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A COMMUTATIVITY THEOREM FOR NEAR-RINGS

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A ring or near-ring R is called *periodic* if for each $x \in R$, there exist distinct positive integers n, m for which $x^n = x^m$. A well-known theorem of Herstein states that a periodic ring is commutative if its nilpotent elements are central [5], and Ligh [6] has asked whether a similar result holds for distributively-generated (d-g) near-rings. It is the purpose of this note to provide an affirmative answer.

Our definition of near-ring assumes *left* distributivity, and the words "center" and "central" refer to multiplication. The term *$R-R$ subgroup* denotes an additive subgroup S of R such that $RS \subseteq S$ and $SR \subseteq S$. An element of a subnear-ring T is called *T-distributive* if it is distributive with respect to addition in T. The near-ring R is said to be the supplementary sum of its subnear-rings A and B—denoted by $R = A \dot{+} B$—if each element of R can be uniquely represented in the form $a+b$, where $a \in A$ and $b \in B$.

THEOREM. *Let R be a distributively-generated near-ring with its nilpotent elements lying in the center. Then the set N of nilpotent elements forms an ideal; and if R/N is periodic, R must be commutative.*

Proof of Theorem.

LEMMA 1. *If R is a d-g near-ring in which nilpotent elements are central, then the set N of nilpotent elements is an ideal.*

Proof. Let $u_1, u_2 \in N$ and $r \in R$. It is obvious that ru_1 and $u_1 r \in N$; and the usual argument for commutative rings, which does not require additive commutativity, shows that $u_1 - u_2 \in N$. It remains to show that N^+ is a normal subgroup of R^+, and this we do by induction on the degree of nilpotence.

If $u^2 = 0$ and $r \in R$, then $(r+u-r)^2 = (r+u-r)r + (r+u-r)u - (r+u-r)r = (r+u-r)r + u(r+u-r) - (r+u-r)r = 0$. Now suppose $r+u-r$ is nilpotent for arbitrary $r \in R$ and $u \in N$ with index of nilpotence less than k, $k \geq 3$; and let $u \in N$ satisfy $u^k = 0$. Then, letting $a = (r+u-r)r$ and proceeding as above, we have $(r+u-r)^2 = a + (ur + u^2 - ur) - a = (a+ur) + u^2 - (a+ur)$, which is nilpotent since $(u^2)^{k-1} = 0$. Thus $r+u-r \in N$.

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LEMMA 2. *Let R be a d-g near-ring with central nilpotent elements, and suppose that R/N is periodic. Then*

(1) *for each $x \in R$, there exists an integer $n(x) > 1$ for which $x - x^{n(x)}$ belongs to N;*

(2) *the commutator subgroup R' of R^+ is contained in N;*

(3) *if $a, b \in R$ and $ab = 0$, then $ba = 0$.*

Proof. The d-g near-ring $\bar{R} = R/N$ is periodic with no non-zero nilpotent elements; and it is easily shown that if $w \in \bar{R}$ and $w^n = w^m$ for $n > m > 1$, then $w = w^{n-m+1}$. Therefore, for each $x \in R$, there exists an integer $n(x) > 1$ for which $x - x^{n(x)} \in N \subseteq Z$–i.e. (1) holds. Since $\bar{R}$ is distributively-generated, Theorem 2 of [1] guarantees that $\bar{R}$ is additively commutative; and (2) follows at once. The proof of (3) is that of part (A) of Lemma 3 in [2].

Part (3) of Lemma 2 guarantees that there is no distinction between left and right annihilators. Henceforth we shall denote the annihilator of an element x of R by $A(x)$.

LEMMA 3. *For any R satisfying the hypotheses of Lemma 2, each of the following statements holds:*

(1) *If e is an idempotent which is S-distributive for some $R-R$ subgroup S, then e is central in R.*

(2) *For each non-zero central idempotent e of R, $R' \subseteq A(e)$.*

(3) *If $e_1, e_2, \ldots, e_k$ are pairwise-orthogonal central idempotents, then $e_1R + \cdots + e_kR$ is a commutative ring.*

(4) *For each central idempotent e of R, $R = eR \dot{+} A(e)$.*

(5) *If $R = A \dot{+} B$ for a pair of orthogonal subnear-rings A and B, and if the distributive element d of R is expressed as $d_1 + d_2$ with $d_1 \in A$ and $d_2 \in B$, then d_2 is B-distributive.*

Proof. (1) Let e be an S-distributive idempotent of the $R-R$ subgroup S, and let x be an arbitrary element of R. Since both ex and exe belong to S, we have $(ex - exe)e = (ex)e - (exe)e = 0$; and by (3) of Lemma 2, $e(ex - exe) = ex - exe = 0$. Similarly, $xe - exe = 0$; thus $e \in Z$.

(2) Since e is a central idempotent, eR is a distributively-generated near-ring with identity e and has the property that $x - x^{n(x)}$ is central for each $x \in eR$. Thus, eR is a commutative *ring* by Theorem 2 of [2]; and $e(x + y - x - y) = 0$ for all $x, y \in R$.

(3) Note that $e_1R + \cdots + e_kR + R'$ is an $R-R$ subgroup containing the idempotent $e = \sum e_i$. Let $x = e_1r_1 + \cdots + e_kr_k + r'$, where $r_1, \ldots r_k \in R$ and $r' \in R'$; and use the distributivity and pairwise-orthogonality of the e_i plus the fact that R' annihilates each e_i, to get the result that $ex = xe = \sum_{i=1}^{k} e_ir_i$. Thus, by (1) of Lemma 3, e is a central idempotent of R; and another appeal to Theorem 2 of [2] gives the result that eR is a commutative ring. Since $e_iR \subseteq eR$ for each i, we are finished.

(4) It is trivial to show that $r = er + (-er + r)$ is the unique representation of $r \in R$ in the form $a + b$, $a \in eR$, $b \in A(e)$.

(5) Let the distributive element d be written in the form $d_1 + d_2$, $d_1 \in A$, $d_2 \in B$. If x, y are arbitrary elements of B, then $(x+y)d = (x+y)d_1 + (x+y)d_2 = (x+y)d_2$ since A and B are orthogonal; on the other hand, by the distributivity of d we get $(x+y)d = xd + yd = xd_1 + xd_2 + yd_1 + yd_2 = xd_2 + yd_2$. Thus d_2 is distributive in B.

LEMMA 4. *Let R satisfy the hypotheses of Lemma 2, and let d be any distributive element of R. Then there exist an integer $n > 1$ and a central idempotent f of R for which $d - d^n \in N$ and $d^n R = fR$.*

Proof. By (1) of Lemma 2, there exists a positive integer j, which we may choose to be at least 3, for which $d - d^j \in N$ and hence $d^k - d^{k+s(j-1)} \in N \subseteq Z$ for all positive integers k and s. Since $d^k - d^{k+s(j-1)}$ commutes with $d + d$, we have $-d^{k+1+s(j-1)} + d^{k+1} = d^{k+1} - d^{k+1+s(j-1)}$, so that d^{k+1} and $d^{k+1+s(j-1)}$ commute additively; and choosing $k+1 = t(j-1)$, $t = 1, 2, \ldots$, shows that all powers of d^{j-1} are additively commutative. Thus the additive subgroup S generated by the powers of d^{j-1} is a d-g near-ring with commutative addition, and hence a ring by a well-known theorem of Fröhlich [4]. Since the ring S has the property that $x - x^{n(x)}$ is nilpotent for each $x \in S$, it is periodic by a theorem of Chacron [3]; therefore, there exist integers p, q with $p > q$, such that $d^p = d^q$ and hence $d^k = d^{k+s(p-q)}$ for all non-negative integers s and all $k \geq q$. In view of the last observation, we may assume that $p - q + 1 \geq q > 1$.

Let $n = p - q + 1$. Then $d - d^n \in N$ by the proof of part (1) of Lemma 2. Moreover, since $n \geq q$ and $n^2 - n$ is divisible by $p - q$, we see that $(d^n)^n = d^n$. It follows at once that $f = d^{n(n-1)}$ is a distributive (hence central) idempotent and that $d^n R = fR$.

Proof of the Theorem. Let x and y be an arbitrary pair of elements of R and $d_1, d_2, \ldots, d_t$ distributive elements of R which generate an additive subgroup containing both x and y; suppose $d_1, \ldots, d_k$ are non-nilpotent and $d_{k+1}, \ldots, d_t$ are nilpotent. For each $i = 1, \ldots, k$, choose an integer $n_i > 1$ and a central idempotent f_i such that $d_i - d_i^{n_i} \in N$ and $f_i R = d_i^{n_i} R$. In view of the fact that $R' \subseteq N$, $\sum f_i R + N$ is a subnear-ring which contains both x and y.

The next step is to construct a set of *pairwise-orthogonal* central idempotents $e_1, \ldots, e_k$ such that $\sum f_i R + N \subseteq \sum e_i R + N$. The case $k = 1$ being immediate, we suppose that we have already obtained pairwise-orthogonal idempotents $e_i, \ldots, e_s$ (some of which may be trivial) such that $\sum_{i=1}^{s} f_i R + N \subseteq \sum_{i=1}^{s} e_i R + N$. By repeated application of (4) of Lemma 3, write $R = A + B$, where $A = \sum_{i=1}^{s} e_i R$ and $B = \bigcap_{i=1}^{s} A(e_i)$; and let $f_{s+1} = g + h$, where $g \in A$ and $h \in B$. Now $(g+h)^2 = (g+h)g + (g+h)h = g(g+h) + h(g+h) = g^2 + h^2 = g + h$, and by uniqueness of representation $h^2 = h$. Since h is B-distributive by (5) of

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Lemma 3, h must be central in R; and the fact that R is distributively-generated and that $R' \subseteq N$ shows that $f_{s+1}R = (g+h)R \subseteq gR + hR + N$. Denoting h by e_{s+1}, and appealing to the inductive hypothesis, we get $\sum_{i=1}^{s+1} f_iR + N \subseteq \sum_{i=1}^{s+1} e_iR + N$; and our construction is finished.

It remains only to show that $\sum_{i=1}^{k} e_iR + N$ is commutative. Accordingly let $u = e_1r_1 + \cdots + e_kr_k + w$ and $v = e_1s_1 + \cdots + e_ks_k + z$, where $w, z \in N$. Then, in view of the additive commutativity asserted by (3) of Lemma 3,

$$uv = \sum_{i=1}^{k} (e_1r_1 + \cdots + e_kr_k + w)e_is_i + \left(\sum_{i=1}^{k} e_ir_i + w\right)z$$

$$= \sum_{i=1}^{k} e_ir_ie_is_i + \sum_{i=1}^{k} e_iwe_is_i + \sum_{i=1}^{k} e_ir_ie_iz + wz; \quad \text{and}$$

$$vu = \sum_{i=1}^{k} e_is_ie_ir_i + \sum_{i=1}^{k} e_ize_ir_i + \sum_{i=1}^{k} e_is_ie_iw + zw.$$

Now using the multiplicative commutativity of $\sum e_iR$ and the centrality of nilpotent elements, we see that $uv = vu$. This completes the proof of the theorem.

REMARK. If R has 1, it must in fact be a ring-this follows from Therorem 2 of [2]; however, in the absence of a multiplicative identity element, R^+ need not be abelian.

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