. Likewise, the right multiplication operator $R_{\theta(x)}$ cannot be rare. Hence $\theta(x)$ is not rare in $\mathcal{B}$. ■

A normed algebra $\mathcal{B}$ is called *simple* if its product is nonzero and there is no nonzero proper ideal in $\mathcal{B}$. A normed algebra $\mathcal{B}$ is said to have a *simple completion* if its completion $\widehat{\mathcal{B}}$ is a simple algebra

Theorem 4.2 *Let $\theta \colon \mathcal{A} \to \mathcal{B}$ be a surjective homomorphism from a non-associative Banach algebra $\mathcal{A}$ to a unital normed algebra $\mathcal{B}$ with a simple completion. Then θ is continuous if and only if $R(\mathcal{B}) = \varnothing$.*

Proof We have already shown in Lemma 4.1 that continuity of θ implies that $\mathcal{B}$ has empty rarity $R(\mathcal{B})$.

Conversely, let $R(\mathcal{B}) = \varnothing$. To show continuity of θ, it suffices to show, by the closed graph theorem, that the separating subspace

$$S(\theta) = \{b \in \widehat{\mathcal{B}} : b = \lim_n \theta(a_n) \text{ for some sequence } a_n \to 0 \text{ in } \mathcal{A}\}$$

reduces to $\{0\}$, where $\widehat{\mathcal{B}}$ is the completion of $\mathcal{B}$. Since $\widehat{\mathcal{B}}$ is simple and $S(\theta)$ is an ideal in $\widehat{\mathcal{B}}$, we only need to show that the identity $e \in \mathcal{B}$ is not in $S(\theta)$.

Indeed, for each $a \in \mathcal{A}$, either the left multiplication $L_{\theta(a)}$ or the right multiplication $R_{\theta(a)}$ is not a rare operator on $\mathcal{B}$, by assumption. We have $\sigma_s^{\mathcal{B}}(L_{\theta(a)}) \subset \sigma_s^{\mathcal{A}}(L_a)$ and $\sigma_s^{\mathcal{B}}(R_{\theta(a)}) \subset \sigma_s^{\mathcal{A}}(R_a)$. Say $L_{\theta(a)}$ is not rare, then neither is $\mathbf{1} - L_{\theta(a)}$, and there exists

$$\lambda \in \sigma_s^{\mathcal{B}}(\mathbf{1} - L_{\theta(a)}) \cap \sigma_s^{\widehat{\mathcal{B}}}(\mathbf{1} - \widehat{L}_{\theta(a)}).$$

Since $1 - \lambda \in \sigma_s^{\mathcal{B}}(L_{\theta(a)}) \subset \sigma^{\mathcal{A}}(L_a)$, we have $|1-\lambda| \le \|a\|$. It follows that

$$1 \le |1-\lambda| + |\lambda| \le \|a\| + \|\mathbf{1} - \widehat{L}_{\theta(a)}\| = \|a\| + \|e - \theta(a)\| \qquad (a \in \mathcal{A}),$$

which implies that $e \notin S(\theta)$. If $R_{\theta(a)}$ is not rare, the same argument also shows that $e \notin S(\theta)$ and hence θ is continuous. ■

Remark 4.3 It can be seen from the proof of the above theorem that it is still valid if we weaken the assumption of an identity e in $\mathcal{B}$ to the requirement that $\widehat{\mathcal{B}}$ contain an identity e.

Since unital associative normed algebras do not have rare element, by Proposition 3.9, we have the following result of automatic continuity.

Corollary 4.4 *Let $\theta\colon \mathcal{A} \to \mathcal{B}$ be a surjective homomorphism from a non-associative Banach algebra $\mathcal{A}$ to a unital associative normed algebra $\mathcal{B}$ with simple completion. Then θ is continuous.*

We say that a normed algebra $(\mathcal{B}, \|\cdot\|)$ admits a *dominating complete norm* if there is a complete algebra norm $|\cdot|$ on $\mathcal{B}$ satisfying $|\cdot| \geq \alpha\|\cdot\|$ for some $\alpha > 0$. This is equivalent to the existence of a continuous surjective homomorphism $\theta\colon \mathcal{A} \to \mathcal{B}$ on a non-associative Banach algebra $\mathcal{A}$. Given a normed algebra $\mathcal{B}$, we now have the following dichotomy.

Theorem 4.5 *Let $\mathcal{B}$ be a unital normed algebra with a simple completion. Then*

(i) *either $\mathcal{B}$ admits a dominating complete norm, in which case all homomorphisms from a non-associative Banach algebra onto $\mathcal{B}$ are continuous;*

(ii) *or $\mathcal{B}$ does not admit a dominating complete norm, in which case all homomorphisms from a non-associative Banach algebra onto $\mathcal{B}$ are discontinuous.*

Proof (i) Let $\mathcal{B}$ have a complete dominating norm $|\cdot|$. Then the identity map $\mathbf{1}\colon (\mathcal{B}, |\cdot|) \to (\mathcal{B}, \|\cdot\|)$ is a continuous surjective homomorphism, and it follows from Theorem 4.2 that $R((\mathcal{B}, \|\cdot\|)) = \varnothing$, and that any other surjective homomorphism $\theta\colon \mathcal{A} \to (\mathcal{B}, \|\cdot\|)$ on a non-associative Banach algebra $\mathcal{A}$ is also continuous.

(ii) This has been already noted above. Indeed, if there is a continuous surjective homomorphism $\theta\colon \mathcal{A} \to \mathcal{B}$ on a non-associative Banach algebra $\mathcal{A}$, then the kernel $\ker\theta$ is closed in $\mathcal{A}$, and the induced continuous isomorphism from the quotient Banach algebra $\mathcal{A}/\ker\theta$ to $\mathcal{B}$ gives a dominating complete algebra norm on $\mathcal{B}$. ∎

Now we consider a weaker condition than continuity on a surjective homomorphism $\theta\colon \mathcal{A} \to \mathcal{B}$, namely, that $\ker\theta$ is closed. This condition is of interest, since the existence of a homomorphism onto $\mathcal{B}$ with closed kernel is equivalent to the existence of a complete algebra norm on $\mathcal{B}$. Let $M(\theta(\overline{\ker\theta}))$ be the subalgebra in $L(\mathcal{B})$ generated by the left and right multiplication operators $\{L_x, R_x : x \in \theta(\overline{\ker\theta}) \subset \mathcal{B}\}$.

Lemma 4.6 *Let $\mathcal{B}$ be a normed algebra and let $\theta\colon \mathcal{A} \to \mathcal{B}$ be a surjective homomorphism on a non-associative Banach algebra $\mathcal{A}$. Let $\mathbf{1} \in L(\mathcal{B})$ be the identity map. If the intersection*

$$B(\mathbf{1}, 1) \cap M\big(\theta(\overline{\ker\theta})\big)$$

is nonempty, then it consists of rare operators on $\mathcal{B}$.

Proof We first note that, for each $x \in \theta(\overline{\ker\theta})$ and for $\varepsilon > 0$, we can choose $a \in \overline{\ker\theta}$ such that $x = \theta(a)$ and $\|a\| < \varepsilon$. Indeed, given $x = \theta(b)$ for some $b \in \overline{\ker\theta}$, there exists $c \in \ker\theta$ such that $\|b - c\| < \varepsilon$ and we have $x = \theta(b - c)$.

The homomorphism θ induces a natural surjective homomorphism

$$\widehat{\theta}\colon M(\mathcal{A}) \longrightarrow M(\mathcal{B}) \subset L(\mathcal{B})$$

between the multiplication algebras satisfying

$$\widehat{\theta}(L_a) = L_{\theta(a)}, \quad \widehat{\theta}(R_a) = R_{\theta(a)}.$$

Indeed, given $F \in M(\mathcal{A})$, one can define $\widehat{\theta}(F)$ to be the unique element in $M(\mathcal{B})$ satisfying $\theta \circ F = \widehat{\theta}(F) \circ \theta$. To see that this is well defined, say that F is of the form

$$F = p(L_{a_1}, \dots, L_{a_k}, R_{b_1}, \dots, R_{b_m}) = q(L_{c_1}, \dots, L_{c_n}, R_{d_1}, \dots, R_{d_s})$$

for some polynomials p and q, then for

$$S_1 = p\big(L_{\theta(a_1)}, \dots, L_{\theta(a_k)}, R_{\theta(b_1)}, \dots, R_{\theta(b_m)}\big),$$
$$S_2 = q\big(L_{\theta(c_1)}, \dots, L_{\theta(c_n)}, R_{\theta(d_1)}, \dots, R_{\theta(d_s)}\big),$$

we have $S_1 \circ \theta = \theta \circ F = S_2 \circ \theta$ and $S_1 = S_2$ by surjectivity of θ. Hence $\widehat{\theta}(F)$ is well defined.

For each $T \in M(\theta(\overline{\ker\theta}))$ with $T = \widehat{\theta}(F)$ and $F \in M(\mathcal{A})$, we have

$$\sigma_s^{\mathcal{B}}(T) = \sigma_s^{\mathcal{B}}(\widehat{\theta}(F)) \subset \sigma_s^{\mathcal{A}}(F),$$

as θ is surjective. Moreover, if $x = \theta(a)$ with $\|a\| < \varepsilon$, then $L_x = \widehat{\theta}(L_a)$ and $\|L_a\| < \varepsilon$. Also, $R_x = \widehat{\theta}(R_a)$ and $\|R_a\| < \varepsilon$.

Now let $T \in B(\mathbf{1}, 1) \cap M(\theta(\overline{\ker\theta}))$. Then T is a polynomial of operators $L \in L(\mathcal{B})$, where L is either a left multiplication L_x or a right multiplication R_x with $x = \theta(a) \in \theta(\overline{\ker\theta})$ and $\|a\|$ can be made arbitrarily small. Therefore, given any $\varepsilon > 0$, we can find $F \in M(\mathcal{A})$ such that $T = \widehat{\theta}(F)$ and $\|F\| < \varepsilon$.

Suppose, for contradiction, that T is not a rare operator on $\mathcal{B}$. Then neither is $\mathbf{1} - T$, and there exists

$$\lambda \in \sigma_s^{\mathcal{B}}(\mathbf{1} - T) \cap \sigma_s^{\widehat{\mathcal{B}}}(\mathbf{1} - \widehat{T}).$$

This gives

$$1 \le |1 - \lambda| + |\lambda| \le |1 - \lambda| + \|\mathbf{1} - T\|$$

where $1 - \lambda \in \sigma_s^{\mathcal{B}}(T) \subset \sigma_s^{\mathcal{A}}(F) \subset \sigma^{\mathcal{A}}(F)$. By the above observation, for each $\varepsilon > 0$ one can choose $F \in M(\mathcal{A})$ with $T = \widehat{\theta}(F)$ and $\|F\| < \varepsilon$. Hence $|1 - \lambda| < \varepsilon$, which implies that $|1 - \lambda| = 0$, as ε is arbitrary. This leads to the contradiction that $1 \le \|\mathbf{1} - T\| < 1$, since $T \in B(\mathbf{1}, 1)$. The proof is now complete. ■

Lemma 4.7 *Let $\mathcal{B}$ be a normed algebra and suppose $\mathcal{B}$ or $\widehat{\mathcal{B}}$ is simple. Let $\mathcal{S}$ be the set of rare operators in $L(\mathcal{B})$. Let $\theta\colon \mathcal{A} \to \mathcal{B}$ be a surjective homomorphism on a non-associative Banach algebra $\mathcal{A}$. Then either* $\ker\theta$ *is closed or*

$$M(\mathcal{B}) \subset \overline{\mathbb{C}\mathbf{1} + M(\theta(\overline{\ker\theta})) \cap \mathcal{S}},$$

where $\mathbf{1} \in L(\mathcal{B})$ is the identity operator.

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Proof Suppose that $\ker\theta$ is not closed in $\mathcal{A}$. Then the simplicity of $\widehat{\mathcal{B}}$ or $\mathcal{B}$ implies that $\theta(\overline{\ker\theta})$ is dense in $\mathcal{B}$.

Let $T \in B(0,1) \cap M(\mathcal{B})$ and $0 < \varepsilon < 1/2$. Pick $\alpha \in (0,\varepsilon)$. Since $\theta(\overline{\ker\theta})$ is dense in $\mathcal{B}$ and $\mathbf{1} \in M(\mathcal{B})$, one can find $S \in M(\theta(\overline{\ker\theta}))$ such that $\|S - (\mathbf{1} - \alpha T)\| < \alpha\varepsilon$. We have

$$\|\mathbf{1} - S\| \le \|\mathbf{1} - (\mathbf{1} - \alpha T)\| + \|S - (\mathbf{1} - \alpha T)\| < \alpha + \alpha\varepsilon < 1.$$

By Lemma 4.6, S is a rare operator on $\mathcal{B}$. Hence $\alpha^{-1}(\mathbf{1} - S)$ is a rare operator on $\mathcal{B}$ satisfying $\|\alpha^{-1}(\mathbf{1} - S) - T\| < \varepsilon$. Since scalar multiples of rare operators are also rare, we conclude that each $T \in M(\mathcal{B})$ is the limit of a sequence of rare operators in $\mathbb{C}\mathbf{1} + M(\theta(\overline{\ker\theta}))$. ■

Theorem 4.8 *Let $\mathcal{B}$ be a normed algebra that is either simple or admits a simple completion. If the closure $\overline{M(\mathcal{B})}$ in $L(\mathcal{B})$ has nonempty interior, then every surjective homomorphism $\theta\colon \mathcal{A} \to \mathcal{B}$ on a non-associative Banach algebra $\mathcal{A}$ has a closed kernel.*

Proof If $\ker\theta$ is not closed in $\mathcal{A}$, then the above lemma implies that $M(\mathcal{B})$ is contained in the closure $\overline{\mathcal{S}}$ of the set $\mathcal{S}$ of rare operators in $L(\mathcal{B})$. Since $\mathcal{S}$ is nowhere dense in $L(\mathcal{B})$ by Theorem 3.6, $M(\mathcal{B})$ is also nowhere dense in $L(\mathcal{B})$. ■

In the above theorem, the existence of a closed kernel $\ker\theta$ need not imply continuity of θ. However, if $\mathcal{B}$ is power associative with a simple completion, then continuity of θ indeed follows from $\ker\theta$ being closed, by [19, Theorem 14]. We recall that an algebra $\mathcal{B}$ is *power-associative* if it satisfies the identity

$$a^{m+n} = a^m a^n \qquad (a \in \mathcal{B},\ m, n \in \mathbb{N}),$$

where the powers are defined inductively by

$$a^1 = a, \quad a^{n+1} = aa^n \qquad (n = 1, 2, \dots).$$

Jordan and Lie algebras are prominent examples of power associative algebras.

Corollary 4.9 *Let $\mathcal{B}$ be a power-associative normed algebra with a simple completion. If $M(\mathcal{B})$ is of second category in $L(\mathcal{B})$, then every surjective homomorphism $\theta\colon \mathcal{A} \to \mathcal{B}$ on a non-associative Banach algebra $\mathcal{A}$ is continuous.*

For a homomorphism with dense range, we have the following result.

Theorem 4.10 *Let $\mathcal{A}$ be a non-associative Banach algebra and $\mathcal{B}$ a unital normed algebra with a simple completion. Let $\theta\colon \mathcal{A} \to \mathcal{B}$ be a homomorphism with a dense range. Among the following conditions, we have* (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv)*:*

(i) *θ is continuous;*
(ii) *$R(\theta(\mathcal{A})) = \varnothing$;*
(iii) *$\overline{R(\theta(\mathcal{A})) + R(\theta(\mathcal{A}))} \neq \mathcal{B}$;*
(iv) *$\ker\theta$ is closed.*

Further, if $\mathcal{B}$ is power-associative, then all the above conditions are equivalent.

Proof (i) $\Rightarrow$ (ii). This follows from Lemma 4.1.

(ii) $\Rightarrow$ (iii). Obvious.

For (iiii) $\Rightarrow$ (iv), if $\ker\theta$ is not closed, then $\theta(\overline{\ker\theta})$ is dense in $\mathcal{B}$, by simplicity of the completion $\widehat{\mathcal{B}}$. Hence the identity $e \in \mathcal{B}$ is the limit of a sequence $(\theta(a_n))$ in $\theta(\overline{\ker\theta})$, and by Lemma 4.6, $\theta(a_n)$ can be chosen in the rarity $R(\theta(\mathcal{A}))$. Given $x \in \theta(\mathcal{A})$, the proof of Lemma 4.7 shows that the left multiplication L_x on $\theta(\mathcal{A})$ is arbitrarily close to a sum $\alpha^{-1}(L_{\theta(a_n)} - L_y)$ of two rare operators with $y \in R(\theta(\mathcal{A}))$. This proves that $R(\theta(\mathcal{A})) + R(\theta(\mathcal{A}))$ is dense in $\theta(\mathcal{A})$.

If $\mathcal{B}$ is power associative, then as noted before, the assertion (iv) $\Rightarrow$ (i) follows from [19, Theorem 14]. ■

In the next example, we construct a simple unital normed algebra $\mathcal{B}$, which is not power associative, and the rarity $R(\mathcal{B})$ is neither empty nor dense.

Example 4.11 Let $\mathcal{B}$ be a complex vector space spanned by $\{e_0, e_1, e_2, \dots\}$. We define a non-associative product on $\mathcal{B}$ by the following table:

	e_0	e_1	e_2	e_3	e_4	e_5	$\cdots$	e_{k-1}	e_k	e_{k+1}	$\cdots$
e_0	e_0	e_1	e_2	e_3	e_4	e_5	$\cdots$	e_{k-1}	e_k	e_{k+1}	$\cdots$
e_1	e_1	0	e_0	0	0	0	$\cdots$	0	0	0	$\cdots$
e_2	e_2	e_0	0	e_1	e_2	e_3	$\cdots$	e_{k-3}	e_{k-2}	e_{k-1}	$\cdots$
e_3	e_3	0	e_1	e_2	0	0	$\cdots$	0	0	0	$\cdots$
e_4	e_4	0	e_2	0	e_3	0	$\cdots$	0	0	0	$\cdots$
e_5	e_5	0	e_3	0	0	e_4	$\cdots$	0	0	0	$\cdots$
e_6	e_6	0	e_4	0	0	0	$\cdots$	0	0	0	$\cdots$
e_7	e_7	0	e_5	0	0	0	$\cdots$	0	0	0	$\cdots$
e_8	e_8	0	e_6	0	0	0	$\cdots$	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	
e_{k-1}	e_{k-1}	0	e_{k-3}	0	0	0	$\cdots$	e_{k-2}	0	0	$\cdots$
e_k	e_k	0	e_{k-2}	0	0	0	$\cdots$	0	e_{k-1}	0	$\cdots$
e_{k+1}	e_{k+1}	0	e_{k-1}	0	0	0	$\cdots$	0	0	e_k	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$

where e_0 is the identity and the marked top left corner is not governed by the multiplication rules for indices $k \geq 4$. This product is not power associative; for example, we have $e_4^2 e_4^2 \neq e_4 e_4^3$.

We equip $\mathcal{B}$ with the ℓ_1-norm:

$$\Big\| \sum_{k=1}^{K} \alpha_{n_k} e_{n_k} \Big\| := \sum_k |\alpha_{n_k}|$$

for $\alpha_{n_1}, \dots, \alpha_{n_K} \in \mathbb{C}$ and $e_{n_1}, \dots, e_{n_K} \in \{e_0, e_1, e_2, \dots\}$.

With this norm, $\mathcal{B}$ is a simple normed algebra with a simple completion $\widehat{\mathcal{B}}$. Indeed, as a Banach space, $\widehat{\mathcal{B}}$ is just the space ℓ_1 with canonical basis $\{e_0, e_1, e_2, \dots\}$. To see that the norm $\|\cdot\|$ is an algebra norm, let $x = \sum_{n=0}^{\infty} \alpha_n e_n \in \widehat{\mathcal{B}}$ and pick any

$m \in \mathbb{N} \cup \{0\}$. Then we have $\|e_m x\| \leq \sum_n |\alpha_n| = \|x\|$ and likewise $\|xe_m\| \leq \|x\|$. It follows that $\|xy\| \leq \|x\|\|y\|$ for every $y \in \widehat{\mathcal{B}}$.

For the simplicity of $\mathcal{B}$ and $\widehat{\mathcal{B}}$, consider any $x = \sum_{n=0}^{\infty} \alpha_n e_n \in \widehat{\mathcal{B}} \backslash \{0\}$. We show that the ideal I_x in $\widehat{\mathcal{B}}$ generated by x contains the identity e_0. First, assume $\alpha_m \neq 0$ for some $m \geq 5$, then we have $(xe_m)e_{m-1} = \alpha_m e_{m-2}$. Since $e_k^2 = e_{k-1}$ for $k \geq 3$ and $e_2 e_1 = e_0$, it follows that $e_0 \in I_x$ in this case. For $x = \sum_{n=0}^{4} \alpha_n e_n$ with $\alpha_n \neq 0$ for some $n \in \{1, 2, 3, 4\}$, we observe that

$$\alpha_4 e_0 = ((xe_4)e_3)e_1, \quad \alpha_3 e_0 = (xe_3)e_1, \quad \alpha_2 e_0 = (xe_4)e_1, \quad \alpha_1 e_0 = \Big(\big((xe_2)e_2\big)\, e_4\Big)\, e_1.$$

From this, one deduces $e_0 \in I_x$ as well. Therefore, $I_x = \widehat{\mathcal{B}}$ for $x \in \widehat{\mathcal{B}} \backslash \{0\}$ and $\widehat{\mathcal{B}}$ is simple. The same argument shows that $\mathcal{B}$ is simple too.

We have $R(\mathcal{B}) \neq \varnothing$. For instance, $e_2 \in R(\mathcal{B})$. In fact, we have $\sigma_s(L_{e_2}) = \varnothing = \sigma_s(R_{e_2})$. To see that the left multiplication $L_{e_2} - \lambda I \colon \mathcal{B} \to \mathcal{B}$ and the right multiplication $R_{e_2} - \lambda I \colon \mathcal{B} \to \mathcal{B}$ are surjective for all $\lambda \in \mathbb{C}$, it suffices to observe the identities

$$\begin{aligned}
(e_2 - \lambda e_0)e_1 &= e_1(e_2 - \lambda e_0) = e_0 - \lambda e_1, \\
(e_2 - \lambda e_0)e_2 &= e_2(e_2 - \lambda e_0) = -\lambda e_2, \\
(e_2 - \lambda e_0)e_m &= e_m(e_2 - \lambda e_0) = e_{m-2} - \lambda e_m \quad (m \geq 3).
\end{aligned}$$

Although $\mathcal{B}$ contains rare elements, we have, nevertheless, $\overline{R(\mathcal{B})} \neq \mathcal{B}$ by Proposition 3.15, since there exist finite rank multiplication operators, for instance, L_{e_4}.

We conclude with an immediate consequence of Theorem 4.10.

Corollary 4.12 *Let $\mathcal{B}$ be a unital power associative normed algebra with a simple completion. If $\mathcal{B}$ admits a complete algebra norm, then it dominates the original norm of $\mathcal{B}$, and every surjective homomorphism $\theta \colon \mathcal{A} \to \mathcal{B}$ on a non-associative Banach algebra $\mathcal{A}$ is continuous.*

Proof Let $\|\cdot\|$ be the original norm on $\mathcal{B}$ and let $|\cdot|$ be a complete algebra norm on $\mathcal{B}$. Then the identity map $\mathbf{1} \colon (\mathcal{B}, |\cdot|) \to (\mathcal{B}, \|\cdot\|)$ is continuous, because its kernel is closed. The last assertion follows from Theorem 4.5. ∎

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