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FURTHER PROPERTIES OF FRACTIONAL STOCHASTIC DOMINANCE

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Abstract

A continuum of stochastic dominance rules, also referred to as fractional stochastic dominance (SD), was introduced by Müller, Scarsini, Tsetlin, and Winkler (2017) to cover preferences from first- to second-order SD. Fractional SD can be used to explain many individual behaviors in economics. In this paper we introduce the concept of fractional pure SD, a special case of fractional SD. We investigate further properties of fractional SD, for example the generating processes of fractional pure SD via γ-transfers of probability, Yaari's dual characterization by utilizing the special class of distortion functions, the separation theorem in terms of first-order SD and fractional pure SD, Strassen's representation, and bivariate characterization. We also establish several closure properties of fractional SD under quantile truncation, under comonotonic sums, and under distortion, as well as its equivalence characterization. Examples of distributions ordered in the sense of fractional SD are provided.

Keywords: Stochastic dominance; dual characterization; distortion; γ-transfer; Strassen's representation

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Secondary 90B50; 91B16

1. Introduction

Stochastic dominance has been studied extensively in applied probability, particularly in the financial and economic literature concerning investment decision-making under uncertainty. The concept of stochastic dominance is quite old and has served as one of the main ways to rank risk prospects or distributions. Of special importance are *first-order stochastic dominance* (FSD) and *second-order stochastic dominance* (SSD). We refer the reader to [15], [24], and [26] for an overview of the SD relations and other stochastic orders. The stochastic dominance relation has an equivalent characterization by a certain class of utility functions. Let X and Y be two random variables (risk prospects). Y dominates X in the FSD means that $\mathbb{E}u(Y) \geq \mathbb{E}u(X)$ for all increasing utility functions u for which the expectations exist, and Y dominates X in the SSD means that $\mathbb{E}u(Y) \geq \mathbb{E}u(X)$ for all increasing and concave utility functions u.

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In the literature of expected utility theory, it is known that a utility function with local convexities is able to explain many individual behaviors, for example, many people buy insurance and also gamble; see [2], [11], [13], [21], among others. Motivated by this, Huang, Tzeng, and Zhao [12] and Müller, Scarsini, Tsetlin, and Winkler [25] proposed two notions of fractional SD, both serving a continuum of FSD and SSD. However, their approaches are different. For $\gamma \in [0, 1]$, Müller *et al.* [25] developed one notion of $(1+\gamma)$-*stochastic dominance*, denoted by $(1+\gamma)$-SD, by adding constraints to the ratio of marginal utilities. The formal definition of $(1+\gamma)$-SD will be given in Definition 2.1. Huang *et al.* [12] introduced another notion of fractional SD, denoted by $(1+\eta)_{\text{HTZ}}$-SD, by adding constraints to the lower bound of the Arrow–Pratt index of absolute risk aversion, where $\eta \in [0, 1]$. Both the degree parameters γ and η have intuitive interpretations. Compared with [25], the approach of [12] can be used to introduce $(n+\eta)$th-degree SD between nth-degree SD and $(n+1)$th-degree SD, where $\eta \in [0, 1]$ and n is any positive integer. It should be pointed out that, for $\gamma \in [0, 1]$, the notion of γ-risk aversion introduced in [20] is equivalent to consistency with $(1+\gamma)$-SD of [25].

The purpose of this paper is to investigate further properties of $(1+\gamma)$-SD in the sense of [25]. The rest of this paper is organized as follows. Section 2 recalls from [25] the definition of $(1+\gamma)$-SD and its basic properties, including the characterization theorem in terms of integral conditions of distribution functions or their inverse functions and closure properties under transformation and mixture. In this section we also introduce the concept of $(1+\gamma)$-pure stochastic dominance, denoted by $(1+\gamma)$-PSD, which will enable one to understand $(1+\gamma)$-SD. Section 3 consists of the main results of this paper, including the generating processes of $(1+\gamma)$-PSD via γ-transfers of probability, Yaari's dual characterization by utilizing the special class of distortion functions, the separation theorem in terms of FSD and $(1+\gamma)$-PSD, Strassen's representation, and bivariate characterization of $(1+\gamma)$-SD. Applications of the main results are given in Section 4. We establish several closure properties of $(1+\gamma)$-SD under p-quantile truncation, under comonotonic sums, and under distortion, as well as its equivalence characterization. Examples of distributions ordered in the sense of $(1+\gamma)$-SD are provided in Section 5.

Throughout this paper, let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let $L^1 = L^1(\Omega, \mathcal{A}, \mathbb{P})$ be the set of all random variables in the probability space with finite expectations. For any distribution function F, the inverse F^{-1} of F is taken to be the left continuous version defined by

$$F^{-1}(\alpha) = \inf\{x\colon F(x) \geq \alpha\} \text{ for } \alpha \in (0, 1],$$

with $F^{-1}(0) = \inf\{x\colon F(x) > 0\}$. For any $x \in \mathbb{R}$, $x_+ = \max\{x, 0\}$ and $x_- = \max\{-x, 0\}$. All expectations are implicitly assumed to exist whenever they are written.

2. Preliminaries

2.1. Definitions

The following definition of stochastic dominance of order $(1+\gamma)$ was given in [25]. We first introduce the following notation. Let $\mathcal{U}$ be the set of all increasing functions on $\mathbb{R}$. For $\gamma \in [0, 1]$, define

$$\mathcal{U}_\gamma = \{u \in \mathcal{U}\colon u \text{ is differentiable, } 0 \leq \gamma u'(y) \leq u'(x) \text{ for all } x \leq y,\ x, y \in \mathbb{R}\}.$$

Definition 2.1. ([25].) Let X and Y be two random variables in $\mathbb{R}$. We say that X is dominated by Y in stochastic dominance of order $(1+\gamma)$, denoted by $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$, if

$$\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)] \quad \text{for all } u \in \mathcal{U}_\gamma.$$

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The order $\preccurlyeq_{(1+\gamma)\text{-SD}}$ cannot be defined for any $\gamma > 1$ because $\mathcal{U}_\gamma$ is empty except for constant functions in this case. For $0 \le \gamma_1 < \gamma_2 \le 1$, $X \preccurlyeq_{(1+\gamma_1)\text{-SD}} Y$ implies $X \preccurlyeq_{(1+\gamma_2)\text{-SD}} Y$ since $\mathcal{U}_{\gamma_2} \subseteq \mathcal{U}_{\gamma_1}$. This means that lower-degree stochastic dominance always implies higher-degree stochastic dominance. In Definition 2.1, the class $\mathcal{U}_\gamma$ of functions can be replaced by $\mathcal{U}^*_\gamma$ defined by

$$\mathcal{U}^*_\gamma = \left\{ u \colon 0 \le \gamma \frac{u(x_4) - u(x_3)}{x_4 - x_3} \le \frac{u(x_2) - u(x_1)}{x_2 - x_1} \text{ for all } x_1 < x_2 \le x_3 < x_4 \right\}.$$

It is obvious that $\preccurlyeq_{1\text{-SD}}$ is equivalent to FSD while $\preccurlyeq_{2\text{-SD}}$ is equivalent to SSD. The orders $\preccurlyeq_{1\text{-SD}}$ and $\preccurlyeq_{2\text{-SD}}$ are also denoted by $\preccurlyeq_{\text{FSD}}$ and $\preccurlyeq_{\text{SSD}}$, respectively. Thus $(1+\gamma)$-SD establishes an interpolation between FSD and SSD. For more properties of FSD, SSD, and other related stochastic orders, refer to [24] and [26].

To investigate the properties of $(1+\gamma)$-SD, we introduce the following $(1+\gamma)$-pure stochastic dominance, denoted by $(1+\gamma)$-PSD.

Definition 2.2. Let $X, Y \in L^1$, and define

$$\gamma = \frac{\int_{-\infty}^{\infty} (G(x) - F(x))_+ \mathrm{d}x}{\int_{-\infty}^{\infty} (G(x) - F(x))_- \mathrm{d}x} \tag{2.1}$$

with the convention that $0/0 = 0$. X is said to be smaller than Y in the pure stochastic dominance of order $(1+\gamma)$ if $\gamma \in [0, 1]$ and $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$. We denote this by $X \preccurlyeq_{(1+\gamma)-\text{PSD}} Y$.

In fact (2.1) can be replaced by

$$\int_0^1 (G^{-1}(\alpha) - F^{-1}(\alpha))_- \mathrm{d}\alpha = \gamma \int_0^1 (G^{-1}(\alpha) - F^{-1}(\alpha))_+ \mathrm{d}\alpha. \tag{2.2}$$

The motivation of the constraint condition (2.1) or (2.1) comes from Proposition 2.1, which gives a characterization of the order $\preccurlyeq_{(1+\gamma)\text{-SD}}$. For $\gamma = 1$, (2.1) is equivalent to $\mathbb{E}[X] = \mathbb{E}[Y]$. Thus 2-PSD is exactly the concave order. Equation (2.1) appears to be similar to that used in [14] to define ϵ-almost FSD as follows: Y dominates X by ϵ-almost FSD, denoted by $X \preccurlyeq_1^{\text{almost}(\epsilon)} Y$, if and only if

$$\frac{\int_{-\infty}^{\infty} (G(x) - F(x))_+ \mathrm{d}x}{\int_{-\infty}^{\infty} (G(x) - F(x))_- \mathrm{d}x} \le \frac{\epsilon}{1-\epsilon},$$

where $0 < \epsilon < 1/2$. Therefore $X \preccurlyeq_{(1+\gamma)-\text{PSD}} Y$ implies $X \preccurlyeq_1^{\text{almost}(\epsilon)} Y$ with $\epsilon = \gamma/(1+\gamma)$. However, the converse is not true.

Hence $(1+\gamma)$-PSD enables one to understand $(1+\gamma)$-SD well. First, $(1+\gamma)$-PSD can be used to characterize $(1+\gamma)$-SD (see Theorem 3.2). Second, if Y dominates X in $(1+\gamma)$-SD, and if there does not exist $Z \in L^1$, not identically distributed with Y, such that Y dominates Z in the FSD and Z dominates X in $(1+\gamma)$-SD, then Y dominates X in $(1+\gamma)$-PSD (see Remark 3.4). These two points are the motivation for us to introduce $(1+\gamma)$-PSD.

It should be pointed out that the order $\preccurlyeq_{(1+\gamma)\text{-PSD}}$ for $\gamma \in (0, 1)$ is not a partial order because it does not possess transitivity, as illustrated by the following example.

Example 2.1. Let X, Y, and Z be three random variables with probability mass functions (PMFs) $\mathbb{P}(X = -1) = 0.4$, $\mathbb{P}(X = 2) = 0.1$, $\mathbb{P}(X = 3) = 0.5$, $\mathbb{P}(Y = 0) = \mathbb{P}(Y = 3) = 1/2$, and

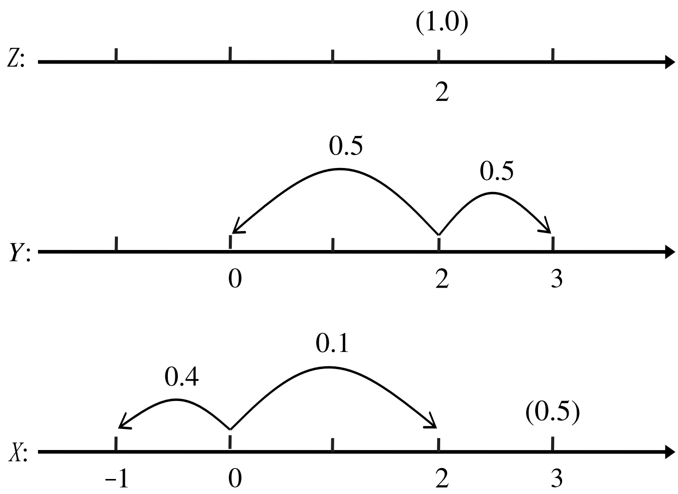

FIGURE 1. Probability mass movement from Z to Y and then to X.

$\mathbb{P}(Z=2)=1$. By Theorem 2.7 in [25], it can be seen from the probability mass movement illustrated in Figure 1 that

$$X \preccurlyeq_{(1+1/2)\text{-PSD}} Y \preccurlyeq_{(1+1/2)\text{-PSD}} Z.$$

However,

$$X \preccurlyeq_{(1+5/12)\text{-PSD}} Z \quad \text{and} \quad X \not\preccurlyeq_{(1+1/2)\text{-PSD}} Z.$$

This means that the order $\preccurlyeq_{(1\,+\,1/2)\text{-PSD}}$ does not possess transitivity.

Let $X \sim F$ and $Y \sim G$. For convenience, we will write $X \preccurlyeq_{\text{order}} Y$ and $F \preccurlyeq_{\text{order}} G$ interchangeably for any order relation $\preccurlyeq_{\text{order}}$.

2.2. Basic properties

In this subsection we list three basic properties of $(1+\gamma)$-SD from [25]. The first characterizes $(1+\gamma)$-SD by using integral conditions, and the second and third are concerned with preservation properties of $(1+\gamma)$-SD under transformations and under mixture, respectively.

Proposition 2.1. ([25].) *Let F and G be the distribution functions of X and Y, respectively. For* $\gamma \in [0, 1]$, $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$ *if and only if*

$$\int_{-\infty}^{t} (G(x)-F(x))_+ \mathrm{d}x \le \gamma \int_{-\infty}^{t} (G(x)-F(x))_- \mathrm{d}x \quad \textit{for all } t \in \mathbb{R}, \tag{2.3}$$

or, equivalently,

$$\int_{0}^{p} (G^{-1}(\alpha)-F^{-1}(\alpha))_- \mathrm{d}\alpha \le \gamma \int_{0}^{p} (G^{-1}(\alpha)-F^{-1}(\alpha))_+ \mathrm{d}\alpha \quad \textit{for all } p \in (0, 1]. \tag{2.4}$$

In view of Proposition 2.1, it is seen that $X \preccurlyeq_{(1+\gamma)\text{-PSD}} Y$ if and only if (2.1) and

$$\int_{t}^{\infty} (G(x)-F(x))_+ \mathrm{d}x \ge \gamma \int_{t}^{\infty} (G(x)-F(x))_- \mathrm{d}x \quad \text{for all } t \in \mathbb{R}.$$

Proposition 2.2. ([25].) *If* $X \preccurlyeq_{(1+\gamma_1\gamma_2)\text{-SD}} Y$ *for* $\gamma_1, \gamma_2 \in [0, 1]$, *then* $u(X) \preccurlyeq_{(1+\gamma_2)\text{-SD}} u(Y)$ *for all* $u \in \mathcal{U}_{\gamma_1}$.

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Proposition 2.3. ([25].) *If random variables X, Y, and* Θ *satisfy*

$$[X \mid \Theta = \theta] \preccurlyeq_{(1+\gamma)\text{-SD}} [Y \mid \Theta = \theta]$$

for some $\gamma \in [0, 1]$ *and all* θ *in the support of* Θ, *then* $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$.

Immediate consequences of Proposition 2.3 are as follows. (i) Let F_i and G_i be the distribution functions of X_i and Y_i, respectively. For $\alpha \in (0, 1)$, assume that $Z_1 \sim \alpha F_1 + (1-\alpha)F_2$ and $Z_2 \sim \alpha G_1 + (1-\alpha)G_2$. If $X_i \preccurlyeq_{(1+\gamma)\text{-SD}} Y_i$ for $i = 1, 2$, then $Z_1 \preccurlyeq_{(1+\gamma)\text{-SD}} Z_2$. (ii) Let X_1 and X_2 be independent, and let Y_1 and Y_2 be independent. If $X_i \preccurlyeq_{(1+\gamma)\text{-SD}} Y_i$ for $i = 1, 2$, then

$$X_1 + X_2 \preccurlyeq_{(1+\gamma)\text{-SD}} Y_1 + Y_2. \tag{2.5}$$

3. Further properties

3.1. Generating processes

For a better understanding of $(1+\gamma)$-SD and $(1+\gamma)$-PSD, we recall the definition of γ-transfer, which is due to [25].

Definition 3.1. (γ*-transfer.*) Let X and Y be two discrete random variables with PMFs f and g, respectively. We say that Y is obtained from X via a γ-transfer if there exist $x_1 < x_2 < x_3 < x_4$ and $\eta_1, \eta_2 > 0$ with $\eta_2(x_4 - x_3) = \gamma\eta_1(x_2 - x_1)$ such that

$$\begin{aligned}
g(x_1) &= f(x_1) - \eta_1,\\
g(x_2) &= f(x_2) + \eta_1,\\
g(x_3) &= f(x_3) + \eta_2,\\
g(x_4) &= f(x_4) - \eta_2,\\
g(z) &= f(z) \quad \text{for all other values } z.
\end{aligned}$$

In the definition of γ-transfer, γ is not necessarily restricted to be in [0,1], which can take any value in $\mathbb{R}_+ = [0, \infty)$. Further, γ-spread is closely related to γ-transfer: X is said to be obtained from Y by a γ-spread if Y is obtained from X by a γ-transfer. In a γ-transfer, a mass of size η_2 is moved to the left from x_4 by $\Delta_2 = x_4 - x_3$, while a mass of size η_1 is moved to the right from x_1 by $\Delta_1 = x_2 - x_1$ such that $\Delta_2\eta_2 = \gamma\Delta_1\eta_1$. A γ transfer increases the mean (i.e. $\mathbb{E}X \leq \mathbb{E}Y$) for $\gamma \in [0, 1]$. In Example 2.1, Y is obtained from X by a $1/2$-transfer, Z is obtained from Y by a $1/2$-transfer, and Z is obtained from X by a $5/12$-transfer.

In Definition 3.1, γ-transfer can also be defined when $x_1 < x_2 = x_3 < x_4$. In this case the conditions $g(x_2) = f(x_2) + \eta_1$ and $g(x_3) = f(x_3) + \eta_2$ were replaced by $g(x_2) = f(x_2) + \eta_1 + \eta_2$.

The following proposition states that γ-transfers account for almost all mass transfers of $(1+\gamma)$-SD. Specifically, part (ii) of Proposition 3.1 can be seen from the proof of Theorem 2.8 of [25]. For two random variables X and Y, we use $X \stackrel{\text{d}}{=} Y$ to denote that X and Y have the same distribution, and let $\|X\|_\infty = \text{ess-sup}(|X|)$.

Proposition 3.1. ([25].) *Let X and Y be two random variables such that* $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$ *for* $\gamma \in [0, 1]$.

(i) *If X and Y both have finite outcomes, then there exist* $X_1, \ldots, X_n$ *such that* $X \stackrel{\text{d}}{=} X_1$, $X_n \leq Y$ *a.s., and* X_i *is a* γ*-transfer of* X_{i-1} *for* $i = 2, \ldots, n$.

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(ii) *If X and Y are bounded, then there exist X_n and Y_n with finite outcomes such that $\|X_n - X\|_\infty \to 0$, $\|Y_n - Y\|_\infty \to 0$ as $n \to \infty$, and $X_n \preccurlyeq_{(1+\gamma)\text{-SD}} Y_n$ for $n \in \mathbb{N}$.*

(iii) *If X and Y are general random variables, then there exist X_n and Y_n with finite outcomes such that $X_n \to X$, $Y_n \to Y$ in distribution, $\mathbb{E}[X_n] \to \mathbb{E}[X]$, $\mathbb{E}[Y_n] \to \mathbb{E}[Y]$ as $n \to \infty$, and $X_n \preccurlyeq_{(1+\gamma)\text{-SD}} Y_n$ for $n \in \mathbb{N}$.*

For $(1+\gamma)$-PSD, we have the following result analogous to Proposition 3.1.

Proposition 3.2. *Let X and Y be two random variables such that $X \preccurlyeq_{(1+\gamma)\text{-PSD}} Y$ for some $\gamma \in [0, 1]$.*

(i) *If X and Y both have finite outcomes, then there exist $X_1, \ldots, X_n$ such that $X \stackrel{\mathrm{d}}{=} X_1$, $X_n \stackrel{\mathrm{d}}{=} Y$ a.s., and X_i is a γ-transfer of X_{i-1} for $i = 2, \ldots, n$.*

(ii) *If X and Y are bounded, then there exist X_n and Y_n with finite outcomes such that $\|X_n - X\|_\infty \to 0$, $\|Y_n - Y\|_\infty \to 0$ as $n \to \infty$, and $X_n \preccurlyeq_{(1+\gamma)\text{-PSD}} Y_n$ for $n \in \mathbb{N}$.*

(iii) *If X and Y are general random variables, then there exist X_n and Y_n with finite outcomes such that $X_n \to X$, $Y_n \to Y$ in distribution, $\mathbb{E}[X_n] \to \mathbb{E}[X]$, $\mathbb{E}[Y_n] \to \mathbb{E}[Y]$ as $n \to \infty$, and $X_n \preccurlyeq_{(1+\gamma)\text{-PSD}} Y_n$ for $n \in \mathbb{N}$.*

Proof. Without loss of generality, assume $\gamma \in (0, 1]$, and let F and G be the distribution functions of X and Y, respectively.

(i) The result can be obtained by modifying the proof of Theorem 2.7 in [25]. We use the same notation as in [25]. Define

$$A^+(p) = \int_0^p (G^{-1}(\alpha) - F^{-1}(\alpha))_+ \mathrm{d}\alpha \quad \text{and} \quad A^-(p) = \int_0^p (G^{-1}(\alpha) - F^{-1}(\alpha))_- \mathrm{d}\alpha.$$

Let $w(a)$ and $v(a)$ be the smallest numbers satisfying

$$A^+(w(a)) = \frac{a}{\gamma} \quad \text{and} \quad A^-(v(a)) = a \quad \text{for } a \in [0, A^-(1)].$$

In view of Proposition 2.1, we have $w(a) \le v(a)$ for all $a \in [0, A^-(1)]$. For each $a \in [0, A^-(1)]$, define

$$x_1(a) = F^{-1}(w(a)), \quad x_2(a) = G^{-1}(w(a)), \quad x_3(a) = G^{-1}(v(a)), \quad x_4(a) = F^{-1}(v(a)).$$

Since X and Y both have finite outcomes, there exist $0 = a_1 < a_2 < \cdots < a_k = A^-(1)$ such that the functions $x_1(a), \ldots, x_4(a)$ are constant on $(a_{i-1}, a_i]$. Denote the corresponding values of these functions as $x_{\ell,i} = x_\ell(a)$ for $a \in (a_{i-1}, a_i]$, $\ell = 1, \ldots, 4$. It was shown in [25] that $x_{1,i} < x_{2,i} \le x_{3,i} < x_{4,i}$, and that for each $i \in \{1, \ldots, k\}$, the probability masses of F at points $x_{1,i}$ and $x_{4,i}$, respectively, are moved to the points $x_{2,i}$ and $x_{3,i}$ of G by a γ-transfer.

Note that when $X \preccurlyeq_{(1+\gamma)\text{-PSD}} Y$, it follows from (2.2) that $A^-(1) = \gamma A^+(1)$. Then $G^{-1}(p) \ge F^{-1}(p)$ for $p > v(a_k)$, and $G^{-1}(p) \le F^{-1}(p)$ for $p > w(a_k)$. This means that $F(x) \ge G(x)$ for $x > x_{4,k}$ and $F(x) \le G(x)$ for $x > x_{1,k}$. Thus $F(x) = G(x)$ for all $x > x_{4,k}$ and the jumps of F and G occur in the points belonging to the set $\{x_{\ell,i} \colon i = 1, \ldots, k; \ell = 1, \ldots, 4\}$. Therefore G can be obtained from F only by a sequence of k γ-transfers.

(ii) First assume that F and G have finite crossings, that is, there exist $-\infty < x_0 < x_1 < \cdots < x_m < \infty$ such that either $F \le G$ or $F \ge G$ holds in (x_{i-1}, x_i), $i = 1, \ldots, m$, where the

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supports of F and G are contained in $[x_0, x_m]$. For $i = 1, \ldots, m$ and $n \in \mathbb{N}$, denote

$$x_{i,j} = \frac{1}{n}[(n-j)x_{i-1} + jx_i], \quad j = 0, \ldots, n.$$

Define two random variables X_n and Y_n with distribution functions F_n and G_n, respectively, where

$$F_n(x) = \frac{1}{x_{i,j} - x_{i,j-1}} \int_{x_{i,j-1}}^{x_{i,j}} F(y)\,\mathrm{d}y, \quad G_n(x) = \frac{1}{x_{i,j} - x_{i,j-1}} \int_{x_{i,j-1}}^{x_{i,j}} G(y)\,\mathrm{d}y$$

for $x \in [x_{i,j-1}, x_{i,j})$. It is easy to see that X_n and Y_n both have finite outcomes. It can be verified that in each interval $[x_{i,j-1}, x_{i,j})$, either $F_n \le G_n$ or $F_n \ge G_n$, and the direction of the inequality is the same as $F \le G$ or $F \ge G$ on the same interval. Hence we have

$$\int_{x_{i,j-1}}^{x_{i,j}} (G_n(y) - F_n(y))_+ \mathrm{d}y = \int_{x_{i,j-1}}^{x_{i,j}} (G(y) - F(y))_+ \mathrm{d}y$$

and

$$\int_{x_{i,j-1}}^{x_{i,j}} (G_n(y) - F_n(y))_- \mathrm{d}w = \int_{x_{i,j-1}}^{x_{i,j}} (G(y) - F(y))_- \mathrm{d}y.$$

Then it follows from $F \preccurlyeq_{(1+\gamma)\text{-PSD}} G$ that $F_n \preccurlyeq_{(1+\gamma)\text{-PSD}} G_n$. On the other hand, it is easy to see that $\max\{|X|, |Y|\} \le M := x_m - x_0 < \infty$,

$$|F^{-1}(\alpha) - F_n^{-1}(\alpha)| \le \frac{1}{n} \max\{x_{i,j} - x_{i,j-1}\} \le \frac{2M}{n}, \quad \alpha \in (0, 1),$$

and

$$|G^{-1}(\alpha) - G_n^{-1}(\alpha)| \le \frac{1}{n} \max\{x_{i,j} - x_{i,j-1}\} \le \frac{2M}{n}, \quad \alpha \in (0, 1).$$

Hence we can easily construct $X_n \sim F_n$ and $Y_n \sim G_n$ such that $|X_n - X| \le 2M/n$ and $|Y_n - Y| \le 2M/n$.

Next, consider that X and Y have infinite crossings, that is, we have infinite intervals $\{(x_{i-1}, x_i), i \in I\}$ such that $G - F$ has the same sign in any one interval. Note that X and Y are both bounded. Then, for $n \in \mathbb{N}$, the number of intervals with length larger than $1/n$ is finite. Then we can merge some of the remaining neighboring intervals to make the lengths smaller than $2/n$ and the number of intervals finite. Without loss of generality, assume that the transformed intervals are still denoted by $\{(x_{i-1}, x_i), i = 1, \ldots, m\}$. In each interval, either $G - F$ has the same sign or the length of the interval is less than $2/n$. For the intervals where $G - F$ has the same sign, we use the same method as in the above case to define the values of F_n and G_n in the intervals. For the other intervals, take (x_{i-1}, x_i) as an example, where $G - F$ has different signs on (x_{i-1}, x_i) and $x_i - x_{i-1} < 2/n$. Let $x^* \in (x_{i-1}, x_i)$ such that $x^* - x_{i-1}$ is equal to the length of $A_i = \{x \in (x_{i-1}, x_i) \colon F(x) \ge G(x)\}$. Denote $A_i^c = (x_{i-1}, x_i) \setminus A_i$ and define

$$F_n(x) = \frac{1}{x^* - x_{i-1}} \int_{A_i} F(y)\,\mathrm{d}y, \quad G_n(x) = \frac{1}{x^* - x_{i-1}} \int_{A_i} G(y)\,\mathrm{d}y$$

for $x \in (x_{i-1}, x^*)$, and

$$F_n(x) = \frac{1}{x_i - x^*} \int_{A_i^c} F(y)\,\mathrm{d}y, \quad G_n(x) = \frac{1}{x_i - x^*} \int_{A_i^c} G(y)\,\mathrm{d}y$$

for $x\in(x^*, x_i)$. Then we have

$$\int_{x_{i-1}}^{x_i}(G_n(y)-F_n(y))_+\mathrm{d}w=\int_{x_{i-1}}^{x_i}(G(y)-F(y))_+\mathrm{d}y$$

and

$$\int_{x_{i-1}}^{x_i}(G_n(y)-F_n(y))_-\mathrm{d}y=\int_{x_{i-1}}^{x_i}(G(y)-F(y))_-\mathrm{d}y.$$

Then it can be checked that $F\preccurlyeq_{(1+\gamma)\text{-PSD}} G$ implies $F_n\preccurlyeq_{(1+\gamma)\text{-PSD}} G_n$. The remaining proof is similar to the above case.

(iii) We modify the proof of Theorem 2.8 in [25] for our purpose. For unbounded random variables X and Y, define

$$\psi(t)=\int_{-\infty}^{t}(G(x)-F(x))_+\mathrm{d}x \quad\text{and}\quad \xi(t)=\gamma\int_{-\infty}^{t}(G(x)-F(x))_-\mathrm{d}x.$$

We approximate X and Y by X_n and Y_n, respectively, as follows. Define

$$X_n=\begin{cases} x_n^* & \text{if } X\le -n,\\ X & \text{if } -n<X\le n,\\ y_n^* & \text{if } X>n,\end{cases}$$

and

$$Y_n=\begin{cases} -n & \text{if } Y\le -n,\\ Y & \text{if } -n<Y\le n,\\ n & \text{if } Y>n,\end{cases}$$

where

$$x_n^*=-n-\frac{\xi(-n)}{\gamma F(-n)} \quad\text{and}\quad y_n^*=n+\frac{\xi(n)-\psi(n)+\psi(-n)}{\overline{F}(n)}.$$

Since $\xi(t)\ge\psi(t)\ge 0$ for all t, it follows that $x_n^*\le -n$ and $y_n^*\ge n$. Let F_n and G_n denote the distribution functions of X_n and Y_n, respectively, and define

$$\psi_n(t)=\int_{-\infty}^{t}(G_n(x)-F_n(x))_+\mathrm{d}x \quad\text{and}\quad \xi_n(t)=\gamma\int_{-\infty}^{t}(G_n(x)-F_n(x))_-\mathrm{d}x.$$

Then it can be checked that

$$\psi_n(t)=\begin{cases} 0 & \text{if } t<-n,\\ \psi(t)-\psi(-n) & \text{if } -n<t\le n,\\ \psi(n)-\psi(-n)+(t-n)\overline{F}(n) & \text{if } n<t\le y_n^*,\\ \xi(n) & \text{if } t>y_n^*,\end{cases}$$

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and

$$\xi_n(t) = \begin{cases} \xi(t) & \text{if } -n \le t \le n, \\ \xi(n) & \text{if } t > n. \end{cases}$$

Thus X_n and Y_n are bounded, $\psi_n(t) \le \xi_n(t)$ for all $t \in \mathbb{R}$, and $\psi_n(+\infty) = \xi(n) = \xi_n(+\infty)$. This means $X_n \preccurlyeq_{(1+\gamma)\text{-PSD}} Y_n$ for all $n \in \mathbb{N}$. On the other hand, $\psi(+\infty) = \xi(+\infty)$ implies that $\mathbb{E}[X_n] \to \mathbb{E}[X]$ and $X_n \to X$ in distribution. Obviously, $\mathbb{E}[Y_n] \to \mathbb{E}[Y]$ and $Y_n \to Y$ in distribution. The desired result now follows from part (ii). This completes the proof. □

3.2. Dual characterization

Let $\mathcal{H}$ denote the set of all probability perception functions h (also referred to as distortion functions in the actuarial literature), that is, $h\colon [0, 1] \to [0, 1]$ is increasing, satisfying $h(0) = 0$ and $h(1) = 1$. For $\gamma \in [0, 1]$, define

$$\mathcal{H}_\gamma = \{h \in \mathcal{H}\colon h \text{ is differentiable, } 0 \le \gamma h'(y) \le h'(x) \text{ for all } 0 \le x \le y \le 1\}$$

and

$$\mathcal{H}_\gamma^* = \left\{h \in \mathcal{H}\colon 0 \le \gamma \frac{h(p_4) - h(p_3)}{p_4 - p_3} \le \frac{h(p_2) - h(p_1)}{p_2 - p_1} \text{ for all } 0 \le p_1 < p_2 \le p_3 < p_4 \le 1\right\},$$

where $h'(0)$ and $h'(1)$ represent the right derivative at 0 and the left derivative at 1, respectively. Obviously, $\mathcal{H}_\gamma$ is the subset of $\mathcal{H}_\gamma^*$ containing all continuously differentiable $h \in \mathcal{H}_\gamma^*$.

In the following theorem, we establish in the framework of Yaari's dual theory that $(1+\gamma)$-SD is equivalent to a common preference among all decision-makers with probability perception function $h \in \mathcal{H}_\gamma$. This is a dual characterization of $(1+\gamma)$-SD as the latter is originally defined via a common preference based on utility functions. For Yaari's dual theory, see [30].

Theorem 3.1. *Let F and G be the distribution functions of X and Y, respectively. For* $\gamma \in [0, 1]$, *the following statements are equivalent:*

(i) $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$,

(ii) $\int_0^1 F^{-1}(\alpha)\,\mathrm{d}h(\alpha) \le \int_0^1 G^{-1}(\alpha)\,\mathrm{d}h(\alpha)$ *for all* $h \in \mathcal{H}_\gamma$,

(iii) $\int_0^1 F^{-1}(\alpha)\,\mathrm{d}h(\alpha) \le \int_0^1 G^{-1}(\alpha)\,\mathrm{d}h(\alpha)$ *for all* $h \in \mathcal{H}_\gamma^*$.

Proof. Part (iii) is equivalent to (ii). It suffices to prove that (iii)⇒ (2.4) ⇒(ii).

To prove (iii)⇒ (2.4), for $p \in (0, 1]$, define a distortion function $h \in \mathcal{H}$ such that

$$h'(\alpha) = \begin{cases} \gamma & \text{if } F^{-1}(\alpha) \le G^{-1}(\alpha) \text{ and } \alpha \le p, \\ 1 & \text{if } F^{-1}(\alpha) > G^{-1}(\alpha) \text{ and } \alpha \le p, \\ 0 & \text{if } \alpha > p. \end{cases}$$

It is easy to verify that $h \in \mathcal{H}_\gamma^*$ and hence (2.4) holds.

To prove the other direction (2.4) ⇒(ii), we use arguments similar to those in the proof of Theorem 2.4 of [25]. For completeness, we give the details. Let $h \in \mathcal{H}_\gamma$, i.e. h

satisfies $0 \le \gamma h'(y) \le h'(x)$ for all $0 \le x \le y \le 1$. Then $R := \sup_{v\in(0,1)} h'(v) \in (0, \infty)$ since $0 \le h'(v) \le h'(1)/\gamma < \infty$. For any fixed $n \ge 2$, define $\epsilon_n = 2^{-n}$ and K as the largest integer k for which

$$R(1 - k\epsilon_n) \ge \inf_{v\in(0,1)} h'(v),$$

and define a partition of [0,1] into intervals $[v_k, v_{k+1}]$ as follows: $v_0 = 0$, $v_{K+1} = 1$, and

$$v_k = \sup\{v\colon h'(v) \ge R(1 - k\epsilon_n)\}, \quad k = 1, \ldots, K.$$

Then we define

$$m_k = \sup\{h'(v)\colon v_{k-1} \le v \le v_k\} = R(1 - (k-1)\epsilon_n).$$

It follows that $\gamma m_{k+1} \le h'(v) \le m_k$ for $v \in [v_{k-1}, v_k]$, i.e. $\gamma(m_k - R\epsilon_n) \le h'(v) \le m_k$ for all $x \in [v_{k-1}, v_k]$ and $k = 1, \ldots, K+1$. This implies that

$$\begin{aligned}
&\int_{v_{k-1}}^{v_k} (G^{-1}(\alpha) - F^{-1}(\alpha))\,\mathrm{d}h(\alpha)\\
&= \int_{v_{k-1}}^{v_k} (G^{-1}(\alpha) - F^{-1}(\alpha))_+ h'(\alpha)\,\mathrm{d}\alpha - \int_{v_{k-1}}^{v_k} (G^{-1}(\alpha) - F^{-1}(\alpha))_- h'(\alpha)\,\mathrm{d}\alpha\\
&\ge \gamma(m_k - R\epsilon_n) \int_{v_{k-1}}^{v_k} (G^{-1}(\alpha) - F^{-1}(\alpha))_+ \mathrm{d}\alpha - m_k \int_{v_{k-1}}^{v_k} (G^{-1}(\alpha) - F^{-1}(\alpha))_- \mathrm{d}\alpha\\
&= m_k T_k - \epsilon_n c_k,
\end{aligned}$$

with

$$T_k = \gamma \int_{v_{k-1}}^{v_k} (G^{-1}(\alpha) - F^{-1}(\alpha))_+ \mathrm{d}\alpha - \int_{v_{k-1}}^{v_k} (G^{-1}(\alpha) - F^{-1}(\alpha))_- \mathrm{d}\alpha$$

and

$$c_k = \gamma R \int_{v_{k-1}}^{v_k} (G^{-1}(\alpha) - F^{-1}(\alpha))_+ \mathrm{d}\alpha.$$

Note that (2.4) implies $\sum_{i=1}^{k} T_i \ge 0$ for all $k = 1, \ldots, K+1$, which in turn implies $\sum_{k=1}^{K+1} m_k T_k \ge 0$ for all decreasing non-negative sequences m_k. Thus

$$\begin{aligned}
\int_0^1 (G^{-1}(\alpha) - F^{-1}(\alpha))\,\mathrm{d}h(\alpha) &\ge \sum_{k=1}^{K+1} (m_k T_k - \epsilon_n c_k)\\
&\ge -\epsilon_n \gamma R \int_0^1 (G^{-1}(\alpha) - F^{-1}(\alpha))_+ \mathrm{d}\alpha.
\end{aligned}$$

Letting $n \to \infty$ yields part (ii). This completes the proof of the theorem. □

Remark 3.1. To get a better understanding of the dual characterization of $(1+\gamma)$-SD, we introduce the following index Q_f of a probability perception function h:

$$Q_h = \sup_{0\le p_1<p_2\le p_3<p_4\le 1} \frac{(h(p_4) - h(p_3))/(p_4 - p_3)}{(h(p_1) - h(p_1))/(p_2 - p_1)}.$$

Here we use the convention that $a/0 = +\infty$ for any real number $a > 0$ and $0/0 = 0$. As mentioned in [5], Q_h is an index of non-concavity of $h \in \mathcal{H}$, $Q_h \ge 1$, and $Q_h = 1$ corresponds exactly

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to concavity. Thus $1/Q_h$ can be regarded as an index of the greediness of a decision-maker with probability perception function h (in short, a decision-maker h). That is, for $h_1, h_2 \in \mathcal{H}$, $Q_{h_1} < Q_{h_2}$ means that h_1 is more greedy than h_2. Therefore, for $\gamma \in [0, 1]$,

$$\mathcal{H}_\gamma^* = \left\{h \in \mathcal{H} \colon Q_h \le \frac{1}{\gamma}\right\}$$

denotes the set of decision-makers with index of greediness larger than or equal to γ. On the other hand, $\mathcal{U}_\gamma^*$ or $\mathcal{U}_\gamma^*$ has a similar interpretation in terms of risk aversion.

The order $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$ is defined in Definition 2.1 by comparing expected utilities $\mathbb{E}[u(X)]$ and $\mathbb{E}[u(Y)]$ for all utility functions $u \in \mathcal{U}_\gamma$, while the dual characterization in Theorem 3.1 compares the expected values $\mathbb{E}[X_h]$ and $\mathbb{E}[Y_h]$ of random variables X_h and Y_h for all probability perception functions $h \in \mathcal{H}_\gamma$, where X_h and Y_h have the distorted distribution functions $h(F(x))$ and $h(G(x)$.

3.3. Separation theorem

We establish a separation theorem similar to the classic separation theorem for 2-SD. That is, a $(1+\gamma)$-SD can be separated by an FSD and a $(1+\gamma)$-PSD.

Theorem 3.2. *For $X, Y \in L^1$, $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$ if and only if there exist $Z_1, Z_2 \in L^1$ such that*

$$X \preccurlyeq_{(1+\gamma)\text{-PSD}} Z_1 \preccurlyeq_{\text{FSD}} Y \tag{3.1}$$

and

$$X \preccurlyeq_{\text{FSD}} Z_2 \preccurlyeq_{(1+\gamma)\text{-PSD}} Y. \tag{3.2}$$

Proof. The sufficiency is trivial. It requires us to prove the necessity. To this end, let F and G denote the distribution functions of X and Y, respectively, and assume that $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$ but $X \not\preccurlyeq_{(1+\gamma)\text{-PSD}} Y$, i.e. (2.1) does not hold. By Proposition 2.1, we have

$$\Delta := \gamma \int_{-\infty}^{\infty} (G(x) - F(x))_- \mathrm{d}x - \int_{-\infty}^{\infty} (G(x) - F(x))_+ \mathrm{d}x > 0.$$

For $t \in \overline{\mathbb{R}} = [-\infty, \infty]$, define

$$\delta_t(x) = (G(x) - F(x))_- 1_{\{x \ge t\}}, \tag{3.3}$$

where $\delta_{-\infty}(x) \equiv (G(x) - F(x))_-$ and $\delta_\infty(x) \equiv 0$. Note that $\delta_t(x)$ is decreasing in $t \in \overline{\mathbb{R}}$ for each fixed x, $\int_{-\infty}^{\infty} \delta_t(x)\, \mathrm{d}x$ is continuous in $t \in \overline{\mathbb{R}}$, and

$$\gamma \int_{-\infty}^{\infty} \delta_\infty(x)\, \mathrm{d}x = 0 < \Delta \le \gamma \int_{-\infty}^{\infty} \delta_{-\infty}(x)\, \mathrm{d}x.$$

Then there exists $t_0 \in \overline{\mathbb{R}}$ such that

$$\gamma \int_{-\infty}^{\infty} \delta_{t_0}(x)\, \mathrm{d}x = \Delta. \tag{3.4}$$

We define

$$H_1(x) = G(x) + \delta_{t_0}(x), \quad x \in \mathbb{R},$$

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which is an increasing and right-continuous function. From (3.3), we have that

$$G(x) \le H_1(x) \le G(x) + (G(x) - F(x))_- = F(x) \vee G(x), \quad x \in \mathbb{R},$$

and hence H_1 is a distribution function on $\mathbb{R}$ such that $H_1 \preccurlyeq_{\mathrm{FSD}} G$. From (3.4), one can verify that

$$\int_{-\infty}^{\infty} (H_1(x) - F(x))_+ \mathrm{d}x = \gamma \int_{-\infty}^{\infty} (H_1(x) - F(x))_- \mathrm{d}x \tag{3.5}$$

and

$$\int_{-\infty}^{t} (H_1(x) - F(x))_+ \mathrm{d}x \le \gamma \int_{-\infty}^{t} (H_1(x) - F(x))_- \mathrm{d}x \quad \text{for all } t \in \mathbb{R}. \tag{3.6}$$

In fact, (3.5) follows from the fact that $(H_1(x) - F(x))_+ = (G(x) - F(x))_+$ and $(H_1(x) - F(x))_- = (G(x) - F(x))_- - \delta_{t_0}(x)$ for all $x \in \mathbb{R}$. Equation (3.6) follows from the facts that $H_1(x) = G(x)$ for $x < t_0$ and $H_1(x) = G(x) \vee F(x)$ for $x \ge t_0$, which implies the inequality

$$\int_{t}^{\infty} (H_1(x) - F(x))_+ \mathrm{d}x \ge \gamma \int_{t}^{\infty} (H_1(x) - F(x))_- \mathrm{d}x \quad \text{for all } t \in \mathbb{R}.$$

This implies that $F \preccurlyeq_{(1+\gamma)\text{-SD}} H_1$. Then (3.1) follows by taking Z_1 as a random variable having distribution function H_1.

A similar argument to the above can be applied to obtain (3.2) by choosing $H_2(x) = F(x) - \eta_{t_1}(x)$, where

$$\eta_t(x) = (F(x) - G(x))_+ 1_{\{x<t\}} \tag{3.7}$$

and $t_1 \in \overline{\mathbb{R}}$ such that

$$\gamma \int_{-\infty}^{\infty} \eta_{t_1}(x)\, \mathrm{d}x = \Delta. \tag{3.8}$$

This completes the proof of the theorem. □

Remark 3.2. For $\gamma = 1$, 2-PSD is the concave order. The separation result in Theorem 3.2 reduces to the separation theorem for the SSD: $X \preccurlyeq_{\mathrm{SSD}} Y$ if and only if there exists a random variable Z such that

$$X \preccurlyeq_{\mathrm{cv}} Z \preccurlyeq_{\mathrm{FSD}} Y \quad \text{or} \quad X \preccurlyeq_{\mathrm{FSD}} Z \preccurlyeq_{\mathrm{cv}} Y.$$

This is a well-known result; see parts (c) and (d) of Theorem 4.A.6 in [26]. There are several proofs of the above separation theorem for the SSD in the literature, for example [19] and [22]. For $\gamma = 1$, the proof of Theorem 3.2 is new and differs from those in the literature.

Remark 3.3. The proof of Theorem 3.2 gives us a method for constructing random variables Z_1 and Z_2 such that (3.1) and (3.2) hold, which is illustrated by Example 5.3.

Remark 3.4. From the proof of Theorem 3.2, we conclude the following.

(i) If $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$, and if there does not exist $Z \in L^1$ such that $Z \overset{\mathrm{d}}{\neq} Y$, $Z \preccurlyeq_{\mathrm{FSD}} Y$ and $X \preccurlyeq_{(1+\gamma)\text{-SD}} Z$, then $X \preccurlyeq_{(1+\gamma)\text{-PSD}} Y$.

(ii) If $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$, and if there does not exist $Z \in L^1$ such that $Z \overset{\mathrm{d}}{\neq} X$, $X \preccurlyeq_{\mathrm{FSD}} Z$ and $Z \preccurlyeq_{(1+\gamma)\text{-SD}} Y$, then $X \preccurlyeq_{(1+\gamma)\text{-PSD}} Y$.

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3.4. Strassen's representation

A famous result of Strassen [28] states that $X \preccurlyeq_{\text{SSD}} Y$ if and only if there exist random variables $\widehat{X}$ and $\widehat{Y}$ defined on a common probability space with the same distributions as X and Y such that $\mathbb{E}[\widehat{X} \mid \widehat{Y}] \le \widehat{Y}$, a.s. Müller and Rüschendorf [23] presented an elementary and constructive proof of this result on the real line. For more details on Strassen's theorem and extensions, see [1], [9], [10], [18], and references therein.

For $(1+\gamma)$-SD, we have the following partial Strassen's representation.

Theorem 3.3. *Let X and Y be two random variables. If there exist $\widehat{X}$ and $\widehat{Y}$ on the same probability space such that $\widehat{X} \overset{\text{d}}{=} X$, $\widehat{Y} \overset{\text{d}}{=} Y$, and*

$$\mathbb{E}[(\widehat{Y}-\widehat{X})_- \mid \widehat{Y}] \le \gamma\, \mathbb{E}[(\widehat{Y}-\widehat{X})_+ \mid \widehat{Y}] \quad a.s. \tag{3.9}$$

for some $\gamma \in [0, 1]$, then $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$.

Proof. First we assert that, for any random variable Z,

$$\mathbb{E}[Z_+] \le \gamma\, \mathbb{E}[Z_-] \Longrightarrow Z \preccurlyeq_{(1+\gamma)\text{-SD}} 0,$$

which can be seen by verifying (2.3). Then it follows from (2.5) that for any $y \in \mathbb{R}$, $Z + y \preccurlyeq_{(1+\gamma)\text{-SD}} y$. Let $\operatorname{supp}(G)$ denote the support of the distribution function of Y. Note that (3.9) implies that, for almost all $y \in \operatorname{supp}(G)$,

$$\mathbb{E}[(\widehat{X}-y)_+ \mid \widehat{Y}=y] \le \gamma\, \mathbb{E}[(\widehat{X}-y)_- \mid \widehat{Y}=y],$$

and hence $[\widehat{X} \mid \widehat{Y}=y] \preccurlyeq_{(1+\gamma)\text{-SD}} y$. Then, for any $\phi \in \mathcal{U}_\gamma^*$, we have $\mathbb{E}[\phi(\widehat{X}) \mid \widehat{Y}=y] \le \phi(y)$ for almost all $y \in \operatorname{supp}(G)$. Hence

$$\mathbb{E}[\phi(X)] = \mathbb{E}[\phi(\widehat{X})] = \mathbb{E}\{\mathbb{E}[\phi(\widehat{X}) \mid \widehat{Y}]\} \le \mathbb{E}[\phi(\widehat{Y})] = \mathbb{E}[\phi(Y)].$$

We thus complete the proof. □

For $\gamma = 1$, (3.9) reduces to $\mathbb{E}[\widehat{X} \mid \widehat{Y}] \le \widehat{Y}$ a.s. In Theorem 3.3, (3.9) is a sufficient condition for $(1+\gamma)$-SD. However, it is not a necessary condition, as illustrated by the following counterexample.

Example 3.1. Let X and Y be two binary random variables with PMFs

$$\mathbb{P}(X=0) = \mathbb{P}(X=4) = 1/2 \quad \text{and} \quad \mathbb{P}(Y=2) = \mathbb{P}(Y=3) = 1/2.$$

Then Y is a 1/2-transfer of X and hence $X \preccurlyeq_{(1+1/2)\text{-SD}} Y$. Assume that there exist $\widehat{X}$ and $\widehat{Y}$ on the same probability space such that $\widehat{X} \overset{\text{d}}{=} X$, $\widehat{Y} \overset{\text{d}}{=} Y$, and (3.9) holds with $\gamma = 1/2$. Denote $b = \mathbb{P}(\widehat{X}=0 \mid \widehat{Y}=3)$. From (3.9) it follows that $b \ge 2$. However, $b \in [0, 1]$. This is a contradiction. Therefore (3.9) is not necessary for $(1+\gamma)$-SD.

To state the next proposition, we recall the definition of comonotonicity. A random vector $(X_1, \ldots, X_n)$ is said to be comonotonic if there exist non-decreasing functions g_i ($i = 1, \ldots, n$), and a random variable W such that $(X_1, \ldots, X_n) \overset{\text{d}}{=} (g_1(W), \ldots, g_n(W))$. For more on comonotonicity, see [7], [8], and references therein.

Proposition 3.3. *Let F and G be two distribution functions. If G is continuous on $\mathbb{R}$, then $F \preccurlyeq_{(1+\gamma)\text{-SD}} G$ for $\gamma \in [0, 1]$ if and only if there exist $X \sim F$ and $Y \sim G$ on the same probability space such that they are comonotonic and*

$$\mathbb{E}[(Y-X)_- \mid Y \le y] \le \gamma\, \mathbb{E}[(Y-X)_+ \mid Y \le y] \quad \textit{for all } y \in \mathbb{R}. \tag{3.10}$$

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Proof. To show sufficiency, let U be a random variable uniformly distributed on (0, 1). Then we have $(X, Y) \stackrel{d}{=} (F^{-1}(U), G^{-1}(U))$ and hence, for $y \in \mathbb{R}$,

$$\begin{aligned}\mathbb{E}[(Y-X)_- \mid Y \le y] &= \mathbb{E}[(G^{-1}(U) - F^{-1}(U))_- \mid G^{-1}(U) \le y] \\ &= \mathbb{E}[(G^{-1}(U) - F^{-1}(U))_- \mid U \le G(y)] \\ &= \frac{1}{G(y)} \int_0^{G(y)} (G^{-1}(\alpha) - F^{-1}(\alpha))_- \mathrm{d}\alpha.\end{aligned}$$

Similarly,

$$\mathbb{E}[(Y-X)_+ \mid Y \le y] = \frac{1}{G(y)} \int_0^{G(y)} (G^{-1}(\alpha) - F^{-1}(\alpha))_+ \mathrm{d}\alpha.$$

Since G is continuous, then for any $p \in (0, 1)$ there exists $y \in \mathbb{R}$ such that $G(y) = p$. It follows from (3.10) that (2.4) holds for all $p \in (0, 1)$. It is obvious to check that (2.4) holds for $p = 1$ by the continuity of the two functions of (2.4). Therefore we have $F \preccurlyeq_{(1+\gamma)\text{-SD}} G$.

Necessity follows immediately by taking $(X, Y) := (F^{-1}(U), G^{-1}(U))$ with U uniformly distributed on (0,1). This completes the proof. □

3.5. Bivariate characterization

To state the bivariate characterization for $(1+\gamma)$-SD, we introduce the following class of bivariate functions:

$$\mathcal{G}_\gamma = \{\phi \colon \mathbb{R}^2 \to \mathbb{R} \mid x \mapsto \phi(x, y) - \phi(y, x) \in \mathcal{U}_\gamma^* \text{ for each } y\}. \tag{3.11}$$

Proposition 3.4. *Let X and Y be two independent random variables. Then $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$ if and only if*

$$\mathbb{E}[\phi(X, Y)] \le \mathbb{E}[\phi(Y, X)] \quad \textit{for all } \phi \in \mathcal{G}_\gamma.$$

Proof. The sufficiency is trivial by noting that, for any $u \in \mathcal{U}_\gamma^*$, the bivariate function $\phi(x, y) := u(x)$ belongs to the set $\mathcal{G}_\gamma$. To see the necessity, for any $\phi \in \mathcal{G}_\gamma$, define

$$u(x) := \mathbb{E}[\phi(x, Y)] - \mathbb{E}[\phi(Y, x)], \quad x \in \mathbb{R}.$$

It can be easily verified that $u \in \mathcal{U}_\gamma^*$ and thus

$$\mathbb{E}[\phi(X, Y)] - \mathbb{E}[\phi(Y, X)] = \mathbb{E}[u(X)] \le \mathbb{E}[u(Y)] = 0.$$

Necessity then follows, and hence we complete the proof. □

It is worth noting that the above result still holds true if all $\mathcal{U}_\gamma^*$ are replaced by $\mathcal{U}_\gamma$. For $\gamma = 1$, the equivalence characterization was implicitly given in [27]; see also Theorem 4.A.7 of [26]. An application of Proposition 3.4 is given in Example 5.6.

4. Applications

4.1. Closure under p-quantile truncation

Proposition 4.1. *Let X and Y be two continuous random variables with respective distribution functions F and G. If $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$, then*

$$[X \mid X \le F^{-1}(p)] \preccurlyeq_{(1+\gamma)\text{-SD}} [Y \mid Y \le G^{-1}(p)], \quad p \in (0, 1). \tag{4.1}$$

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Proof. Let $\phi \in \mathcal{U}_\gamma$, and suppose that $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$. From Proposition 2.2 it follows that $\phi(X) \preccurlyeq_{\text{SSD}} \phi(Y)$. Since $F^{-1}_{\phi(X)}(\alpha) = \phi(F^{-1}(\alpha))$ for each α, by Theorems 4.A.1 and 4.A.3 in [26], we have

$$\int_0^p \phi(F^{-1}(\alpha))\, \mathrm{d}\alpha \le \int_0^p \phi(G^{-1}(\alpha))\, \mathrm{d}\alpha, \quad p \in (0, 1),$$

or, equivalently, $\mathbb{E}[\phi(X) \mid X \le F^{-1}(p)] \le \mathbb{E}[\phi(Y) \mid Y \le G^{-1}(p)]$ since F and G are continuous. This means that (4.1) holds. □

From the proof of Proposition 4.1, we conclude that if X and Y are general random variables (not necessarily continuous), then $F \preccurlyeq_{(1+\gamma)\text{-SD}} G$ implies that $F^{-1}(U)1_{\{U\le p\}} \preccurlyeq_{(1+\gamma)\text{-SD}} G^{-1}(U)1_{\{U\le p\}}$ for $p \in (0, 1)$, where U is a random variable uniformly distributed on (0,1).

Remark 4.1. When $\gamma = 1$, the result of Proposition 4.1 was implicitly given by Theorem 4.A.42 of [26] without the constraint of continuity. However, we point out that the condition of continuity is necessary. To see it, we give a counter-example. Define two random variables X and Y with PMFs given by $\mathbb{P}(X = 0) = 0.625$, $\mathbb{P}(X = 4) = 0.375$ and $\mathbb{P}(Y = 1) = 0.7$, $\mathbb{P}(Y = 2) = 0.1$, $\mathbb{P}(Y = 3) = 0.2$. It is easy to verify that $\mathbb{E}[X] = 1.5 = \mathbb{E}[Y]$, and hence $X \preccurlyeq_{\text{SSD}} Y$. Let F and G denote the distribution functions of X and Y, respectively. Note that, for $p = 0.7$,

$$[X \mid X \le F^{-1}(p)] = [X \mid X \le 4] = X$$

and

$$[Y \mid Y \le G^{-1}(p)] = [Y \mid Y \le 1] = 1.$$

Thus $[X \mid X \le F^{-1}(p)] \not\preccurlyeq_{\text{SSD}} [Y \mid Y \le G^{-1}(p)]$.

4.2. Closure under comonotonic sums

Equation (2.5) states that $(1 + \gamma)$-SD is closed under independent sums. With Theorem 3.1 we can prove that $(1 + \gamma)$-SD is closed under comonotonic sums.

Proposition 4.2. *Let X_i and Y_i be two random variables such that $X_i \preccurlyeq_{(1+\gamma)\text{-SD}} Y_i$ for $i = 1, 2$ and $\gamma \in [0, 1]$. If X_1 and X_2 are comonotonic and Y_1 and Y_2 are comonotonic, then $X_1 + X_2 \preccurlyeq_{(1+\gamma)\text{-SD}} Y_1 + Y_2$.*

Proof. Let F_i and F denote the distribution functions of X_i and $X_1 + X_2$, respectively. Similarly, let G_i and G denote the distribution functions of Y_i and $Y_1 + Y_2$, respectively. Since X_1 and X_2 are comonotonic, it follows from [8] that $F^{-1}(\alpha) = F_1^{-1}(\alpha) + F_2^{-1}(\alpha)$ for all $\alpha \in (0, 1)$. Similarly, $G^{-1}(\alpha) = G_1^{-1}(\alpha) + G_2^{-1}(\alpha)$ for all $\alpha \in (0, 1)$. By Theorem 3.1 (iii), $X_i \preccurlyeq_{(1+\gamma)\text{-SD}} Y_i$ implies that

$$\int_0^1 F_i^{-1}(\alpha)\, \mathrm{d}h(\alpha) \le \int_0^1 G_i^{-1}(\alpha)\, \mathrm{d}h(\alpha) \quad \text{for all } h \in \mathcal{H}^*_\gamma,\ i = 1, 2.$$

Thus

$$\int_0^1 F^{-1}(\alpha)\, \mathrm{d}h(\alpha) \le \int_0^1 G^{-1}(\alpha)\, \mathrm{d}h(\alpha) \quad \text{for all } h \in \mathcal{H}^*_\gamma,$$

implying $X_1 + X_2 \preccurlyeq_{(1+\gamma)\text{-SD}} Y_1 + Y_2$ by applying Theorem 3.1 (iii) again. This completes the proof of the proposition. □

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4.3. Closure under minima

We first present a general result concerning the preservation of $(1+\gamma)$-SD under increasing and concave transforms.

Proposition 4.3. *Let* $X_1, \ldots, X_n$ *be a set of independent random variables, and let* $Y_1, \ldots, Y_n$ *be another set of independent random variables. If* $X_i \preccurlyeq_{(1+\gamma)\text{-SD}} Y_i$ *for* $i = 1, \ldots, n$ *and* $\gamma \in [0, 1]$*, then*

$$g(X_1, \ldots, X_n) \preccurlyeq_{(1+\gamma)\text{-SD}} g(Y_1, \ldots, Y_n) \tag{4.2}$$

for every increasing and component-wise concave function g.

Proof. Without loss of generality, assume that all random variables X_i and Y_i are independent. The proof is by induction on n. For $n = 1$, the result is just Proposition 2.2. Assume that (4.2) holds true for $n = m - 1 \geq 1$. Let $g\colon \mathbb{R}^m \to \mathbb{R}$ be an increasing and component-wise concave function, and let $u \in \mathcal{U}_\gamma$. Then $u(g(x_1, \ldots, x_m)) \in \mathcal{U}_\gamma$ with respect to x_j with other x_i fixed, and hence

$$\begin{aligned}\mathbb{E}[u(g(X_1, X_2, \ldots, X_m)) \mid X_1 = x] &= \mathbb{E}[u(g(x, X_2, \ldots, X_m))] \\ &\leq \mathbb{E}[u(g(x, Y_2, \ldots, Y_m))] \\ &= \mathbb{E}[u(g(X_1, Y_2, \ldots, Y_m)) \mid X_1 = x],\end{aligned}$$

where the equality follows from the independence of all random variables, and the inequality follows from the induction assumption. Thus

$$\mathbb{E}[u(g(X_1, X_2, \ldots, X_m))] \leq \mathbb{E}[u(g(X_1, Y_2, \ldots, Y_m))].$$

Similarly, we have

$$\mathbb{E}[u(g(X_1, Y_2, \ldots, Y_m))] \leq \mathbb{E}[u(g(Y_1, Y_2, \ldots, Y_m))].$$

This proves the desired result. □

From Proposition 4.3 we obtain the following corollary by observing that $\min\{x_1, \ldots, x_n\}$ is an increasing and component-wise concave function.

Corollary 4.1. *Let* $X_1, \ldots, X_n$ *be a set of independent random variables, and let* $Y_1, \ldots, Y_n$ *be another set of independent random variables. If* $X_i \preccurlyeq_{(1+\gamma)\text{-SD}} Y_i$ *for* $i = 1, \ldots, n$ *and* $\gamma \in [0, 1]$*, then*

$$\min\{X_1, X_2, \ldots, X_n\} \preccurlyeq_{(1+\gamma)\text{-SD}} \min\{Y_1, Y_2, \ldots, Y_n\}.$$

For $\gamma = 1$, Corollary 4.1 for SSD was implicitly given in [17]; see, for example, the paragraph after Corollary 4.A.16 in [26].

4.4. Closure under distortion

Under suitable conditions, $(1+\gamma)$-SD is preserved under a distortion transformation on the space of distribution functions.

Proposition 4.4. *Let F and G be two distribution functions such that* $F \preccurlyeq_{(1+\gamma)\text{-SD}} G$*, and right continuous* $h \in \mathcal{H}_\beta$ *with* $\gamma \in [0, 1]$*,* $\beta \in (0, 1]$*, and* $\gamma \leq \beta$*. Then* $h(F) \preccurlyeq_{(1+\gamma/\beta)\text{-SD}} h(G)$*.*

Proof. Denote $F_h = h(F)$ and $G_h = h(G)$. Then their left inverse functions are $F_h^{-1}(p) = F^{-1}(h^{-1}(p))$ and $G_h^{-1}(p) = G^{-1}(h^{-1}(p))$, where all the inversion functions are left inverse.

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Then, for any $\phi \in \mathcal{H}^*_{\gamma/\beta}$, we have

$$\int_0^1 F_h^{-1}(\alpha)\,\mathrm{d}\phi(\alpha) = \int_0^1 F^{-1}(h^{-1}(\alpha))\,\mathrm{d}\phi(\alpha) = \int_0^1 F^{-1}(\alpha)\,\mathrm{d}\phi(h(\alpha)).$$

Similarly,

$$\int_0^1 G_h^{-1}(\alpha)\,\mathrm{d}\phi(\alpha) = \int_0^1 G^{-1}(\alpha)\,\mathrm{d}\phi(h(\alpha)).$$

Note that for any $\phi \in \mathcal{H}^*_{\gamma/\beta}$ it can be verified that $\phi(h) \in \mathcal{H}^*_{\gamma}$. To see this, for any $0 \le p_1 < p_2 \le p_3 < p_4 \le 1$ we have

$$\begin{aligned}\gamma\frac{\phi(h(p_4)) - \phi(h(p_3))}{p_4 - p_3} &= \frac{\gamma}{\beta}\frac{\phi(h(p_4)) - \phi(h(p_3))}{h(p_4) - h(p_3)} \cdot \beta\frac{h(p_4) - h(p_3)}{p_4 - p_3}\\ &\le \frac{\phi(h(p_2)) - \phi(h(p_1))}{h(p_2) - h(p_1)} \cdot \frac{h(p_2) - h(p_1)}{p_2 - p_1}\\ &= \frac{\phi(h(p_2)) - \phi(h(p_1))}{p_2 - p_1},\end{aligned}$$

where the inequality follows from the fact that $\phi \in \mathcal{H}^*_{\gamma/\beta}$, $h \in \mathcal{H}^*_{\beta}$, and h is increasing. Then the desired result follows immediately from Theorem 3.1 (iii). □

For SSD ($\gamma = \beta = 1$), Proposition 4.4 was implicitly given in Theorem 4.2 of [29], which was proved by using the fact that any concave $h \in \mathcal{H}$ can be approximated by a sequence of piecewise linear concave distortion functions of the form $h_\alpha(t) = \min\{t/\alpha, 1\}$, $0 < \alpha \le 1$.

4.5. Equivalence characterization

In the expected utility theory, a decision-maker is risk-averse if she has an increasing and concave utility function. The next proposition states that, for two risks X and Y satisfying $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$, if a risk-averse decision-maker is indifferent between X and Y, then X and Y are identically distributed.

Proposition 4.5. *Let $\gamma \in [0, 1)$, and let X and Y be two random variables such that $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$. If $\mathbb{E}[\phi(X)] = \mathbb{E}[\phi(Y)]$ for some strictly increasing and concave function ϕ, then $X \overset{\mathrm{d}}{=} Y$.*

Proof. By Proposition 4.3, it suffices to show the case when $\phi(x) = x$ for $x \in \mathbb{R}$, i.e. $\mathbb{E}[X] = \mathbb{E}[Y]$. Let F and G denote the distribution functions of X and Y, respectively. By Proposition 2.1, we have that $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$ if and only if (2.3) holds, that is,

$$(1-\gamma)\int_{-\infty}^{t} (G(x) - F(x))_+ \mathrm{d}x \le \gamma \int_{-\infty}^{t} (F(x) - G(x))\,\mathrm{d}x \quad \text{for all } t \in \mathbb{R}. \tag{4.3}$$

Note that

$$\mathbb{E}[Y] - \mathbb{E}[X] = \int_{-\infty}^{\infty} (F(x) - G(x))\,\mathrm{d}x.$$

Then, taking $t \to \infty$ in (4.3) yields

$$(1-\gamma)\int_{-\infty}^{\infty} (G(x) - F(x))_+ \mathrm{d}x \le \gamma(\mathbb{E}[Y] - \mathbb{E}[X]) = 0,$$

which implies that $(G(x)-F(x))_+=0$ for all $x\in\mathbb{R}$, i.e. $G(x)\le F(x)$ for all $x\in\mathbb{R}$. Thus we have $X\preccurlyeq_{\rm st} Y$. By $\mathbb{E}[X]=\mathbb{E}[Y]$, it follows from Theorem 1.A.8 of [26] that $X\overset{\rm d}{=}Y$. This completes the proof. □

In the literature, several authors have investigated conditions under which ordered random variables are equal in distribution; see, for example, [3], [4], [6], and [16].

An immediate consequence of Proposition 4.5 is the following corollary.

Corollary 4.2. *Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be two collections of independent and identically distributed random variables. If $X_1\preccurlyeq_{(1+\gamma)\text{-SD}} Y_1$ and*

$$\mathbb{E}\Big[\min_{1\le i\le n} X_i\Big]=\mathbb{E}\Big[\min_{1\le i\le n} Y_i\Big],$$

then $X_1\overset{\rm d}{=}Y_1$.

Proof. From Corollary 4.1 it follows that $\min_{1\le i\le n} X_i\}\preccurlyeq_{(1+\gamma)\text{-SD}} \min_{1\le i\le n} Y_i$. By Proposition 4.5, $\mathbb{E}[\min_{1\le i\le n} X_i]=\mathbb{E}[\min_{1\le i\le n} Y_i]$ implies that $\min_{1\le i\le n} X_i\overset{\rm d}{=}\min_{1\le i\le n} Y_i$. Therefore, by the relation between the survival functions of X_1 and $\min_{1\le i\le n} X_i$, we have $X_1\overset{\rm d}{=}Y_1$. □

5. Examples

In this section we present several examples of distributions ordered with respect to the order $\preccurlyeq_{(1+\gamma)\text{-SD}}$ except for those given in [25], and also give some applications of the main results in the previous section.

Example 5.1. (*Binary distribution.*) Let X and Y be two binary random variables with PMFs given by

$$\mathbb{P}(X=x_1)=p=1-\mathbb{P}(X=x_2)\quad\text{and}\quad \mathbb{P}(Y=y_1)=q=1-\mathbb{P}(Y=y_2),$$

where $x_1<x_2$ and $y_1<y_2$, and assume that $X\preccurlyeq_{(1+\gamma)\text{-SD}} Y$ for some $\gamma\in(0,1]$. Denote $X\sim F$ and $Y\sim G$. From (2.3), it follows that $x_1\le y_1$ and

$$x_1p+x_2(1-p)\le y_1q+y_2(1-q). \tag{5.1}$$

If $y_2>x_2$, then define a new random variable Y^* such that $\mathbb{P}(Y^*=y_1)=q=1-\mathbb{P}(Y^*=x_2)$. Then $X\preccurlyeq_{(1+\gamma)\text{-SD}} Y^*$ if and only if $X\preccurlyeq_{(1+\gamma)\text{-SD}} Y$. So, without loss of generality, assume that $x_1\le y_1<y_2\le x_2$ and (5.1) holds. Then

$$G(x)-F(x)=\begin{cases}-p & \text{for } x_1\le x<y_1,\\ q-p & \text{for } y_1\le x<y_2,\\ 1-p & \text{for } y_2\le x<x_2,\\ 0 & \text{otherwise.}\end{cases}$$

It follows that $X\preccurlyeq_{(1+\gamma)\text{-SD}} Y$ if and only if

$$(q-p)_+(y_2-y_1)+(1-p)(x_2-y_2)\le\gamma[p(y_1-x_1)+(q-p)_-(y_2-y_1)]$$

or, equivalently,

$$\gamma\ge\frac{(q-p)_+(y_2-y_1)+(1-p)(x_2-y_2)}{p(y_1-x_1)+(q-p)_-(y_2-y_1)}.$$

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Example 5.2. (*Special transfer.*) Let Y be a discrete random variable with PMF given by

Y	x_1	x_2	$\cdots$	x_n
Prob.	p_1	p_2	$\cdots$	p_n

where $x_1 < x_2 < \cdots < x_n$. Let $0 < \alpha < 1$ and $P_i = \sum_{j=1}^{i} p_i$ for each i. There exists a k such that $P_{k-1} < \alpha \le P_k$, where $P_0 = 0$. Define a random variable X with PMF given by

X	$x_1 - a$	$\cdots$	$x_{k-1} - a$	$x_k - a$	$x_k + b$	$x_{k+1} + b$	$\cdots$	$x_n + b$
Prob.	p_1	$\cdots$	p_{k-1}	$a - P_{k-1}$	$P_k - \alpha$	p_{k+1}	$\cdots$	p_n

where $a, b > 0$ such that $(1-\alpha)b \le \alpha a$. Denote $X \sim F$ and $Y \sim G$. Obviously, F single-crosses G from above at point x_k. It is easy to see that

$$A := \int_{-\infty}^{x_k} [F(x) - G(x)]\,\mathrm{d}x = \alpha a$$

and

$$B := \int_{x_k}^{\infty} [G(x) - F(x)]\,\mathrm{d}x = (1-\alpha)b.$$

From Corollary 2.5 in [25], it follows that $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$ if and only if

$$\gamma \ge \frac{B}{A} = \frac{(1-\alpha)b}{\alpha a},$$

and that $X \preccurlyeq_{(1+\gamma)\text{-PSD}} Y$ if and only if $\gamma = (1-\alpha)b/(\alpha a)$.

Example 5.3. (*Application of the separation theorem.*) Let X be a discrete random variable with PMF given by

X	0	1	2	3	4	5
Prob.	1/8	1/12	1/4	1/8	7/24	1/8

and let Y be another random variable with $\mathbb{P}(Y=2) = \mathbb{P}(Y=4) = 1/2$. We claim that $X \preccurlyeq_{(1\,+\,1/2)\text{-SD}} Y$. To verify this assertion, define a discrete random variable X_1 with PMF given by

X_1	0	2	3	4
Prob.	1/8	1/4	1/8	1/2

It can be checked that X_1 is a $1/2$-transfer of X, and Y is a $1/2$-transfer of X_1, and hence $X \preccurlyeq_{(1\,+\,1/2)\text{-PSD}} X_1$ and $X_1 \preccurlyeq_{(1+1/2)\text{-PSD}} Y$. This implies $X \preccurlyeq_{(1+1/2)\text{-SD}} Y$ (but $X \not\preccurlyeq_{(1+1/2)\text{-PSD}} Y$). Now we apply the method in the proof of Theorem 3.2 to construct two random variables Z_1 and Z_2 such that

$$X \preccurlyeq_{(1+1/2)\text{-PSD}} Z_1 \preccurlyeq_{\text{FSD}} Y, \qquad X \preccurlyeq_{\text{FSD}} Z_2 \preccurlyeq_{(1+1/2)\text{-PSD}} Y.$$

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Let $X \sim F$, $Y \sim G$, $Z_1 \sim H_1$, and $Z_2 \sim H_2$, and let $\delta_t(x)$ and $\eta_t(x)$ be defined by (3.3) and (3.7), respectively. Then $t_0 = 3$ in (3.4) and $t_1 = 2/3$ in (3.8). Hence

$$H_1(x) = G(x) + \delta_{t_0}(x), \quad x \in \mathbb{R},$$

and

$$H_2(x) = F(x) - \eta_{t_1}(x), \quad x \in \mathbb{R}.$$

Therefore the PMFs of Z_1 and Z_2, respectively, are given by

Z_1	2	3	4
Prob.	1/2	1/12	5/12

and

Z_2	2/3	1	2	3	4	5
Prob.	1/8	1/12	1/4	1/8	7/24	1/8

Example 5.4. (*Uniform distribution.*) Let X and Y be random variables uniformly distributed over the intervals (a,b) and (c,d), respectively, and assume that $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$ for some $\gamma \in (0, 1]$. From (2.3) it follows that $a \le c$ and $X \preccurlyeq_{\text{SSD}} Y$, which implies $\mathbb{E}[X] \le \mathbb{E}[Y]$ (i.e. $a + b \le c + d$). If $d > b$, then $X \preccurlyeq_{\text{FSD}} Y$. Without loss of generality, assume that $a < c < d \le b$ and $a + b \le c + d$. Denote $X \sim F$ and $Y \sim G$. Then F single-crosses G at $x_0 \in (c, d)$ from above, where

$$x_0 = a + \frac{(b-a)(c-a)}{b+c-a-d} = c + \frac{(d-c)(c-a)}{b+c-a-d}.$$

It is easy to check that

$$A := \int_{-\infty}^{x_0} [F(x) - G(x)]\, \mathrm{d}x = \frac{(c-a)^2}{2(b+c-a-d)}$$

and

$$B := \int_{x_0}^{\infty} [G(x) - F(x)]\, \mathrm{d}x = \frac{(c-a)^2}{2(b+c-a-d)} + \frac{1}{2}(b+a-c-d).$$

From Corollary 2.5 in [25] it follows that $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$ if and only if

$$\gamma \ge \frac{B}{A} = \left(\frac{b-d}{c-a}\right)^2.$$

Example 5.5. (*Shifted exponential distribution.*) Let X and Y be two random variables with respective density functions given by $f(x) = \lambda \mathrm{e}^{-\lambda(x-a)}$ for $x > a$ and $g(y) = \mu \mathrm{e}^{-\mu(y-b)}$ for $y > b$, where $a, b \in \mathbb{R}$ and $\lambda, \mu > 0$. Assume that $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$ for $\gamma \in (0, 1]$. It is known that $X \preccurlyeq_{\text{SSD}} Y$ if and only if $a \le b$ and $\delta := b + 1/\mu - a - 1/\lambda \ge 0$. If $a \le b$ and $\lambda \ge \mu$, then $X \preccurlyeq_{\text{FSD}} Y$. So, assume without loss of generality that $a \le b$, $\lambda < \mu$ and $\delta \ge 0$. Then F single-crosses G at $x_0 \in (c, d)$ from above, where

$$x_0 = a + \frac{\mu(b-a)}{\mu-\lambda} = b + \frac{\lambda(b-a)}{\mu-\lambda}.$$

It is easy to check that

$$A := \int_{-\infty}^{x_0} [F(x) - G(x)]\, \mathrm{d}x = \delta + \Delta \quad \text{and} \quad B := \int_{x_0}^{\infty} [G(x) - F(x)]\, \mathrm{d}x = \Delta,$$

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where

$$\Delta := \left(\frac{1}{\lambda} - \frac{1}{\mu}\right) \exp\left\{-\frac{\lambda\mu(b-a)}{\mu-\lambda}\right\} > 0.$$

From Corollary 2.5 in [25] it follows that $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$ if and only if

$$\gamma \geq \frac{B}{A} = \frac{\Delta}{\delta + \Delta}.$$

Example 5.6. (*Application of bivariate characterization.*) Choose two real numbers $a < b$ such that $2a \geq b$, and define

$$\gamma := \frac{2a-b}{2b-a} \in [0, 1].$$

Let ψ be any differentiable function with $a \leq \psi'(x) \leq b$ for all $x \in \mathbb{R}$. Then $\phi(x, y) := \psi(2x + y) \in \mathcal{G}_\gamma$, where $\mathcal{G}_\gamma$ is defined by (3.11). If $X \preccurlyeq_{(1+\gamma)\text{-SD}} Y$, and X and Y are independent, then

$$\mathbb{E}[\psi(X + 2Y)] \leq \mathbb{E}[\psi(2X + Y)]$$

by Proposition 3.4.

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