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SPECTRALITY OF SELF-AFFINE MEASURES ON THE THREE-DIMENSIONAL SIERPINSKI GASKET

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Abstract The self-affine measure $\mu_{M,D}$ corresponding to $M = \mathrm{diag}[p_1, p_2, p_3]$ ($p_j \in \mathbb{Z} \setminus \{0, \pm 1\}$, $j = 1, 2, 3$) and $D = \{0, e_1, e_2, e_3\}$ in the space $\mathbb{R}^3$ is supported on the three-dimensional Sierpinski gasket $T(M, D)$, where e_1, e_2, e_3 are the standard basis of unit column vectors in $\mathbb{R}^3$. We shall determine the spectrality and non-spectrality of $\mu_{M,D}$, and show that if $p_j \in 2\mathbb{Z} \setminus \{0, 2\}$ for $j = 1, 2, 3$, then $\mu_{M,D}$ is a spectral measure, and if $p_j \in (2\mathbb{Z} + 1) \setminus \{\pm 1\}$ for $j = 1, 2, 3$, then $\mu_{M,D}$ is a non-spectral measure and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, where the number 4 is the best possible. This generalizes the known results on the spectrality of self-affine measures.

Keywords: iterated function system; self-affine measure; orthogonal exponentials; spectrality

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1. Introduction

This paper is concerned with harmonic analysis on certain fractals. This class falls within more general fractal constructions. In a variety of different contexts, this analysis typically involves notions of repeated similarity and, in certain cases, similarity may be defined by a fixed scale number or scaling matrix that is then iterated 'in the small' through a prescribed algorithm. In each iteration step, 'small' components are then repeated according to fixed rules. Since the general situation can be complicated, it is helpful to look for patterns in special examples. An important class of fractals is defined from a system of *affine mappings* in Euclidean space of a fixed dimension: the so-called *affine fractals*. In general, the affine mappings in $\mathbb{R}^n$ can be written in the following form:

$$\phi_d(x) = M^{-1}(x + d), \quad d \in D,\ x \in \mathbb{R}^n,$$

where $M \in M_n(\mathbb{Z})$ is an expanding integer matrix and $D \subset \mathbb{Z}^n$ is a finite digit set of cardinality $|D|$. For a given pair (M, D), the affine fractal is a unique non-empty compact subset $T := T(M, D) \subset \mathbb{R}^n$ such that $T = \bigcup_{d \in D} \phi_d(T)$. More precisely, $T(M, D)$ is an *attractor* (or *invariant set*) of the affine iterated function system (IFS) $\{\phi_d(x)\}_{d \in D}$. In this affine case, the scaling law is defined by a fixed matrix M^{-1}, and at each level in the iteration a system of translations is applied, and of course the translation vectors scale as well. Each fractal defined by a contractive scale is naturally equipped with a unique *invariant probability measure* $\mu := \mu_{M,D}$; invariance is defined from the given affine system $\{\phi_d(x)\}_{d\in D}$ by

$$\mu = \frac{1}{|D|}\sum_{d\in D} \mu \circ \phi_d^{-1}. \tag{1.1}$$

The existence of $\mu_{M,D}$ is a construction due to Hutchinson [**12**], and $\mu_{M,D}$ is supported on $T(M, D)$. The invariant measure $\mu_{M,D}$ is also called the *self-affine measure*, and it may be thought of as a generalization of the more familiar Haar measure for compact groups. Since the affine fractals can be realized in $\mathbb{R}^n$ for some n, it is natural to ask whether or not the Hilbert space $L^2(\mu_{M,D})$ has an orthogonal family of complex exponentials: a Fourier basis. In successful constructions, it has been shown that Fourier bases may indeed be obtained by a dual scaling in the large. However, many open questions remain.

The question of the existence of an orthogonal family of complex exponentials was first raised, and answered, for $n = 1$ in the case of families of Cantor constructions. The results were somewhat surprising. It turned out that, when the affine fractal is specified, the answer to the existence question for a Fourier basis in the corresponding $L^2(\mu)$ is sensitive to choice of scaling number. Jorgensen and Pedersen [**15**] showed that a Fourier basis exists when the scaling number is an even integer, but not in the odd case. In later papers, by a number of different authors, the same questions were addressed for Sierpinski constructions in two and three dimensions (i.e. $n = 2, 3$). The case when $n = 3$ is also called the fractal Eiffel tower. Indeed, for a number of reasons, the case when $n = 3$ has received relatively more attention in the literature, but so far only scaling by the same number in each of the three directions has been considered (see, for example, [**15**, Example 7.1], [**26**], [**27**, Example 2.9 (e)], [**14**], [**4**, Theorem 5.1 (iii)], [**23**, Theorem 1]). The present paper contains interesting results (see Theorem 1.1, below) when a different scale is allowed in the separate directions.

The question proposed above also has its origin in analysis and geometry. For a probability measure μ of compact support on $\mathbb{R}^n$, we call μ a *spectral measure* if there exists a discrete set $\Lambda \subset \mathbb{R}^n$ such that $E(\Lambda) := \{\exp(2\pi i\langle\lambda, x\rangle) \colon \lambda \in \Lambda\}$ forms an orthogonal basis (Fourier basis) for $L^2(\mu)$. The set Λ is then called a *spectrum* for μ; we also say that (μ, Λ) is a *spectral pair*. Spectral measure is a generalization of the *spectral set*. The notion of the spectral set was introduced by Fuglede [**10**], whose famous spectrum-tiling conjecture motivated the previous research (see, for example, [**6**, **8**, **17**]). In recent years, research on the spectrality or non-spectrality of a self-affine measure $\mu_{M,D}$ has received much attention, following the pioneering work of Jorgensen and Pedersen [**15**] (see, for example, [**1**–**3**, **5**, **7**, **9**, **11**, **13**, **16**, **18**–**22**, **24**, **25**]). The Fourier series associated with a spectrum of spectral Cantor measure or spectral self-affine measure is called a *mock Fourier series*, and has some interesting properties compared with the ordinary Fourier series [**27**, **28**]. Until now, the spectral or non-spectral problem of a self-affine measure has focused mainly on the conditions under which $\mu_{M,D}$ is a spectral measure or a non-spectral measure. There are two conjectures related to the problem (see, for example, [**5**, Conjecture 2.5], [**7**, Conjecture 1.1], [**6**, Problem 1] and [**21**, Conjectrue 1]), which are still open even for the Sierpinski family.

Recall that the self-affine measure $\mu_{M,D}$ corresponding to

$$\left.\begin{aligned} M &= \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}, \quad p_1, p_2, p_3 \in \mathbb{Z} \setminus \{0, \pm 1\}, \\ D &= \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \end{aligned}\right\} \tag{1.2}$$

is supported on the three-dimensional Sierpinski gasket $T(M, D)$. Because of the efforts of Jorgensen and Pedersen (see [**15**, Example 7.1] and [**14**]), Strichartz (see [**26**] and [**27**, Example 2.9 (e)]), Dutkay and Jorgensen (see [**4**, Theorem 5.1 (iii)]) and Li (see [**23**, Theorem 1]), the spectrality and non-spectrality of the self-affine measure $\mu_{M,D}$ are known only in the case when $p_1 = p_2 = p_3 = p$: if $p \in (2\mathbb{Z}) \setminus \{0\}$, then $\mu_{M,D}$ is a spectral measure; if $p \in (2\mathbb{Z} + 1) \setminus \{\pm 1\}$, then $\mu_{M,D}$ is a non-spectral measure, and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, where the number 4 is the best possible. The general case for the spectrality or non-spectrality of the self-affine measure $\mu_{M,D}$ is not known.

Motivated by the above problem on the self-affine measure, we shall determine the spectrality and non-spectrality of self-affine measure $\mu_{M,D}$ for the three-dimensional Sierpinski gasket (1.2). The main result of the paper is the following.

Theorem 1.1. *For the self-affine measure $\mu_{M,D}$ corresponding to (1.2), the following spectrality and non-spectrality hold:*

(i) *if $p_j \in 2\mathbb{Z} \setminus \{0, 2\}$ for $j = 1, 2, 3$, then $\mu_{M,D}$ is a spectral measure;*

(ii) *if $p_j \in (2\mathbb{Z} + 1) \setminus \{\pm 1\}$ for $j = 1, 2, 3$, then $\mu_{M,D}$ is a non-spectral measure, and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. Furthermore, it is possible to construct examples for which there are four mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. This shows that the number 4 is the best possible (i.e. smallest) upper bound for the number of mutually orthogonal exponential functions in $L^2(\mu_{M,D})$.*

This extends the known results on the spectrality of the self-affine measure $\mu_{M,D}$, which is supported on the three-dimensional Sierpinski gasket (see [**15**, Example 7.1], [**26**], [**27**, Example 2.9 (e)], [**14**], [**4**, Theorem 5.1 (iii)] and [**23**, Theorem 1]).

The paper is organized as follows. In § 2 we review some known results on the spectral self-affine measures. We then prove Theorem 1.1 (i) by applying a result of Strichartz [**26**]. In § 3 we prove Theorem 1.1 (ii) in several typical cases. This proof is based on the detailed analysis of the zero set $Z(\hat{\mu}_{M,D}(\xi))$ of the Fourier transform $\hat{\mu}_{M,D}(\xi)$ and the careful choice of each case. We obtain a method to deal with each case (only the typical cases are presented). Note that our method is different from the previously known method. Finally, we give a supplement to the planar Sierpinski family in § 4.

2. Proof of Theorem 1.1 (i)

Let B and P be finite subsets of $\mathbb{R}^n$ of the same cardinality q. We say that (B,P) is a *compatible pair* if the $q\times q$ matrix $H_{B,P} := [q^{-1/2}\exp(2\pi \mathrm{i}\langle b,p\rangle)]_{b\in B,p\in P}$ is unitary, i.e. $H_{B,P}H^*_{B,P} = I_q$. Here we use $*$ to denote the transposed conjugate. For a given pair (M,D), the spectrality or non-spectrality of $\mu_{M,D}$ is directly connected with the Fourier transform $\hat{\mu}_{M,D}(\xi)$. From (1.1), we have

$$\hat{\mu}_{M,D}(\xi) := \int \exp(2\pi \mathrm{i}\langle x,\xi\rangle)\,\mathrm{d}\mu_{M,D}(x) = \prod_{j=1}^{\infty} m_D(M^{*-j}\xi), \quad \xi\in\mathbb{R}^n, \tag{2.1}$$

where

$$m_D(x) = \frac{1}{|D|}\sum_{d\in D}\exp(2\pi \mathrm{i}\langle d,x\rangle), \quad x\in\mathbb{R}^n. \tag{2.2}$$

It has been conjectured by Dutkay and Jorgensen (see [**5**, Conjecture 2.5], [**7**, Conjecture 1.1] and also [**6**, Problem 1]) that for an expanding integer matrix $M\in M_n(\mathbb{Z})$ and a finite digit set $D\subset\mathbb{Z}^n$ with $0\in D$, if there exists a subset $S\subset\mathbb{Z}^n$, $0\in S$, such that $(M^{-1}D,S)$ is a compatible pair (or (M,D,S) is a Hadamard triple), then $\mu_{M,D}$ is a spectral measure. This conjecture holds in the dimension $n=1$ [**16**], in the case when $|D|=|S|=|\det(M)|$ [**18**] and in the case when $|\det(M)|$ is a prime number [**19**]; it also holds in higher dimensions with additional conditions (cf. [**3**, **5**, **7**]). With the above existence S, the dual IFS $\{\psi_s(x)=M^*x+s\}_{s\in S}$ and its invariant set $\Lambda(M,S)$ (fractals in the large) play an important role. Usually, we use $\Lambda(M,S)$ to denote the expansive orbit of 0 under $\{\psi_s(x)\}_{s\in S}$, that is

$$\Lambda(M,S) := \left\{\sum_{j=0}^{k-1} M^{*j}s_j : k\geqslant 1 \text{ and all } s_j\in S\right\}, \tag{2.3}$$

Since $M^*\Lambda(M,S)+S=\Lambda(M,S)$, we also call $\Lambda(M,S)$ the *invariant set* of the dual IFS $\{\psi_s(x)\}_{s\in S}$. In the above conjecture of Dutkay and Jorgensen, we know that $E(\Lambda(M,S))$ is an infinite orthogonal system in $L^2(\mu_{M,D})$ [**14**], but it need not be an orthogonal basis (Fourier basis) for $L^2(\mu_{M,D})$ [**18**, Example 4.4]. To ensure the completeness of $E(\Lambda(M,S))$ in $L^2(\mu_{M,D})$, Strichartz [**26**] obtained the following theorem, which can be applied to prove Theorem 1.1 (i).

Theorem 2.1. *Let $M\in M_n(\mathbb{Z})$ be expanding, let D and S be finite subsets of $\mathbb{Z}^n$ such that $(M^{-1}D,S)$ is a compatible pair and let $0\in D\cap S$. Suppose that the zero set $Z(m_{M^{-1}D}(x))$ of the function $m_{M^{-1}D}(x)$ is disjoint from the set $T(M^*,S)$. Then $\Lambda(M,S)$ is a spectrum for $\mu_{M,D}$.*

Proof of Theorem 1.1 (i). Firstly, for the pair (M,D) given by (1.2) with even numbers p_1, p_2 and p_3, we can choose the set

$$S = \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}p_1/2\\p_2/2\\0\end{pmatrix}, \begin{pmatrix}0\\p_2/2\\p_3/2\end{pmatrix}, \begin{pmatrix}p_1/2\\0\\p_3/2\end{pmatrix}\right\} \tag{2.4}$$

in $\mathbb{Z}^3$ such that $(M^{-1}D, S)$ is a compatible pair. Then, the invariant set $T(M^*, S)$ is given by

$$\begin{aligned}
T(M^*, S) &= \left\{ \sum_{j=1}^{\infty} M^{*-j} s_j \colon s_j \in S \right\} \\
&= \left\{ \sum_{j=1}^{\infty} \begin{bmatrix} 1/p_1^j & 0 & 0 \\ 0 & 1/p_2^j & 0 \\ 0 & 0 & 1/p_3^j \end{bmatrix} \begin{pmatrix} s_{1,j} \\ s_{2,j} \\ s_{3,j} \end{pmatrix} : \begin{pmatrix} s_{1,j} \\ s_{2,j} \\ s_{3,j} \end{pmatrix} \in S \right\} \\
&= \left\{ \sum_{j=1}^{\infty} \begin{pmatrix} s_{1,j}/p_1^j \\ s_{2,j}/p_2^j \\ s_{3,j}/p_3^j \end{pmatrix} : \begin{pmatrix} s_{1,j} \\ s_{2,j} \\ s_{3,j} \end{pmatrix} \in S \right\},
\end{aligned} \tag{2.5}$$

which shows that for any $x = (x_1, x_2, x_3)^{\mathrm{T}} \in T(M^*, S)$ we have

$$|x_1| \leqslant \frac{|p_1|}{2(|p_1| - 1)}, \qquad |x_2| \leqslant \frac{|p_2|}{2(|p_2| - 1)}, \qquad |x_3| \leqslant \frac{|p_3|}{2(|p_3| - 1)}. \tag{2.6}$$

So, if $p_1, p_2, p_3 \in (2\mathbb{Z}) \setminus \{0, \pm 2\}$, then $T(M^*, S) \subseteq [-\frac{2}{3}, \frac{2}{3}]^3$. Furthermore, it follows from (2.5) that if $p_j = -2$ for a fixed $j \in \{1, 2, 3\}$, then the point $x = (x_1, x_2, x_3)^{\mathrm{T}} \in T(M^*, S)$ satisfies $x_j \in [-\frac{1}{3}, \frac{2}{3}]$ for the same $j \in \{1, 2, 3\}$. Hence,

$$T(M^*, S) \subseteq [-\tfrac{2}{3}, \tfrac{2}{3}]^3 \quad \text{if } p_1, p_2, p_3 \in (2\mathbb{Z}) \setminus \{0, 2\}. \tag{2.7}$$

Secondly, for the given digit set D in (1.2), we have

$$Z(m_D(x)) := \{x \in \mathbb{R}^3 \colon m_D(x) = 0\} = A_1 \cup A_2 \cup A_3, \tag{2.8}$$

where

$$m_D(x) = \tfrac{1}{4}\{1 + \exp(2\pi \mathrm{i} x_1) + \exp(2\pi \mathrm{i} x_2) + \exp(2\pi \mathrm{i} x_3)\}, \quad x = (x_1, x_2, x_3)^{\mathrm{T}} \in \mathbb{R}^3,$$

and A_1, A_2, A_3 are given by

$$\left.\begin{aligned}
A_1 &= \left\{ \begin{pmatrix} \frac{1}{2} + k_1 \\ a + k_2 \\ \frac{1}{2} + a + k_3 \end{pmatrix} : a \in \mathbb{R}, k_1, k_2, k_3 \in \mathbb{Z} \right\} \subset \mathbb{R}^3, \\
A_2 &= \left\{ \begin{pmatrix} \frac{1}{2} + a + k_1 \\ \frac{1}{2} + k_2 \\ a + k_3 \end{pmatrix} : a \in \mathbb{R}, k_1, k_2, k_3 \in \mathbb{Z} \right\} \subset \mathbb{R}^3, \\
A_3 &= \left\{ \begin{pmatrix} a + k_1 \\ \frac{1}{2} + a + k_2 \\ \frac{1}{2} + k_3 \end{pmatrix} : a \in \mathbb{R},\ k_1, k_2, k_3 \in \mathbb{Z} \right\} \subset \mathbb{R}^3.
\end{aligned}\right\} \tag{2.9}$$

So, the zero set $Z(m_{M^{-1}D}(x))$ of the function $m_{M^{-1}D}(x)=m_D(M^{*-1}x)$ is given by

$$Z(m_{M^{-1}D}(x))=M^*Z(m_D(x))=M^*A_1\cup M^*A_2\cup M^*A_3. \tag{2.10}$$

Now, if $p_1,p_2,p_3\in(2\mathbb{Z})\setminus\{0,2\}$, then for any $x=(x_1,x_2,x_3)^{\mathrm{T}}\in Z(m_{M^{-1}D}(x))$, we observe from (2.9) and (2.10) that

$$\left.\begin{aligned}|x_1|=|(\tfrac{1}{2}+k_1)p_1|\geqslant 1 \quad &\text{if } k_1\in\mathbb{Z} \text{ and } x\in M^*A_1,\\ |x_2|=|(\tfrac{1}{2}+k_2)p_2|\geqslant 1 \quad &\text{if } k_2\in\mathbb{Z} \text{ and } x\in M^*A_2,\\ |x_3|=|(\tfrac{1}{2}+k_3)p_3|\geqslant 1 \quad &\text{if } k_3\in\mathbb{Z} \text{ and } x\in M^*A_3.\end{aligned}\right\} \tag{2.11}$$

This shows that

$$Z(m_{M^{-1}D}(x))\cap[-\tfrac{2}{3},\tfrac{2}{3}]^3=\emptyset. \tag{2.12}$$

Finally, it follows from (2.7), (2.12) and Theorem 2.1 that $\Lambda(M,S)$ with $S\subset\mathbb{Z}^3$ given by (2.4) is a spectrum for $\mu_{M,D}$. That is, $\mu_{M,D}$ is a spectral measure with spectrum $\Lambda(M,S)$. This completes the proof of Theorem 1.1 (i). □

Since the spectrality or non-spectrality of a self-affine measure is invariant under the similarity [**18**, p. 208], Theorem 1.1 (i) can be stated in a more general form. For example, we have the following corollary.

Corollary 2.2. *If M and D are given by*

$$M=\begin{bmatrix}p_1&0&0\\0&p_2&0\\0&0&p_3\end{bmatrix} \quad \textit{and} \quad D=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}d_1\\0\\0\end{pmatrix},\begin{pmatrix}0\\d_2\\0\end{pmatrix},\begin{pmatrix}0\\0\\d_3\end{pmatrix}\right\}, \tag{2.13}$$

where $p_1,p_2,p_3\in 2\mathbb{Z}\setminus\{0,2\}$ and $d_1,d_2,d_3\in\mathbb{R}\setminus\{0\}$, then $\mu_{M,D}$ is a spectral measure.

With the same method as above, we also have the following.

Corollary 2.3. *If M and D are given by*

$$M=\begin{bmatrix}0&0&p_1\\0&p_2&0\\p_3&0&0\end{bmatrix} \quad \textit{and} \quad D=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}d_1\\0\\0\end{pmatrix},\begin{pmatrix}0\\d_2\\0\end{pmatrix},\begin{pmatrix}0\\0\\d_1\end{pmatrix}\right\}, \tag{2.14}$$

where $p_1,p_2,p_3\in 2\mathbb{Z}\setminus\{0,2\}$ and $d_1,d_2\in\mathbb{R}\setminus\{0\}$, then $\mu_{M,D}$ is a spectral measure.

3. Proof of Theorem 1.1 (ii)

Firstly, we know from (2.1) and (2.8) that the zero set $Z(\hat{\mu}_{M,D}(\xi))$ of the Fourier transform $\hat{\mu}_{M,D}(\xi)$ is

$$Z(\hat{\mu}_{M,D}(\xi))=\bigcup_{j=1}^{\infty}M^{*j}Z(m_D(\xi)):=B_1\cup B_2\cup B_3, \tag{3.1}$$

where

$$B_1 = \bigcup_{j=1}^{\infty} M^{*j} A_1 = \bigcup_{j=1}^{\infty} \left\{ \begin{pmatrix} (\frac{1}{2}+k_1)p_1^j \\ (a+k_2)p_2^j \\ (\frac{1}{2}+a+k_3)p_3^j \end{pmatrix} : a \in \mathbb{R},\ k_1,k_2,k_3 \in \mathbb{Z} \right\} \subset \mathbb{R}^3, \tag{3.2}$$

$$B_2 = \bigcup_{j=1}^{\infty} M^{*j} A_2 = \bigcup_{j=1}^{\infty} \left\{ \begin{pmatrix} (\frac{1}{2}+a+k_1)p_1^j \\ (\frac{1}{2}+k_2)p_2^j \\ (a+k_3)p_3^j \end{pmatrix} : a \in \mathbb{R},\ k_1,k_2,k_3 \in \mathbb{Z} \right\} \subset \mathbb{R}^3 \tag{3.3}$$

and

$$B_3 = \bigcup_{j=1}^{\infty} M^{*j} A_3 = \bigcup_{j=1}^{\infty} \left\{ \begin{pmatrix} (a+k_1)p_1^j \\ (\frac{1}{2}+a+k_2)p_2^j \\ (\frac{1}{2}+k_3)p_3^j \end{pmatrix} : a \in \mathbb{R},\ k_1,k_2,k_3 \in \mathbb{Z} \right\} \subset \mathbb{R}^3. \tag{3.4}$$

Since $p_1, p_2, p_3 \in (2\mathbb{Z}+1) \setminus \{\pm 1\}$, we can verify directly that the following two lemmas hold.

Lemma 3.1. *The sets B_j, $j = 1,2,3$, given by (3.2)–(3.4) satisfy the following properties:*

(a) $\xi \in B_j$ *if and only if* $-\xi \in B_j$, $j = 1,2,3$;

(b) $Z(\hat{\mu}_{M,D}(\xi)) \cap \mathbb{Z}^3 = Z(\hat{\mu}_{M,D}(\xi)) \cap ((\frac{1}{2},\frac{1}{2},\frac{1}{2})^{\mathrm{T}} + \mathbb{Z}^3) = \emptyset$;

(c) *if* $\xi = (\xi_1,\xi_2,\xi_3)^{\mathrm{T}} \in B_1 \pm B_1$, *then* $\xi_1 \in \mathbb{Z}$;

(d) *if* $\xi = (\xi_1,\xi_2,\xi_3)^{\mathrm{T}} \in B_2 \pm B_2$, *then* $\xi_2 \in \mathbb{Z}$;

(e) *if* $\xi = (\xi_1,\xi_2,\xi_3)^{\mathrm{T}} \in B_3 \pm B_3$, *then* $\xi_3 \in \mathbb{Z}$.

Lemma 3.2. *Let* $\xi = (\xi_1,\xi_2,\xi_3)^{\mathrm{T}} \in Z(\hat{\mu}_{M,D}(\xi)) = B_1 \cup B_2 \cup B_3$. *Then the following statements hold:*

(i) *if* $\xi \in B_j$, *then* $\xi_j \in \frac{1}{2} + \mathbb{Z}$, *where* $j = 1,2,3$;

(ii) *if* $\xi_1 \in \mathbb{Z}$, *then* $\xi_2 \notin \mathbb{Z}$, $\xi_3 \notin \mathbb{Z}$ and $\xi \in B_2 \cup B_3$;

(iii) *if* $\xi_2 \in \mathbb{Z}$, *then* $\xi_1 \notin \mathbb{Z}$, $\xi_3 \notin \mathbb{Z}$ and $\xi \in B_1 \cup B_3$;

(iv) *if* $\xi_3 \in \mathbb{Z}$, *then* $\xi_1 \notin \mathbb{Z}$, $\xi_2 \notin \mathbb{Z}$ and $\xi \in B_1 \cup B_2$.

Secondly, if $\lambda_j \in \mathbb{R}$, $j = 1,2,3,4,5$, are such that the five exponential functions

$$\exp(2\pi\mathrm{i}\langle\lambda_1,x\rangle),\quad \exp(2\pi\mathrm{i}\langle\lambda_2,x\rangle),\quad \exp(2\pi\mathrm{i}\langle\lambda_3,x\rangle),\quad \exp(2\pi\mathrm{i}\langle\lambda_4,x\rangle),\quad \exp(2\pi\mathrm{i}\langle\lambda_5,x\rangle) \tag{3.5}$$

are mutually orthogonal in $L^2(\mu_{M,D})$, then the differences $\lambda_j - \lambda_k$, $1 \leqslant j \neq k \leqslant 5$, are in the zero set $Z(\hat{\mu}_{M,D}(\xi))$ of the Fourier transform $\hat{\mu}_{M,D}(\xi)$ (see, for example, [**20**, p. 163], [**21**, p. 3140]). That is, we have

$$\lambda_j - \lambda_k \in Z(\hat{\mu}_{M,D}(\xi)) = B_1 \cup B_2 \cup B_3, \quad 1 \leqslant j \neq k \leqslant 5. \tag{3.6}$$

Define $\lambda_j - \lambda_k$ by

$$\lambda_j - \lambda_k = (x_{j,k}, y_{j,k}, z_{j,k})^{\mathrm{T}} \in \mathbb{R}^3 \quad \text{for } 1 \leqslant j \neq k \leqslant 5.$$

We shall apply the above two lemmas to deduce a contradiction below.

Observe that the following 10 differences:

$$\left.\begin{aligned} \lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1, \\ \lambda_3 - \lambda_2, \lambda_4 - \lambda_2, \lambda_5 - \lambda_2, \\ \lambda_4 - \lambda_3, \lambda_5 - \lambda_3, \\ \lambda_5 - \lambda_4 \end{aligned}\right\} \tag{3.7}$$

belong to the union of the three sets B_1, B_2 and B_3. In particular, we have

$$\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1 \in B_1 \cup B_2 \cup B_3. \tag{3.8}$$

Claim 3.3. *Each set B_1 (or B_2 or B_3) cannot contain any three differences of the form $\lambda_{j_1} - \lambda_j$, $\lambda_{j_2} - \lambda_j$, $\lambda_{j_3} - \lambda_j$, where $1 \leqslant j_1 \neq j_2 \neq j_3 \neq j \leqslant 5$.*

Claim 3.3 can be checked directly. For example, if $\lambda_{j_1} - \lambda_j, \lambda_{j_2} - \lambda_j, \lambda_{j_3} - \lambda_j \in B_1$, then, by applying Lemma 3.1 (c) and (3.6), we have

$$\begin{gathered} \lambda_{j_1} - \lambda_{j_3} = (\lambda_{j_1} - \lambda_j) - (\lambda_{j_3} - \lambda_j) \in (B_1 - B_1) \cap Z(\hat{\mu}_{M,D}(\xi)), \\ \lambda_{j_2} - \lambda_{j_3} = (\lambda_{j_2} - \lambda_j) - (\lambda_{j_3} - \lambda_j) \in (B_1 - B_1) \cap Z(\hat{\mu}_{M,D}(\xi)), \\ \lambda_{j_1} - \lambda_{j_2} = (\lambda_{j_1} - \lambda_j) - (\lambda_{j_2} - \lambda_j) \in (B_1 - B_1) \cap Z(\hat{\mu}_{M,D}(\xi)), \\ x_{j_1,j_3}, x_{j_2,j_3}, x_{j_1,j_2} \in \mathbb{Z}, \end{gathered}$$

and by Lemma 3.2 (ii),

$$\lambda_{j_1} - \lambda_{j_3}, \lambda_{j_2} - \lambda_{j_3}, \lambda_{j_1} - \lambda_{j_2} \in B_2 \cup B_3,$$

which shows that at least one of the two sets B_2 and B_3, say B_3, contains two differences, say $\lambda_{j_2} - \lambda_{j_3}$ and $\lambda_{j_1} - \lambda_{j_2}$. Then

$$\lambda_{j_1} - \lambda_{j_3} = (\lambda_{j_1} - \lambda_{j_2}) + (\lambda_{j_2} - \lambda_{j_3}) \in B_3 + B_3.$$

This shows (by Lemma 3.1 (e)) that $z_{j_1,j_3} \in \mathbb{Z}$: a contradiction of Lemma 3.2 (ii).

From (3.8) and Claim 3.3, we only need to deal with the following two typical cases.

Case 1. $\lambda_2 - \lambda_1 \in B_1$, $\lambda_3 - \lambda_1 \in B_2$, $\lambda_4 - \lambda_1 \in B_3$ and $\lambda_5 - \lambda_1 \in B_1$.

Case 2. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1 \in B_1$ and $\lambda_4 - \lambda_1, \lambda_5 - \lambda_1 \in B_2$.

Note that Case 1 denotes the $2-1-1$ distribution in (3.8), it also denotes the $1-2-1$ or $1-1-2$ distribution in (3.8), while Case 2 denotes the $2-2-0$ (or $2-0-2$ or $0-2-2$) distribution in (3.8). If Case 1 and Case 2 can be proved, then the other cases can be proved in the same way.

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In Case 1, we have

$$\lambda_5 - \lambda_2 = (\lambda_5 - \lambda_1) - (\lambda_2 - \lambda_1) \in B_1 - B_1.$$

Applying Lemma 3.1 (c), (3.6) and Lemma 3.2 (ii), we also have

$$\lambda_5 - \lambda_2 = (x_{5,2}, y_{5,2}, z_{5,2})^{\mathrm{T}} \in B_2 \cup B_3 \colon x_{5,2} \in \mathbb{Z},\ y_{5,2} \notin \mathbb{Z},\ z_{5,2} \notin \mathbb{Z}. \tag{3.9}$$

From $\lambda_5 - \lambda_2 \in B_2 \cup B_3$, the discussion here can be divided into two cases: $\lambda_5 - \lambda_2 \in B_2$ and $\lambda_5 - \lambda_2 \in B_3$. That is, we have the following two subcases.

Case 1.1. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_1 \in B_1$, $\lambda_3 - \lambda_1, \lambda_5 - \lambda_2 \in B_2$, $\lambda_4 - \lambda_1 \in B_3$.

Case 1.2. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_1 \in B_1$, $\lambda_3 - \lambda_1 \in B_2$, $\lambda_4 - \lambda_1, \lambda_5 - \lambda_2 \in B_3$.

The discussion of Case 1.2 is analogous to that of Case 1.1: it denotes the $2-2-1$ or $2-1-2$ or $1-2-2$ distribution. So we only need to deal with Case 1.1. In this case, by Lemma 3.2 (i), we see that (3.9) becomes

$$\lambda_5 - \lambda_2 = (x_{5,2}, y_{5,2}, z_{5,2})^{\mathrm{T}} \in B_2 \colon x_{5,2} \in \mathbb{Z},\ y_{5,2} \in \tfrac{1}{2} + \mathbb{Z},\ z_{5,2} \notin \mathbb{Z}. \tag{3.10}$$

From $\lambda_3 - \lambda_2 \in B_1 \cup B_2 \cup B_3$, Case 1.1 can be divided into three cases.

Case 1.1.1. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_1, \lambda_3 - \lambda_2 \in B_1$, $\lambda_3 - \lambda_1, \lambda_5 - \lambda_2 \in B_2$, $\lambda_4 - \lambda_1 \in B_3$.

Case 1.1.2. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_1 \in B_1$, $\lambda_3 - \lambda_1, \lambda_5 - \lambda_2, \lambda_3 - \lambda_2 \in B_2$, $\lambda_4 - \lambda_1 \in B_3$.

Case 1.1.3. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_1 \in B_1$, $\lambda_3 - \lambda_1, \lambda_5 - \lambda_2 \in B_2$, $\lambda_4 - \lambda_1, \lambda_3 - \lambda_2 \in B_3$.

The above three cases denote the $3-2-1$ (or $2-3-1$) distribution and $2-2-2$ distribution. The first two cases are similar. We shall give a method to deal with each case by considering the remainder differences in (3.7). Note that each case is concluded with a contradiction.

3.1. Case 1.1.1

In Case 1.1.1, we have

$$\lambda_3 - \lambda_1 = (\lambda_3 - \lambda_2) + (\lambda_2 - \lambda_1) \in B_1 + B_1,$$

which shows (by applying Lemma 3.1 (c), (3.6) and parts (i) and (ii) of Lemma 3.2) that

$$\lambda_3 - \lambda_1 = (x_{3,1}, y_{3,1}, z_{3,1})^{\mathrm{T}} \in B_2 \colon x_{3,1} \in \mathbb{Z},\ y_{3,1} \in \tfrac{1}{2} + \mathbb{Z},\ z_{3,1} \notin \mathbb{Z}. \tag{3.11}$$

By Lemma 3.1 (a) and $\lambda_1 - \lambda_2, \lambda_3 - \lambda_2 \in B_1$, we know from Claim 3.3 that $\lambda_4 - \lambda_2 \notin B_1$. So $\lambda_4 - \lambda_2 \in B_2$ or $\lambda_4 - \lambda_2 \in B_3$, and Case 1.1.1 can be divided into two cases.

Case 1.1.1.1. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_1, \lambda_3 - \lambda_2 \in B_1$, $\lambda_3 - \lambda_1, \lambda_5 - \lambda_2, \lambda_4 - \lambda_2 \in B_2$, $\lambda_4 - \lambda_1 \in B_3$.

Case 1.1.1.2. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_1, \lambda_3 - \lambda_2 \in B_1$, $\lambda_3 - \lambda_1, \lambda_5 - \lambda_2 \in B_2$, $\lambda_4 - \lambda_1, \lambda_4 - \lambda_2 \in B_3$.

The above two cases denote the $3-3-1$ distribution and $3-2-2$ distribution. By considering the remainder differences in (3.7), we apply Lemmas 3.1 and 3.2 to deal with each case.

Step 1. In Case 1.1.1.1, we have

$$\lambda_5 - \lambda_4 = (\lambda_5 - \lambda_2) - (\lambda_4 - \lambda_2) \in B_2 - B_2,$$

which shows (by applying Lemma 3.1 (d), (3.6) and Lemma 3.2 (iii)) that

$$\lambda_5 - \lambda_4 = (x_{5,4}, y_{5,4}, z_{5,4})^{\mathrm{T}} \in B_1 \cup B_3 \colon x_{5,4} \notin \mathbb{Z},\ y_{5,4} \in \mathbb{Z},\ z_{5,4} \notin \mathbb{Z}. \tag{3.12}$$

(i) If $\lambda_5 - \lambda_4 \in B_1$, then (3.12) becomes

$$\lambda_5 - \lambda_4 = (x_{5,4}, y_{5,4}, z_{5,4})^{\mathrm{T}} \in B_1 \colon x_{5,4} \in \tfrac{1}{2} + \mathbb{Z},\ y_{5,4} \in \mathbb{Z},\ z_{5,4} \notin \mathbb{Z}. \tag{3.13}$$

From $\lambda_4 - \lambda_1 = (\lambda_5 - \lambda_1) - (\lambda_5 - \lambda_4) \in B_1 - B_1$, we have (by applying Lemma 3.1 (c), (3.6) and parts (i) and (ii) of Lemma 3.2) that

$$\lambda_4 - \lambda_1 = (x_{4,1}, y_{4,1}, z_{4,1})^{\mathrm{T}} \in B_3 \colon x_{4,1} \in \mathbb{Z},\ y_{4,1} \notin \mathbb{Z},\ z_{4,1} \in \tfrac{1}{2} + \mathbb{Z}. \tag{3.14}$$

It follows from (3.11), (3.14) and $\lambda_4 - \lambda_3 = (\lambda_4 - \lambda_1) - (\lambda_3 - \lambda_1)$ that

$$\lambda_4 - \lambda_3 = (x_{4,3}, y_{4,3}, z_{4,3})^{\mathrm{T}} \colon x_{4,3} \in \mathbb{Z},\ y_{4,3} \notin \mathbb{Z},\ z_{4,3} \notin \mathbb{Z}, \tag{3.15}$$

which shows (by Lemma 3.2 (ii)) that $\lambda_4 - \lambda_3 \in B_2 \cup B_3$. If $\lambda_4 - \lambda_3 \in B_2$, then

$$\lambda_4 - \lambda_1 = (\lambda_4 - \lambda_3) + (\lambda_3 - \lambda_1) \in B_2 + B_2,$$

which shows (by Lemma 3.1 (d)) that $y_{4,1} \in \mathbb{Z}$: a contradiction of (3.14). If $\lambda_4 - \lambda_3 \in B_3$, then

$$\lambda_3 - \lambda_1 = (\lambda_4 - \lambda_1) - (\lambda_4 - \lambda_3) \in B_3 - B_3,$$

which shows (by Lemma 3.1 (e)) that $z_{3,1} \in \mathbb{Z}$: a contradiction of (3.11).

Note that in (3.15), we write $y_{4,3} \notin \mathbb{Z}$ and $z_{4,3} \notin \mathbb{Z}$ by Lemma 3.2 (ii) after obtaining $x_{4,3} = x_{4,1} - x_{3,1} \in \mathbb{Z}$ (where $x_{4,1}, x_{3,1} \in \mathbb{Z}$). From $y_{4,3} = y_{4,1} - y_{3,1}$, $y_{4,1} \notin \mathbb{Z}$ and $y_{3,1} \in \frac{1}{2} + \mathbb{Z}$, we cannot assert that $y_{4,3} \notin \mathbb{Z}$. If $y_{4,3} \in \mathbb{Z}$ or $z_{4,3} \in \mathbb{Z}$, then the discussion will immediately conclude with a contradiction.

(ii) If $\lambda_5 - \lambda_4 \in B_3$, then (3.12) becomes

$$\lambda_5 - \lambda_4 = (x_{5,4}, y_{5,4}, z_{5,4})^{\mathrm{T}} \in B_3 \colon x_{5,4} \notin \mathbb{Z},\ y_{5,4} \in \mathbb{Z},\ z_{5,4} \in \tfrac{1}{2} + \mathbb{Z}. \tag{3.16}$$

From $\lambda_5 - \lambda_1 = (\lambda_5 - \lambda_4) + (\lambda_4 - \lambda_1) \in B_3 + B_3$, we have (by applying Lemma 3.1 (e), (3.6) and parts (i) and (iv) of Lemma 3.2)

$$\lambda_5 - \lambda_1 = (x_{5,1}, y_{5,1}, z_{5,1})^{\mathrm{T}} \in B_1 \colon x_{5,1} \in \tfrac{1}{2} + \mathbb{Z},\ y_{5,1} \notin \mathbb{Z},\ z_{5,1} \in \mathbb{Z}. \tag{3.17}$$

Now, consider the remainder difference $\lambda_4 - \lambda_3$ in (3.7): by Lemma 3.1 (a) and Claim 3.3, we have $\lambda_4 - \lambda_3 \notin B_3$, so $\lambda_4 - \lambda_3 \in B_1 \cup B_2$. If $\lambda_4 - \lambda_3 \in B_1$, then

$$\lambda_4 - \lambda_2 = (\lambda_4 - \lambda_3) + (\lambda_3 - \lambda_2) \in B_1 + B_1,$$

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which shows (by applying Lemma 3.1 (c), (3.6) and parts (i) and (ii) of Lemma 3.2) that

$$\lambda_4-\lambda_2=(x_{4,2},y_{4,2},z_{4,2})^{\mathrm{T}}\in B_2\colon x_{4,2}\in\mathbb{Z},\ y_{4,2}\in\tfrac{1}{2}+\mathbb{Z},\ z_{4,2}\notin\mathbb{Z}. \tag{3.18}$$

It follows from (3.10), (3.18) and $\lambda_5-\lambda_4=(\lambda_5-\lambda_2)-(\lambda_4-\lambda_2)$ that $x_{5,4}\in\mathbb{Z}$: a contradiction of (3.16). If $\lambda_4-\lambda_3\in B_2$, then

$$\lambda_4-\lambda_1=(\lambda_4-\lambda_3)+(\lambda_3-\lambda_1)\in B_2+B_2,$$

which shows (by applying Lemma 3.1 (d), (3.6) and parts (i) and (iii) of Lemma 3.2) that

$$\lambda_4-\lambda_1=(x_{4,1},y_{4,1},z_{4,1})^{\mathrm{T}}\in B_3\colon x_{4,1}\notin\mathbb{Z},\ y_{4,1}\in\mathbb{Z},\ z_{4,1}\in\tfrac{1}{2}+\mathbb{Z}. \tag{3.19}$$

It follows from (3.16), (3.19) and $\lambda_5-\lambda_1=(\lambda_5-\lambda_4)+(\lambda_4-\lambda_1)$ that $y_{5,1}\in\mathbb{Z}$ and $z_{5,1}\in\mathbb{Z}$: a contradiction of (3.17) and parts (iii) and (iv) of Lemma 3.2, respectively.

Parts (i) and (ii) above and (3.12) illustrate that Case 1.1.1.1 is proved.

Step 2. In Case 1.1.1.2, we have

$$\lambda_2-\lambda_1=(\lambda_4-\lambda_1)-(\lambda_4-\lambda_2)\in B_3-B_3,$$

which shows (by applying Lemma 3.1 (e), (3.6) and parts (i) and (iv) of Lemma 3.2) that

$$\lambda_2-\lambda_1=(x_{2,1},y_{2,1},z_{2,1})^{\mathrm{T}}\in B_1\colon x_{2,1}\in\tfrac{1}{2}+\mathbb{Z},\ y_{2,1}\notin\mathbb{Z},\ z_{2,1}\in\mathbb{Z}. \tag{3.20}$$

Now, consider the three remainder differences $\lambda_4-\lambda_3$, $\lambda_5-\lambda_3$ and $\lambda_5-\lambda_4$ in (3.7). By Claim 3.3, we have $\lambda_4-\lambda_3\notin B_3$, so $\lambda_4-\lambda_3\in B_1\cup B_2$.

(i) If $\lambda_4-\lambda_3\in B_1$, then

$$\lambda_4-\lambda_2=(\lambda_4-\lambda_3)+(\lambda_3-\lambda_2)\in B_1+B_1,$$

which shows (by applying Lemma 3.1 (c), (3.6) and parts (i) and (ii) of Lemma 3.2) that

$$\lambda_4-\lambda_2=(x_{4,2},y_{4,2},z_{4,2})^{\mathrm{T}}\in B_3\colon x_{4,2}\in\mathbb{Z},\ y_{4,2}\notin\mathbb{Z},\ z_{4,2}\in\tfrac{1}{2}+\mathbb{Z}. \tag{3.21}$$

Consider the remainder difference $\lambda_5-\lambda_4$ in (3.7). By Claim 3.3, we have $\lambda_5-\lambda_4\notin B_3$, so $\lambda_5-\lambda_4\in B_1\cup B_2$. If $\lambda_5-\lambda_4\in B_1$, then

$$\lambda_4-\lambda_1=(\lambda_5-\lambda_1)-(\lambda_5-\lambda_4)\in B_1-B_1,$$

which shows (by applying Lemma 3.1 (c), (3.6) and parts (i) and (ii) of Lemma 3.2) that

$$\lambda_4-\lambda_1=(x_{4,1},y_{4,1},z_{4,1})^{\mathrm{T}}\in B_3\colon x_{4,1}\in\mathbb{Z},\ y_{4,1}\notin\mathbb{Z},\ z_{4,1}\in\tfrac{1}{2}+\mathbb{Z}. \tag{3.22}$$

It follows from (3.21), (3.22) and $\lambda_2-\lambda_1=(\lambda_4-\lambda_1)-(\lambda_4-\lambda_2)$ that $x_{2,1}\in\mathbb{Z}$ and $z_{2,1}\in\mathbb{Z}$: a contradiction of (3.20). If $\lambda_5-\lambda_4\in B_2$, then

$$\lambda_4-\lambda_2=(\lambda_5-\lambda_2)-(\lambda_5-\lambda_4)\in B_2-B_2,$$

which shows (by applying Lemma 3.1 (d), (3.6) and Lemma 3.2 (iii)) that $y_{4,2}\in\mathbb{Z}$: a contradiction of (3.21).

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(ii) If $\lambda_4 - \lambda_3 \in B_2$, then

$$\lambda_4 - \lambda_1 = (\lambda_4 - \lambda_3) + (\lambda_3 - \lambda_1) \in B_2 + B_2,$$

which shows (by applying Lemma 3.1 (d), (3.6) and parts (i) and (iii) of Lemma 3.2) that

$$\lambda_4 - \lambda_1 = (x_{4,1}, y_{4,1}, z_{4,1})^{\mathrm{T}} \in B_3 \colon x_{4,1} \notin \mathbb{Z},\ y_{4,1} \in \mathbb{Z},\ z_{4,1} \in \tfrac{1}{2} + \mathbb{Z}. \tag{3.23}$$

Consider the remainder difference $\lambda_5 - \lambda_4$ in (3.7). By Claim 3.3, we have $\lambda_5 - \lambda_4 \notin B_3$, so $\lambda_5 - \lambda_4 \in B_1 \cup B_2$. If $\lambda_5 - \lambda_4 \in B_1$, then

$$\lambda_4 - \lambda_1 = (\lambda_5 - \lambda_1) - (\lambda_5 - \lambda_4) \in B_1 - B_1,$$

which shows (by applying Lemma 3.1 (c), (3.6) and Lemma 3.2 (ii)) that $x_{4,1} \in \mathbb{Z}$: a contradiction of (3.23). If $\lambda_5 - \lambda_4 \in B_2$, then

$$\lambda_4 - \lambda_2 = (\lambda_5 - \lambda_2) - (\lambda_5 - \lambda_4) \in B_2 - B_2,$$

which shows (by applying Lemma 3.1 (d), (3.6) and parts (i) and (iii) of Lemma 3.2) that

$$\lambda_4 - \lambda_2 = (x_{4,2}, y_{4,2}, z_{4,2})^{\mathrm{T}} \in B_3 \colon x_{4,2} \notin \mathbb{Z},\ y_{4,2} \in \mathbb{Z},\ z_{4,1} \in \tfrac{1}{2} + \mathbb{Z}. \tag{3.24}$$

It follows from (3.23), (3.24) and $\lambda_2 - \lambda_1 = (\lambda_4 - \lambda_1) - (\lambda_4 - \lambda_2)$ that $y_{2,1} \in \mathbb{Z}$: a contradiction of (3.20).

Step 2 illustrates that Case 1.1.1.2 is proved. Hence, Case 1.1.1 is proved.

3.2. Case 1.1.2

This case is analogous to, but easier than, Case 1.1.1; we mainly give a method here. In Case 1.1.2, we have

$$\lambda_2 - \lambda_1 = (\lambda_3 - \lambda_1) - (\lambda_3 - \lambda_2) \in B_2 - B_2$$

and

$$\lambda_5 - \lambda_3 = (\lambda_5 - \lambda_2) - (\lambda_3 - \lambda_2) \in B_2 - B_2,$$

which shows (by applying Lemma 3.1 (d), (3.6) and parts (i) and (iii) of Lemma 3.2) that

$$\lambda_2 - \lambda_1 = (x_{2,1}, y_{2,1}, z_{2,1})^{\mathrm{T}} \in B_1 \colon x_{2,1} \in \tfrac{1}{2} + \mathbb{Z},\ y_{2,1} \in \mathbb{Z},\ z_{2,1} \notin \mathbb{Z} \tag{3.25}$$

and

$$\lambda_5 - \lambda_3 = (x_{5,3}, y_{5,3}, z_{5,3})^{\mathrm{T}} \in B_1 \cup B_3 \colon x_{5,3} \notin \mathbb{Z},\ y_{5,3} \in \mathbb{Z},\ z_{5,3} \notin \mathbb{Z}. \tag{3.26}$$

Also, by Lemma 3.1 (a) and Claim 3.3, we have

$$\lambda_4 - \lambda_2 \notin B_2 \quad \text{and} \quad \lambda_4 - \lambda_3 \notin B_2. \tag{3.27}$$

If $\lambda_4 - \lambda_2 \in B_3$, then $\lambda_2 - \lambda_1 = (\lambda_4 - \lambda_1) - (\lambda_4 - \lambda_2) \in B_3 - B_3$ and $z_{2,1} \in \mathbb{Z}$: a contradiction of (3.25); hence,

$$\lambda_4 - \lambda_2 \notin B_3 \quad \text{and} \quad \lambda_4 - \lambda_2 \in B_1.$$

From $\lambda_4 - \lambda_1 = (\lambda_4 - \lambda_2) + (\lambda_2 - \lambda_1) \in B_1 + B_1$, we have

$$\lambda_4 - \lambda_1 = (x_{4,1}, y_{4,1}, z_{4,1})^{\mathrm{T}} \in B_3\colon x_{4,1} \in \mathbb{Z},\ y_{4,1} \notin \mathbb{Z},\ z_{4,1} \in \tfrac{1}{2} + \mathbb{Z}. \tag{3.28}$$

If $\lambda_4 - \lambda_3 \in B_1$, then $\lambda_3 - \lambda_2 = (\lambda_4 - \lambda_2) - (\lambda_4 - \lambda_3) \in B_1 - B_1$ and

$$\lambda_3 - \lambda_2 = (x_{3,2}, y_{3,2}, z_{3,2})^{\mathrm{T}} \in B_2\colon x_{3,2} \in \mathbb{Z},\ y_{3,2} \in \tfrac{1}{2} + \mathbb{Z},\ z_{3,2} \notin \mathbb{Z},$$

which, combined with (3.10) and $\lambda_5 - \lambda_3 = (\lambda_5 - \lambda_2) - (\lambda_3 - \lambda_2)$, shows that $x_{5,3} \in \mathbb{Z}$: a contradiction of (3.26); hence,

$$\lambda_4 - \lambda_3 \notin B_1 \quad \text{and} \quad \lambda_4 - \lambda_3 \in B_3.$$

From $\lambda_3 - \lambda_1 = (\lambda_4 - \lambda_1) - (\lambda_4 - \lambda_3) \in B_3 - B_3$, we have

$$\lambda_3 - \lambda_1 = (x_{3,1}, y_{3,1}, z_{3,1})^{\mathrm{T}} \in B_2\colon x_{3,1} \notin \mathbb{Z},\ y_{3,1} \in \tfrac{1}{2} + \mathbb{Z},\ z_{3,1} \in \mathbb{Z}. \tag{3.29}$$

If $\lambda_5 - \lambda_3 \in B_1$, then $\lambda_3 - \lambda_1 = (\lambda_5 - \lambda_1) - (\lambda_5 - \lambda_3) \in B_1 - B_1$ and $x_{3,1} \in \mathbb{Z}$: a contradiction of (3.29); hence, $\lambda_5 - \lambda_3 \notin B_1$. From (3.26), we have $\lambda_5 - \lambda_3 \in B_3$, and (3.26) becomes

$$\lambda_5 - \lambda_3 = (x_{5,3}, y_{5,3}, z_{5,3})^{\mathrm{T}} \in B_3\colon x_{5,3} \notin \mathbb{Z},\ y_{5,3} \in \mathbb{Z},\ z_{5,3} \in \tfrac{1}{2} + \mathbb{Z}. \tag{3.30}$$

From $\lambda_5 - \lambda_4 = (\lambda_5 - \lambda_3) - (\lambda_4 - \lambda_3) \in B_3 - B_3$, we have

$$\lambda_5 - \lambda_4 = (x_{5,4}, y_{5,4}, z_{5,4})^{\mathrm{T}} \in B_1 \cup B_2\colon x_{5,4} \notin \mathbb{Z},\ y_{5,4} \notin \mathbb{Z},\ z_{5,4} \in \mathbb{Z}. \tag{3.31}$$

If $\lambda_5 - \lambda_4 \in B_2$, then $\lambda_4 - \lambda_2 = (\lambda_5 - \lambda_2) - (\lambda_5 - \lambda_4) \in B_2 - B_2$ and

$$\lambda_4 - \lambda_2 = (x_{4,2}, y_{4,2}, z_{4,2})^{\mathrm{T}} \in B_1\colon x_{4,2} \in \tfrac{1}{2} + \mathbb{Z},\ y_{4,2} \in \mathbb{Z},\ z_{4,2} \notin \mathbb{Z},$$

which, combined with (3.25) and $\lambda_4 - \lambda_1 = (\lambda_4 - \lambda_2) + (\lambda_2 - \lambda_1)$, yields $y_{4,1} \in \mathbb{Z}$: a contradiction of (3.28); hence, $\lambda_5 - \lambda_4 \notin B_2$. This shows that $\lambda_5 - \lambda_4 \in B_1$, and (3.31) becomes

$$\lambda_5 - \lambda_4 = (x_{5,4}, y_{5,4}, z_{5,4})^{\mathrm{T}} \in B_1\colon x_{5,4} \in \tfrac{1}{2} + \mathbb{Z},\ y_{5,4} \notin \mathbb{Z},\ z_{5,4} \in \mathbb{Z}. \tag{3.32}$$

With the above method and $\lambda_5 - \lambda_4 \in B_1$, we have $\lambda_5 - \lambda_2 = (\lambda_5 - \lambda_4) + (\lambda_4 - \lambda_2) \in B_1 + B_1$ and $\lambda_4 - \lambda_1 = (\lambda_5 - \lambda_1) - (\lambda_5 - \lambda_4) \in B_1 - B_1$, which yield (3.10) and (3.28), respectively.

Now, the 10 differences in (3.7) satisfy

$$\left.\begin{aligned} \lambda_2 - \lambda_1, \lambda_5 - \lambda_1, \lambda_4 - \lambda_2, \lambda_5 - \lambda_4 &\in B_1, \\ \lambda_3 - \lambda_1, \lambda_5 - \lambda_2, \lambda_3 - \lambda_2 &\in B_2, \\ \lambda_4 - \lambda_1, \lambda_4 - \lambda_3, \lambda_5 - \lambda_3 &\in B_3, \end{aligned}\right\} \tag{3.33}$$

and we have (3.10), (3.25), (3.28)–(3.30) and (3.32). There are several contradictions implied in them. For example, from (3.10) and (3.25), we have

$$\lambda_5 - \lambda_1 = (x_{5,1}, y_{5,1}, z_{5,1})^{\mathrm{T}} = (\lambda_5 - \lambda_2) + (\lambda_2 - \lambda_1)\colon x_{5,1} \in \tfrac{1}{2} + \mathbb{Z},\ y_{5,1} \in \tfrac{1}{2} + \mathbb{Z}.$$

From (3.29) and (3.30), we have

$$\lambda_5 - \lambda_1 = (x_{5,1}, y_{5,1}, z_{5,1})^{\mathrm{T}} = (\lambda_5 - \lambda_3) + (\lambda_3 - \lambda_1)\colon y_{5,1} \in \tfrac{1}{2} + \mathbb{Z},\ z_{5,1} \in \tfrac{1}{2} + \mathbb{Z}.$$

Thus, we have

$$\lambda_5 - \lambda_1 = (x_{5,1}, y_{5,1}, z_{5,1})^{\mathrm{T}} \in (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})^{\mathrm{T}} + \mathbb{Z}^3,$$

which contradicts Lemma 3.1 (b). This proves Case 1.1.2.

3.3. Case 1.1.3

According to $\lambda_4 - \lambda_2 \in B_1 \cup B_2 \cup B_3$, Case 1.1.3 can be divided into the following three cases.

Case 1.1.3.1. $\lambda_2-\lambda_1, \lambda_5-\lambda_1, \lambda_4-\lambda_2 \in B_1$, $\lambda_3-\lambda_1, \lambda_5-\lambda_2 \in B_2$, $\lambda_4-\lambda_1, \lambda_3-\lambda_2 \in B_3$.

Case 1.1.3.2. $\lambda_2-\lambda_1, \lambda_5-\lambda_1 \in B_1$, $\lambda_3-\lambda_1, \lambda_5-\lambda_2, \lambda_4-\lambda_2 \in B_2$, $\lambda_4-\lambda_1, \lambda_3-\lambda_2 \in B_3$.

Case 1.1.3.3. $\lambda_2-\lambda_1, \lambda_5-\lambda_1 \in B_1$, $\lambda_3-\lambda_1, \lambda_5-\lambda_2 \in B_2$, $\lambda_4-\lambda_1, \lambda_3-\lambda_2, \lambda_4-\lambda_2 \in B_3$.

The above three cases denote that the seven differences in the first two rows of (3.7) have the $3-2-2$ or $2-3-2$ or $2-2-3$ distribution. In each case, we have (3.10). In addition, in Case 1.1.3.1, we have $\lambda_4 - \lambda_1 \in B_1 + B_1$; in Case 1.1.3.2, we have $\lambda_5 - \lambda_4 \in B_2 - B_2$; in Case 1.1.3.3, we have $\lambda_4 - \lambda_3 \in B_3 - B_3$ and $\lambda_2 - \lambda_1 \in B_3 - B_3$. So, the discussion of the first two cases is analogous, and the discussion of the third case is easier than the first two cases. In the following, we use the above method to deal with Case 1.1.3.1. The other two cases can be proved in the same manner.

In Case 1.1.3.1, we have

$$\lambda_4 - \lambda_1 = (\lambda_4 - \lambda_2) + (\lambda_2 - \lambda_1) \in B_1 + B_1,$$

which shows (by applying Lemma 3.1 (c), (3.6) and parts (i) and (ii) of Lemma 3.2) that

$$\lambda_4 - \lambda_1 = (x_{4,1}, y_{4,1}, z_{4,1})^{\mathrm{T}} \in B_3\colon x_{4,1} \in \mathbb{Z},\ y_{4,1} \notin \mathbb{Z},\ z_{4,1} \in \tfrac{1}{2} + \mathbb{Z}. \tag{3.34}$$

Note that the three remainder differences $\lambda_4 - \lambda_3$, $\lambda_5 - \lambda_3$ and $\lambda_5 - \lambda_4$ are also in the set $B_1 \cup B_2 \cup B_3$. Firstly, (3.10) and (3.34) give us the information that

$$\lambda_4 - \lambda_3 \notin B_2 \quad \text{or} \quad \lambda_4 - \lambda_3 \in B_1 \cup B_3 \tag{3.35}$$

and

$$\lambda_5 - \lambda_3 \notin B_3 \quad \text{or} \quad \lambda_5 - \lambda_3 \in B_1 \cup B_2. \tag{3.36}$$

In fact, by Lemma 3.1, if $\lambda_4-\lambda_3 \in B_2$, then $\lambda_4-\lambda_1 = (\lambda_4-\lambda_3)+(\lambda_3-\lambda_1) \in B_2+B_2$ gives $y_{4,1} \in \mathbb{Z}$: a contradiction of (3.34). If $\lambda_5 - \lambda_3 \in B_3$, then $\lambda_5 - \lambda_2 = (\lambda_5 - \lambda_3) + (\lambda_3 - \lambda_2) \in B_3 + B_3$ gives $z_{5,2} \in \mathbb{Z}$: a contradiction of (3.10). Hence, (3.35) and (3.36) hold.

From (3.35), we consider the following two cases.

(i) If $\lambda_4 - \lambda_3 \in B_1$, then, from $\lambda_3 - \lambda_2 = (\lambda_4 - \lambda_2) - (\lambda_4 - \lambda_3) \in B_1 - B_1$, we have (by applying Lemma 3.1 (c), (3.6) and parts (i) and (ii) of Lemma 3.2)

$$\lambda_3 - \lambda_2 = (x_{3,2}, y_{3,2}, z_{3,2})^{\mathrm{T}} \in B_3 \colon x_{3,2} \in \mathbb{Z},\ y_{3,2} \notin \mathbb{Z},\ z_{3,2} \in \tfrac{1}{2} + \mathbb{Z}, \tag{3.37}$$

which also shows $\lambda_5 - \lambda_3 \notin B_2$ (otherwise, by Lemma 3.1, $\lambda_3 - \lambda_2 = (\lambda_5 - \lambda_2) - (\lambda_5 - \lambda_3) \in B_2 - B_2$ gives $y_{3,2} \in \mathbb{Z}$: a contradiction of (3.37)). From (3.36), we have $\lambda_5 - \lambda_3 \in B_1$. From $\lambda_3 - \lambda_1 = (\lambda_5 - \lambda_1) - (\lambda_5 - \lambda_3) \in B_1 - B_1$, we have (by applying Lemma 3.1 (c), (3.6) and parts (i) and (ii) of Lemma 3.2)

$$\lambda_3 - \lambda_1 = (x_{3,1}, y_{3,1}, z_{3,1})^{\mathrm{T}} \in B_2 \colon x_{3,1} \in \mathbb{Z},\ y_{3,1} \in \tfrac{1}{2} + \mathbb{Z},\ z_{3,1} \notin \mathbb{Z}. \tag{3.38}$$

From (3.37), (3.38) and $\lambda_2 - \lambda_1 = (\lambda_3 - \lambda_1) - (\lambda_3 - \lambda_2)$, we have $x_{2,1} \in \mathbb{Z}$: a contradiction of Lemma 3.2 (i) (for $\lambda_2 - \lambda_1 \in B_1$ gives $x_{2,1} \in \frac{1}{2} + \mathbb{Z}$).

(ii) If $\lambda_4 - \lambda_3 \in B_3$, then, from $\lambda_4 - \lambda_2 = (\lambda_4 - \lambda_3) + (\lambda_3 - \lambda_2) \in B_3 + B_3$ and $\lambda_3 - \lambda_1 = (\lambda_4 - \lambda_1) - (\lambda_4 - \lambda_3) \in B_3 - B_3$, we have (by applying Lemma 3.1 (e), (3.6) and parts (i) and (iv) of Lemma 3.2)

$$\lambda_4 - \lambda_2 = (x_{4,2}, y_{4,2}, z_{4,2})^{\mathrm{T}} \in B_1 \colon x_{4,2} \in \tfrac{1}{2} + \mathbb{Z},\ y_{4,2} \notin \mathbb{Z},\ z_{4,2} \in \mathbb{Z} \tag{3.39}$$

and

$$\lambda_3 - \lambda_1 = (x_{3,1}, y_{3,1}, z_{3,1})^{\mathrm{T}} \in B_2 \colon x_{3,1} \notin \mathbb{Z},\ y_{3,1} \in \tfrac{1}{2} + \mathbb{Z},\ z_{3,1} \in \mathbb{Z}. \tag{3.40}$$

Now we consider (3.36). If $\lambda_5 - \lambda_3 \in B_1$, then, by Lemma 3.1 and $\lambda_3 - \lambda_1 = (\lambda_5 - \lambda_1) - (\lambda_5 - \lambda_3) \in B_1 - B_1$, we have $x_{3,1} \in \mathbb{Z}$: a contradiction of (3.40). If $\lambda_5 - \lambda_3 \in B_2$, then, from $\lambda_3 - \lambda_2 = (\lambda_5 - \lambda_2) - (\lambda_5 - \lambda_3) \in B_2 - B_2$ and $\lambda_5 - \lambda_1 = (\lambda_5 - \lambda_3) + (\lambda_3 - \lambda_1) \in B_2 + B_2$, we have (by applying Lemma 3.1 (d), (3.6) and parts (i) and (iii) of Lemma 3.2)

$$\lambda_3 - \lambda_2 = (x_{3,2}, y_{3,2}, z_{3,2})^{\mathrm{T}} \in B_3 \colon x_{3,2} \notin \mathbb{Z},\ y_{3,2} \in \mathbb{Z},\ z_{3,2} \in \tfrac{1}{2} + \mathbb{Z} \tag{3.41}$$

and

$$\lambda_5 - \lambda_1 = (x_{5,1}, y_{5,1}, z_{5,1})^{\mathrm{T}} \in B_1 \colon x_{5,1} \in \tfrac{1}{2} + \mathbb{Z},\ y_{5,1} \in \mathbb{Z},\ z_{5,1} \notin \mathbb{Z}. \tag{3.42}$$

From the equality

$$\lambda_2 - \lambda_1 = (\lambda_5 - \lambda_1) - (\lambda_5 - \lambda_2) = (\lambda_3 - \lambda_1) - (\lambda_3 - \lambda_2) = (\lambda_4 - \lambda_1) - (\lambda_4 - \lambda_2),$$

we see, from (3.42) and (3.10), (3.40) and (3.41), (3.34) and (3.39) that

$$\lambda_2 - \lambda_1 = (x_{2,1}, y_{2,1}, z_{2,1})^{\mathrm{T}} \in (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) + \mathbb{Z}^3,$$

which contradicts (3.6) and Lemma 3.1 (b).

Parts (i) and (ii) above and (3.35) illustrate that Case 1.1.3.1 is proved, and Case 1.1.3 is proved.

Thus, the proof of Case 1 is completed.

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3.4. Case 2

In Case 2, we have $\lambda_3 - \lambda_2 = (\lambda_3 - \lambda_1) - (\lambda_2 - \lambda_1) \in B_1 - B_1$, and $\lambda_5 - \lambda_4 = (\lambda_5 - \lambda_1) - (\lambda_4 - \lambda_1) \in B_2 - B_2$, which shows (by Lemmas 3.1 and 3.2) that

$$\lambda_3 - \lambda_2 = (x_{3,2}, y_{3,2}, z_{3,2})^{\mathrm{T}} \in B_2 \cup B_3 \colon x_{3,2} \in \mathbb{Z},\ y_{3,2} \notin \mathbb{Z},\ z_{3,2} \notin \mathbb{Z} \tag{3.43}$$

and

$$\lambda_5 - \lambda_4 = (x_{5,4}, y_{5,4}, z_{5,4})^{\mathrm{T}} \in B_1 \cup B_3 \colon x_{5,4} \notin \mathbb{Z},\ y_{5,4} \in \mathbb{Z},\ z_{5,4} \notin \mathbb{Z}. \tag{3.44}$$

According to $\lambda_3 - \lambda_2 \in B_2 \cup B_3$, Case 2 can be divided into the following two cases.

Case 2.1. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1 \in B_1$ and $\lambda_4 - \lambda_1, \lambda_5 - \lambda_1, \lambda_3 - \lambda_2 \in B_2$.

Case 2.2. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1 \in B_1$, $\lambda_4 - \lambda_1, \lambda_5 - \lambda_1 \in B_2$ and $\lambda_3 - \lambda_2 \in B_3$.

Case 2.2 is similar to Case 1.1, so we only need to deal with Case 2.1. In this case, (3.43) becomes

$$\lambda_3 - \lambda_2 = (x_{3,2}, y_{3,2}, z_{3,2})^{\mathrm{T}} \in B_2 \colon x_{3,2} \in \mathbb{Z},\ y_{3,2} \in \tfrac{1}{2} + \mathbb{Z},\ z_{3,2} \notin \mathbb{Z}. \tag{3.45}$$

According to $\lambda_5 - \lambda_4 \in B_1 \cup B_3$, Case 2.1 can be divided into the following two cases.

Case 2.1.1. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_5 - \lambda_4 \in B_1$ and $\lambda_4 - \lambda_1, \lambda_5 - \lambda_1, \lambda_3 - \lambda_2 \in B_2$.

Case 2.1.2. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1 \in B_1$, $\lambda_4 - \lambda_1, \lambda_5 - \lambda_1, \lambda_3 - \lambda_2 \in B_2$ and $\lambda_5 - \lambda_4 \in B_3$.

Case 2.1.2 is similar to Case 1.1.2, so we only need to deal with Case 2.1.1. In this case, (3.44) becomes

$$\lambda_5 - \lambda_4 = (x_{5,4}, y_{5,4}, z_{5,4})^{\mathrm{T}} \in B_1 \colon x_{5,4} \in \tfrac{1}{2} + \mathbb{Z},\ y_{5,4} \in \mathbb{Z},\ z_{5,4} \notin \mathbb{Z}. \tag{3.46}$$

Consider the remainder difference $\lambda_4 - \lambda_2$ in (3.7). According to $\lambda_4 - \lambda_2 \in B_1 \cup B_2 \cup B_3$, Case 2.1.1 can be divided into three cases.

Case 2.1.1.1. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_5 - \lambda_4, \lambda_4 - \lambda_2 \in B_1$, $\lambda_4 - \lambda_1, \lambda_5 - \lambda_1, \lambda_3 - \lambda_2 \in B_2$.

Case 2.1.1.2. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_5 - \lambda_4 \in B_1$, $\lambda_4 - \lambda_1, \lambda_5 - \lambda_1, \lambda_3 - \lambda_2, \lambda_4 - \lambda_2 \in B_2$.

Case 2.1.1.3. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_5 - \lambda_4 \in B_1$, $\lambda_4 - \lambda_1, \lambda_5 - \lambda_1, \lambda_3 - \lambda_2 \in B_2$, $\lambda_4 - \lambda_2 \in B_3$.

Case 2.1.1.3 is similar to Case 1.1.1.1. Case 2.1.1.2 is similar to Case 2.1.1.1, so we only need to deal with Case 2.1.1.1. In this case, from $\lambda_5 - \lambda_2 = (\lambda_5 - \lambda_4) + (\lambda_4 - \lambda_2) \in B_1 + B_1$ and $\lambda_4 - \lambda_1 = (\lambda_4 - \lambda_2) + (\lambda_2 - \lambda_1) \in B_1 + B_1$, we have

$$\lambda_5 - \lambda_2 = (x_{5,2}, y_{5,2}, z_{5,2})^{\mathrm{T}} \in B_2 \cup B_3 \colon x_{5,2} \in \mathbb{Z},\ y_{5,2} \notin \mathbb{Z},\ z_{5,2} \notin \mathbb{Z} \tag{3.47}$$

and

$$\lambda_4 - \lambda_1 = (x_{4,1}, y_{4,1}, z_{4,1})^{\mathrm{T}} \in B_2 \colon x_{4,1} \in \mathbb{Z},\ y_{4,1} \in \tfrac{1}{2} + \mathbb{Z},\ z_{4,1} \notin \mathbb{Z}. \tag{3.48}$$

If $\lambda_5 - \lambda_2 \in B_2$, then, (3.47) becomes

$$\lambda_5 - \lambda_2 = (x_{5,2}, y_{5,2}, z_{5,2})^{\mathrm{T}} \in B_2 \colon x_{5,2} \in \mathbb{Z},\ y_{5,2} \in \tfrac{1}{2} + \mathbb{Z},\ z_{5,2} \notin \mathbb{Z}, \tag{3.49}$$

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which, combined with (3.45), yields

$$\lambda_5 - \lambda_3 = (\lambda_5 - \lambda_2) - (\lambda_3 - \lambda_2) = (x_{5,3}, y_{5,3}, z_{5,3})^{\mathrm{T}} \colon x_{5,3} \in \mathbb{Z},\ y_{5,3} \in \mathbb{Z},\ z_{5,3} \notin \mathbb{Z}, \quad (3.50)$$

which contradicts parts (ii) and (iii) of Lemma 3.2, respectively. Hence, $\lambda_5 - \lambda_2 \in B_3$, and Case 2.1.1.1 can be treated as before. So the same method can be applied here to complete the proof of Case 2.

Summing up the above discussion, we know that any set of orthogonal exponentials in $L^2(\mu_{M,D})$ contains at most 4 elements. One can obtain many such orthogonal systems which contain four elements. For instance, the exponential function system $E(S)$ with $S \subset \mathbb{R}^3$ given by (2.4) is an orthogonal system in $L^2(\mu_{M,D})$ consisting of 4 exponential functions. This shows that the number 4 is the best possible. The proof of Theorem 1.1 (ii) is complete.

Note that Theorem 1.1 (ii) can be stated in a more general form. For example, we also have the following corollary.

Corollary 3.4. *If M and D are given by (2.13) with $p_1, p_2, p_3 \in (2\mathbb{Z}+1) \setminus \{\pm 1\}$ and $d_1, d_2, d_3 \in \mathbb{R} \setminus \{0\}$, then $\mu_{M,D}$ is a non-spectral measure, and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, where the number 4 is the best possible.*

To conclude this section, we point out that the method used to prove Theorem 1.1 (ii) essentially gives us the following useful hints on the non-spectrality of self-affine measures.

If $Z(\hat{\mu}_{M,D}(\xi)) = \bigcup_{j=1}^{k} B_j$ is a finite union of sets $B_1, B_2, \dots, B_k$ in $\mathbb{R}^n$ such that for each orthogonal system $E(\Lambda)$ in $L^2(\mu_{M,D})$ with $\Lambda \subset \mathbb{R}^n$, B_j, $j = 1, 2, \dots, k$, contain finite elements of set $\Lambda - \{\lambda_i\}$ for a fixed $\lambda_i \in \Lambda$, then $\mu_{M,D}$ is a non-spectral measure and Λ is a finite set. Furthermore, to obtain the best upper bound of $|\Lambda|$, one needs some combinative techniques along with more properties on the sets B_j, $j = 1, 2, \dots, k$.

4. A supplement to the planar Sierpinski family

The generalized planar Sierpinski family corresponds to the expanding integer matrix $M \in M_2(\mathbb{Z})$ and the three-element digit set D given by

$$M = \begin{bmatrix} a & b \\ d & c \end{bmatrix} \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}. \quad (4.1)$$

Li [**22**] proved that if $ac - bd \notin 3\mathbb{Z}$, then there exist at most 3 mutually orthogonal exponentials in $L^2(\mu_{M,D})$, and the number 3 is the best possible. Note that the proof of this result in [**22**] contains an incomplete statement. For the sake of completeness, we add here a supplement to the proof of the main result in [**22**].

The statement 'the remainder three elements (3.7) will be in the three different small boxes also' on [**22**, p. 548] is not correct. It needs the fact that we can deal with the case when these three remainder elements (3.7) are in two different small boxes. (It is clear

Table 1. *Distribution of differences*

Z_1	Z_2	Z_3	Z_4	$\tilde{Z}_1$	$\tilde{Z}_2$	$\tilde{Z}_3$	$\tilde{Z}_4$
$\lambda_1-\lambda_2$	$\lambda_1-\lambda_3$	$\lambda_1-\lambda_4$					
				$\lambda_2-\lambda_1$	$\lambda_3-\lambda_1$	$\lambda_4-\lambda_1$	
			$\lambda_2-\lambda_3$				$\lambda_3-\lambda_2$
			$\lambda_3-\lambda_4$				$\lambda_4-\lambda_3$
			$\lambda_4-\lambda_2$				$\lambda_2-\lambda_4$

that they cannot be in one small box.) Here we provide a proof of this fact. The three remainder elements (3.7) of [**22**] may be in two small boxes, such as

$$\lambda_2-\lambda_3\in Z_4,\qquad \lambda_2-\lambda_4\in\tilde{Z}_4,\qquad \lambda_3-\lambda_4\in Z_4 \tag{4.2}$$

or $\lambda_2-\lambda_3\in\tilde{Z}_4$, $\lambda_2-\lambda_4\in Z_4$, $\lambda_3-\lambda_4\in\tilde{Z}_4$. We use the same symbol as in [**22**]. If this case happens, then there exists another typical case among the cases (3.14) of [**22**] that should be treated separately. This typical case is

$$\left.\begin{aligned}&\lambda_1-\lambda_2\in Z_1,\quad \lambda_1-\lambda_3\in Z_2,\quad \lambda_1-\lambda_4\in Z_3,\\&\lambda_2-\lambda_3\in Z_4,\quad \lambda_2-\lambda_4\in\tilde{Z}_4,\quad \lambda_3-\lambda_4\in Z_4,\end{aligned}\right\} \tag{4.3}$$

and the corresponding Box 1 in [**22**] becomes Table 1 by [**22**, Proposition 2].

For this typical case, we can deduce a contradiction in the following manner. Note that the proof of this typical case, though implicit in the author's previous research (the previous research illustrates that a small box such as Z_4 in Table 1 cannot contain many elements), does not appear there explicitly. We present it here briefly.

From the definition of Z_l $(l=1,2,3,4)$ (the same method applies to $\tilde{Z}_l$ $(l=1,2,3,4)$) in [**22**], we see that there are integer numbers $k_{ij},k'_{ij}\in\mathbb{Z}$, $i=1,2,3,4$, $j=2,3,4$, such that

$$\begin{aligned}\lambda_1-\lambda_2&=M^*\begin{pmatrix}\frac13+k_{12}\\ \frac23+k'_{12}\end{pmatrix}, &\qquad \lambda_1-\lambda_3&=M^{*2}\begin{pmatrix}\frac13+k_{13}\\ \frac23+k'_{13}\end{pmatrix},\\ \lambda_1-\lambda_4&=M^{*3}\begin{pmatrix}\frac13+k_{14}\\ \frac23+k'_{14}\end{pmatrix}, &\qquad \lambda_2-\lambda_3&=M^{*4}\begin{pmatrix}\frac13+k_{23}\\ \frac23+k'_{23}\end{pmatrix},\\ \lambda_3-\lambda_4&=M^{*4}\begin{pmatrix}\frac13+k_{34}\\ \frac23+k'_{34}\end{pmatrix}, &\qquad \lambda_4-\lambda_2&=M^{*4}\begin{pmatrix}\frac13+k_{42}\\ \frac23+k'_{42}\end{pmatrix}\end{aligned}$$

hold. Then, from $(\lambda_1-\lambda_3)-(\lambda_1-\lambda_2)=\lambda_2-\lambda_3$, we have

$$M^*\begin{pmatrix}\frac13+k_{13}\\ \frac23+k'_{13}\end{pmatrix}-\begin{pmatrix}\frac13+k_{12}\\ \frac23+k'_{12}\end{pmatrix}=M^{*3}\begin{pmatrix}\frac13+k_{23}\\ \frac23+k'_{23}\end{pmatrix}. \tag{4.4}$$

Similarly, we have

$$M^*\begin{pmatrix}\frac{1}{3}+k_{14}\\ \frac{2}{3}+k'_{14}\end{pmatrix}-\begin{pmatrix}\frac{1}{3}+k_{13}\\ \frac{2}{3}+k'_{13}\end{pmatrix}=M^{*2}\begin{pmatrix}\frac{1}{3}+k_{34}\\ \frac{2}{3}+k'_{34}\end{pmatrix}, \tag{4.5}$$

$$\begin{pmatrix}\frac{1}{3}+k_{12}\\ \frac{2}{3}+k'_{12}\end{pmatrix}-M^{*2}\begin{pmatrix}\frac{1}{3}+k_{14}\\ \frac{2}{3}+k'_{14}\end{pmatrix}=M^{*3}\begin{pmatrix}\frac{1}{3}+k_{42}\\ \frac{2}{3}+k'_{42}\end{pmatrix}. \tag{4.6}$$

In type 4 of [**22**], the expanding matrix M is given by

$$M^*=\begin{bmatrix}a & d\\ b & c\end{bmatrix}=3\begin{bmatrix}l_1 & l_4\\ l_2 & l_3\end{bmatrix}+M_\alpha := 3\tilde{M}+M_\alpha \tag{4.7}$$

for $\alpha=3,5,10,12,19,22,26,28,37,42,44,48$, where

$$\left.\begin{aligned}
&M_3=\begin{bmatrix}0&1\\1&1\end{bmatrix}, \quad M_5=\begin{bmatrix}0&1\\1&2\end{bmatrix}, \quad M_{10}=\begin{bmatrix}0&2\\2&1\end{bmatrix}, \quad M_{12}=\begin{bmatrix}0&2\\2&2\end{bmatrix},\\
&M_{19}=\begin{bmatrix}1&1\\1&0\end{bmatrix}, \quad M_{22}=\begin{bmatrix}1&2\\1&1\end{bmatrix}, \quad M_{26}=\begin{bmatrix}1&2\\2&0\end{bmatrix}, \quad M_{28}=\begin{bmatrix}1&1\\2&1\end{bmatrix},\\
&M_{37}=\begin{bmatrix}2&1\\1&0\end{bmatrix}, \quad M_{42}=\begin{bmatrix}2&2\\1&2\end{bmatrix}, \quad M_{44}=\begin{bmatrix}2&2\\2&0\end{bmatrix}, \quad M_{48}=\begin{bmatrix}2&1\\2&2\end{bmatrix}.
\end{aligned}\right\} \tag{4.8}$$

From (4.7) and (4.8), we can verify that in each case of type 4, none of the equalities (4.4)–(4.6) holds. For example, consider the case (M_{22}); we see that the left-hand side of (4.4) is in $(\frac{1}{3},\frac{1}{3})^{\mathrm{T}}+\mathbb{Z}^2$, but the right-hand side of (4.4) is in $(0,\frac{1}{3})^{\mathrm{T}}+\mathbb{Z}^2$, so (4.4) does not hold; the left-hand side of (4.5) is in $(\frac{1}{3},\frac{1}{3})^{\mathrm{T}}+\mathbb{Z}^2$, but the right-hand side of (4.5) is in $(\frac{2}{3},\frac{2}{3})^{\mathrm{T}}+\mathbb{Z}^2$, so (4.5) does not hold; the left-hand side of (4.6) is in $(\frac{2}{3},0)^{\mathrm{T}}+\mathbb{Z}^2$, but the right-hand side of (4.6) is in $(0,\frac{1}{3})^{\mathrm{T}}+\mathbb{Z}^2$, so (4.6) does not hold. The other cases of type 4 can be proved in the same way. This completes the proof of the main result of [**22**] in the case similar to (4.3) or Table 1.

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