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HEIGHTS AND L-SERIES

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0. Introduction. Let $f(z) = \sum_{m \geqq 1} a_m e^{2\pi i m z}$ be a cusp form of weight $2k$ and trivial character for $\Gamma_0(N)$, where N is prime, which is orthogonal with respect to the Petersson product to all forms $g(dz)$, where g is of level $L < N$, $dL|N$. Let K be an imaginary quadratic field of discriminant $-D$ where the prime N is inert. Denote by ϵ the quadratic character of $(\mathbb{Z}/D\mathbb{Z})^*$ determined by $\epsilon(p) = (-D/p)$ for primes p not dividing D. For A an ideal class in K, let $r_A(m)$ be the number of integral ideals of norm m in A. We will be interested in the Dirichlet series $L(f, A, s)$ defined by

$$L(f, A, s) = \sum_{\substack{m \geqq 1 \\ (m,N)=1}} \left(\frac{p\epsilon(m)}{m^{2s-2k+1}} \right) \sum_{m \geqq 1} \left(\frac{a_m r_A(m)}{m^s} \right).$$

It is known that $L(f, A, s)$ admits an analytic continuation to the entire plane. The main identity of this paper is an equation representing the value of $L(f, A, s)$ at $s = k$ in terms of height pairings of special points on a vector bundle V which is associated with the quaternion algebra over $\mathbb{Q}$ ramified at N and ∞.

In the first section, we define quaternion algebras over $\mathbb{Q}$ and discuss some of their properties. We also define the Brandt matrices associated with the quaternion algebra B ramified at N and ∞. Some properties of the Brandt matrices are introduced in section two.

In the third section, the vector bundle V associated with the $2k-1$ dimensional representation of the group B^* is defined. Special points of discriminant $-D$ on V and an action by elements of $\mathrm{Pic}(O)$ on these points are also defined. Here, O is the order of K of discriminant $-D$. We also define a height pairing $\langle \ , \ \rangle$ on V.

Section four introduces an operator t_m on V whose action on $\mathrm{Pic}(V)$ is given by the Brandt matrix $B_{2k-2}(m)$.

After defining modular forms, the Petersson product and the Dirichlet series $L(f, A, s)$ in section five, we are ready to state and prove the main identity in sections six and seven. The proof makes use of the result of Gross and Zagier [**5**, p. 291, Theorem 5.6]. An important part of the proof is the computation of the height pairing $\langle v_B, t_m v_{AB} \rangle$, where v_B and v_{AB} are special points of discriminant $-D$. The computation of $\langle v_B, t_m v_{AB} \rangle$ for $2k = 2$ was done in [**4**].

In section eight, we use the main identity to evaluate the L-series $L(f, \chi, s) = \sum_A \chi(A) L(f, A, s)$ at the value $s = k$, where χ is a complex character of the group $\mathrm{Pic}(O)$, f is a normalized eigenform for the Hecke algebra $\mathbb{T}$, and the sum

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is over all A in $\mathrm{Pic}(O)$. Note that the L-series $L(f,\chi,s)$ always has an Euler product, and when $\chi = 1$, it is the product of two Hecke L-series. We then derive some arithmetic corollaries, some of which are in the special case $\chi = 1$.

1. Quaternion Algebras and the Brandt Matrices. A quaternion algebra B over $\mathbb{Q}$ is a central simple algebra of dimension 4 over $\mathbb{Q}$. Any such B has a basis 1, i, j, k over $\mathbb{Q}$, and multiplication in B is defined by the relations

$$\begin{aligned} i^2 &= a \\ j^2 &= b \\ ij &= -ji = k, \end{aligned} \tag{1.1}$$

where a and b are nonzero elements of $\mathbb{Q}$. Conversely, given any two nonzero elements a and b of $\mathbb{Q}$, the relations in (1.1) define a quaternion algebra over $\mathbb{Q}$. We will denote this quaternion algebra by $B = (a, b)$.

B is said to ramify at a prime p of $\mathbb{Q}$ if $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a division algebra. Similarly, we say that B ramifies at ∞ if $B \otimes_{\mathbb{Q}} \mathbb{R}$ is a division algebra. If B ramifies at ∞, then it is called a definite quaternion algebra. Otherwise, it is called indefinite. The set of primes which ramify in B, including the infinite prime, is finite and has even cardinality. Conversely, given any set consisting of an even number of primes, there exists a quaternion algebra over $\mathbb{Q}$ which ramifies at exactly those primes in the set, and further, the quaternion algebra is unique up to isomorphism.

If $\alpha = x + yi + zj + wk \in B$ with $x, y, z, w \in \mathbb{Q}$, then the conjugate $\bar{\alpha}$ of α is defined to be $\bar{\alpha} = x - yi - zj - wk$. The reduced norm $\mathbb{N}$ of B is defined by $\mathbb{N}(\alpha) = \alpha\bar{\alpha}$, and the reduced trace Tr is given by $\mathrm{Tr}(\alpha) = \alpha + \bar{\alpha}$.

A lattice on B is a free $\mathbb{Z}$ submodule of B of rank 4. An order of B is a lattice of B which is also a subring containing the identity element. An order is said to be maximal if it is not properly contained in any other order.

We will be interested in the quaternion algebra over $\mathbb{Q}$ which is ramified only at the rational prime N and at ∞. For the remainder, let B denote this definite quaternion algebra.

The following results are from [**7**, pp. 368–369, Propositions 5.1 and 5.2].

PROPOSITION 1.2. *Let N be a rational prime. Then the unique quaternion algebra B over $\mathbb{Q}$ ramified precisely at N and ∞ is given by*

$$\begin{aligned} B &= (-1,-1) && \text{if } N = 2 \\ B &= (-1,-N) && \text{if } N \equiv 3(4) \\ B &= (-2,-N) && \text{if } N \equiv 5(8) \\ B &= (-N,-q) && \text{if } N \equiv 1(8), \end{aligned}$$

where q is a prime with $q \equiv 3(4)$ and $\left(\dfrac{N}{q}\right) = -1$.

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PROPOSITION 1.3. *Let N be a rational prime. Let $B = (a, b)$ be the quaternion algebra over $\mathbb{Q}$ ramified precisely at N and ∞. Then a maximal order of B is given by the $\mathbb{Z}$ basis*

$$\begin{array}{ll} i, \quad j, \quad k, \quad \dfrac{1+i+j+k}{2} & \text{if } N = 2 \\ j, \quad k, \quad \dfrac{1+j}{2}, \quad \dfrac{i+k}{2} & \text{if } N \equiv 3(4) \\ j, \quad k, \quad \dfrac{1+j+k}{2}, \quad \dfrac{i+2j+k}{4} & \text{if } N \equiv 5(8) \\ k, \quad \dfrac{1+j}{2}, \quad \dfrac{i+k}{2}, \quad \dfrac{j+nk}{q} & \text{if } N \equiv 1(8), \end{array}$$

where n is some integer such that $q|(n^2p+1)$.

Let R be a maximal order of B. A left ideal of R is a lattice I on B such that $RI = I$. The right order of I is the set $\{\alpha \in B : I\alpha \subset I\}$, and it is also a maximal order of B. Similarly, we define a right ideal and a left order. The set $I^{-1} = \{\alpha \in B : I\alpha I \subset I\}$ is a right ideal for R, whose left order is the right order of I.

The norm $\mathbb{N}(I)$ of an ideal I of B is defined to be the unique positive rational number such that the quotients $\mathbb{N}(\alpha)/\mathbb{N}(I)$, $\alpha \in I$, are all integers with no common factor.

Two left ideals I and J of R are said to be in the same class if $I = J\alpha$ for some $\alpha \in B^*$. The set of left ideal classes is finite and its order n is independent of the choice of maximal order R. The number n is called the class number of B.

Let $\{I_1, I_2, \ldots, I_n\}$ be a set of left ideals which represent the distinct ideal classes. Let R_i denote the right order of the ideal I_i. Each isomorphism class of maximal orders in B is represented once or twice in the set $\{R_1, R_2, \ldots, R_n\}$. Let t be the number of distinct isomorphism classes of maximal orders in B. The number t is called the type number of B.

An element $\alpha \in R_i$ is a unit if and only if $\mathbb{N}(\alpha) = 1$. Further, $\mathbb{N}$ is a positive definite quadratic form on B. Hence, the number of units in R_i is finite. Let $\Gamma_i = R_i^*/\langle \pm 1\rangle$, then Γ_i is finite. Set $w_i = |\Gamma_i|$. Eichler's mass formula is given by

$$\sum_{i=1}^{n} \frac{1}{w_i} = \frac{N-1}{12}. \tag{1.4}$$

For a proof of this fact see [**1**, p. 147]. Also, the integer $\prod_{i=1}^{n} w_i$ is independent of the choice of R and equal to the exact denominator of the number $(N-1)/12$. We can use these facts to determine n.

There is a useful connection between the quaternion algebra B over $\mathbb{Q}$ which is ramified at the rational prime N and ∞, and the isogenies between supersingular elliptic curves in characteristic N. Let $\mathbb{F}$ denote an algebraically closed field of characteristic N. Then there are exactly n isomorphism classes of supersingular elliptic curves over $\mathbb{F}$, where n is the class number of B. These isomorphism classes can be ordered $E_1, E_2, \ldots, E_n$ so that $\mathrm{End}(E_i) \cong R_i$. We have $I_j^{-1}I_i = \{\sum a_k b_k : a_k \in I_j^{-1}, b_k \in I_i\}$. Then $I_j^{-1}I_i$ is a left ideal of R_j with right order R_i, and we have the isomorphism

$$I_j^{-1}I_i \cong \mathrm{Hom}(E_i, E_j)$$

as a left R_j module and a right R_i module. Let $\phi_b : E_i \longrightarrow E_j$ be the isogeny in $\mathrm{Hom}(E_i, E_j)$ corresponding to a nonzero element $b \in I_j^{-1}I_i$. Then

$$\deg \phi_b = \frac{\mathbb{N}(b)}{\mathbb{N}(I_j^{-1}I_i)} = \frac{(\mathbb{N}(b))(\mathbb{N}(I_j))}{\mathbb{N}(I_i)}.$$

The orders R_i and R_j are conjugate in B if and only if the corresponding elliptic curves are conjugate by an automorphism of the field $\mathbb{F}$.

Now, we will consider representations of B which will lead to the definition of Brandt matrices.

Suppose $B = (a, b)$. Then, by Proposition 1.2, we know $a < 0$ and $b < 0$. B can be represented by a subset of $GL_2(\mathbb{C})$ as follows. Consider the matrices

$$M_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad M_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where the i above is the usual element of $\mathbb{C}$. We can represent $i, j, k \in B$ by

$$i \longmapsto \sqrt{-a}M_1 \qquad j \longmapsto \sqrt{-b}M_2 \qquad k \longmapsto \sqrt{ab}M_3.$$

Thus, the general element $\alpha = x_0 + x_1 i + x_2 j + x_3 k$ with $x_i \in \mathbb{Q}$ can be represented by

$$\alpha \longmapsto x_0 I + x_1\sqrt{-a}M_1 + x_2\sqrt{-b}M_2 + x_3\sqrt{ab}M_3,$$

where I is the 2×2 identity matrix. In other words, B allows a matrix representation

$$\alpha \longmapsto X_1(\alpha) = \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix} \in GL_2(\mathbb{C}).$$

Hence, we have a representation Φ_1 of B^* on $V_2 = \mathbb{C}^2$ with corresponding matrix representation X_1 in terms of the basis $e_1 = (1, 0)$, $e_2 = (0, 1)$ of V_2.

The representation Φ_1 induces a representation Φ_s of B^* on the s^{th} symmetric power of V_2

$$\mathrm{Sym}^s(V_2) = V_2 \otimes \cdots \otimes V_2 / K,$$

the product s times, where K is the symmetric kernel. The set

$$\{\underbrace{e_1 \otimes \cdots \otimes e_1}_{s-i \text{ times}} \otimes \underbrace{e_2 \otimes \cdots \otimes e_2}_{i \text{ times}} (\text{mod } K) : i = 0, \ldots, s\},$$

which we will write as

$$\{e_1^{s-i} e_2^i : i = 0, \ldots, s\},$$

forms a basis for $\mathrm{Sym}^s(V_2)$. The representation Φ_s is given by

$$\begin{aligned}&\Phi_s(\alpha)(e_1 \otimes \cdots \otimes e_1 \otimes e_2 \otimes \cdots \otimes e_2)\\ &= (\Phi_1(\alpha)e_1) \otimes \cdots \otimes (\Phi_1(\alpha)e_1) \otimes (\Phi_1(\alpha)e_2) \otimes \cdots \otimes (\Phi_1(\alpha)e_2),\end{aligned}$$

all read modulo K. Denote the matrix representation corresponding to Φ_s with respect to the basis $\{e_1^{s-i}e_2^i : i = 0, \ldots, s\}$ by X_s. Let X_0 denote the trivial one dimensional representation of B^*. Then, for all $s \geqq 0$, X_s is an $s+1$ dimensional representation of B^*.

We can now define the Brandt matrices. As before, let R be a maximal order of B, let $\{I_1, I_2, \ldots, I_n\}$ be a set of left ideals of R representing the distinct ideal classes, and let R_i denote the right order of I_i. For any integers $s \geqq 0$, $m \geqq 1$, and $1 \leqq i, j \leqq n$, set

$$(1.5) \qquad b_{ij}^s(m) = \frac{1}{2w_j} \sum_{\substack{\alpha \in I_j^{-1} I_i \\ \mathbb{N}(\alpha) = \frac{m\mathbb{N}(I_i)}{\mathbb{N}(I_j)}}} X_s^t(\alpha),$$

where t means transpose. Let $b_{ij}^0(0) = 1/2w_j$ and $b_{ij}^s(0)$ be the $(s+1) \times (s+1)$ zero matrix for $s > 0$. Then for any integers $s \geqq 0$, $m \geqq 0$, the Brandt matrix $B_s(m)$ is defined by

$$B_s(m) = (b_{ij}^s(m)).$$

2. Some Properties of the Brandt Matrices. The Brandt Matrix $B_s(m)$ is an $n(s+1) \times n(s+1)$ matrix with complex entries. For s odd, we have the following result.

PROPOSITION 2.1. *If s is odd, then for all m, $B_s(m)$ is the zero matrix.*

Proof. For s odd, $X_s(-\alpha) = -X_s(\alpha)$ for all $\alpha \in B$. Thus, from (1.5), it follows that $b_{ij}^s(m) = 0$ for all m. Hence, $B_s(m)$ is the zero matrix. □

For a different choice of ideal class representatives $I_1, I_2, \ldots, I_n$, the Brandt matrices change only by conjugation by a matrix with complex coefficients. Further, the Brandt matrices do not depend on the choice of maximal order R

used to define them. For proof of these facts see [**8**, p. 189, Propositions 4.2 and 4.3].

Next, we will derive a formula for the trace of the Brandt matrices. By the above comments, the trace of a Brandt matrix does not depend on the choice of ideal class representatives or maximal order R used to define them. The formula we arrive at will involve modified Hurwitz class numbers, so we will first define these.

For d a negative discriminant, let $h(d)$ be the class number of binary quadratic forms of discriminant d. Define $u(d)$

$$u(d) = \begin{cases} 1 & \text{for } d < -4 \\ 2 & \text{for } d = -4 \\ 3 & \text{for } d = -3. \end{cases}$$

Notice that if O is the order of discriminant d and rank 2 over $\mathbb{Z}$, then $u(d)$ is the order of the group $O^*/\langle \pm 1 \rangle$ and $h(d)$ is the order of the group $\operatorname{Pic}(O)$. For $D > 0$, define the Hurwitz class number $H(D)$ by

$$H(D) = \sum_{df^2 = -D} \frac{h(d)}{u(d)}.$$

For N prime, we now define the modified invariant of the Hurwitz class number, denoted $H_N(D)$, as follows.

$$H_N(D) = \begin{cases} \dfrac{N-1}{24} & \text{if } D = 0 \\ 0 & \text{if } N \text{ splits in } O = O_{-D} \\ H(D) & \text{if } N \text{ is inert in } O \\ \frac{1}{2}H(D) & \text{if } N \text{ ramifies in } O \text{ and does not divide} \\ & \text{the conductor of } O \\ H_N\left(\dfrac{D}{N^2}\right) & \text{if } N \text{ divides the conductor of } O. \end{cases}$$

Since the Brandt matrix $B_s(m)$ for s odd is the zero matrix, its trace is zero. We now want to find a formula for the Brandt matrices $B_s(m)$ for s even. First, we set some notation. For $t \in \mathbb{Z}$, with $t^2 \leqq 4m$, set

$$\lambda = \tfrac{1}{2}(t + \sqrt{t^2 - 4m}),$$

and for s nonnegative and even define

$$P_s(t,m) = \lambda^s + \lambda^{s-1}\bar{\lambda} + \cdots + \bar{\lambda}^s.$$

Note that in the case $s = 0$, we have $P_0(t,m) = 1$. We also have

$$P_2(t,m) = t^2 - m$$
$$P_4(t,m) = t^4 - 3t^2m + m^2.$$

In general, $P_s(t,m)$ is a polynomial in t and m. The trace formula for the Brandt matrices $B_s(m)$ for s even, which is a form of Eichler's trace formula, is given by the following proposition.

PROPOSITION 2.2. *For all $m \geqq 0$, s nonnegative and even*

$$\operatorname{Trace}(B_s(m)) = \sum_{\substack{t\in\mathbb{Z} \\ t^2\leqq 4m}} H_N(4m-t^2)P_s(t,m).$$

We will need the following lemma in the proof of Proposition 2.2.

LEMMA 2.3. *Suppose $\alpha \in B^*$ with* $\operatorname{Tr}(\alpha) = t$, $\mathbb{N}(\alpha) = m$. *Then* $\operatorname{Trace}(X_s^t(\alpha)) = P_s(t,m)$.

Proof. The matrix $X_1(\alpha)$ can be transformed into

$$\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$$

by conjugation by an element of $GL_2(\mathbb{C})$. Hence,

$$\operatorname{Trace}(X_1(\alpha)) = \operatorname{Trace}\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} = \lambda + \bar{\lambda}.$$

Therefore, $\operatorname{Trace}(X_s(\alpha)) = \lambda^s + \lambda^{s-1}\bar{\lambda} + \cdots + \bar{\lambda}^s = P_s(t,m)$, and the lemma follows since $\operatorname{Trace}(X_s^t(\alpha)) = \operatorname{Trace}(X_s(\alpha))$. □

Define $A_i(t,m)$ = cardinality of $\{\alpha \in R_i : \operatorname{Tr}(\alpha) = t, \mathbb{N}(\alpha) = m\}$. Since every $\alpha \in R_i$ has discriminant $t^2 - 4m \leqq 0$, it follows that $A_i(t,m) = 0$ for $t^2 > 4m$. We will need the equality

$$\sum_{i=1}^{n} \left(\frac{A_i(t,m)}{2w_i}\right) = H_N(4m-t^2), \tag{2.4}$$

for $t^2 \leqq 4m$. This is proved in [**4**].

We can now prove Proposition 2.2. For s nonnegative and even,

$$\begin{aligned}
\operatorname{Trace}(B_s(m)) &= \sum_{i=1}^{n} \operatorname{Trace}(b_{ii}^s(m)) \\
&= \sum_{i=1}^{n} \operatorname{Trace}\left(\frac{1}{2w_i} \sum_{\substack{\alpha\in I_i^{-1}I_i=R_i \\ \mathbb{N}(\alpha)=\frac{m\mathbb{N}(I_i)}{\mathbb{N}(I_i)}=m}} X_s^t(\alpha)\right) \\
&= \sum_{i=1}^{n} \left(\frac{1}{2w_i}\right) \sum_{\substack{\alpha\in R_i \\ \mathbb{N}(\alpha)=m}} \operatorname{Trace}(X_s^t(\alpha)) \\
&= \sum_{\substack{t\in\mathbb{Z} \\ t^2\leqq 4m}} \sum_{i=1}^{n} \left(\frac{1}{2w_i}\right) \sum_{\substack{\alpha\in R_i \\ \mathbb{N}(\alpha)=m \\ \operatorname{Tr}(\alpha)=t}} \operatorname{Trace}(X_s^t(\alpha)).
\end{aligned}$$

Applying Lemma 2.3 and (2.4), we have

$$\begin{aligned}
\operatorname{Trace}(B_s(m)) &= \sum_{\substack{t\in\mathbb{Z}\\ t^2\leqq 4m}} \sum_{i=1}^{n} \left(\frac{1}{2w_i}\right) \sum_{\substack{\alpha\in R_i\\ \mathbb{N}(\alpha)=m\\ \operatorname{Tr}(\alpha)=t}} P_s(t,m) \\
&= \sum_{\substack{t\in\mathbb{Z}\\ t^2\leq 4m}} \sum_{i=1}^{n} \left(\frac{A_i(t,m)}{2w_i}\right) P_s(t,m) \\
&= \sum_{\substack{t\in\mathbb{Z}\\ t^2\leqq 4m}} H_N(4m-t^2)P_s(t,m).
\end{aligned}$$

This completes the proof of Proposition 2.2.

The Brandt matrices have several other properties. Some of these are listed in the following two propositions.

PROPOSITION 2.5. *Let B be the quaternion algebra over $\mathbb{Q}$ ramified at N and ∞. Then the Brandt matrices associated to B satisfy the following properties.*

(1) *For $m > 0$,*

$$B_s(m)^t = \begin{pmatrix} I\left(\frac{2w_1}{\mathbb{N}(I_1)}\right) & & 0 \\ & \ddots & \\ 0 & & I\left(\frac{2w_n}{\mathbb{N}(I_n)}\right) \end{pmatrix}^{-1}$$

$$\times B_s(m) \begin{pmatrix} I\left(\frac{2w_1}{\mathbb{N}(I_1)}\right) & & 0 \\ & \ddots & \\ 0 & & I\left(\frac{2w_n}{\mathbb{N}(I_n)}\right) \end{pmatrix}$$

where I is the $(s+1)\times(s+1)$ identity matrix

(2) $B_s(m_1)B_s(m_2) = B_s(m_1m_2)$ *for* $(m_1,m_2)=1$

(3) $B_s(p^i)B_s(p^j) = \sum_{k=1}^{\min(i,j)} p^{(s+1)k}B_s(p^{i+j-2k})$ *for* $p\nmid N$

(4) $B_s(p^i)B_s(p^j) = B_s(p^{i+j})$ *for* $p\mid N$

(5) *The Brandt matrices generate a semisimple commutative ring.*

The proof of Proposition 2.5 can be found in [**2**, p. 106, Theorem 2].

PROPOSITION 2.6. *The entries of the Brandt matrix series*

$$\sum_{m=0}^{\infty} B_s(m)q^m,$$

where $q = e^{2\pi i z}$, are modular forms of weight $s+2$ on $\Gamma_0(N)$. If $s>0$, they are cusp forms.

In fact, the entries of the Brandt matrix series above are of the form

$$\theta(z) = \frac{1}{2w_j} \sum_{\alpha \in I_j^{-1} I_i} p(\alpha) q^{\mathbb{N}(\alpha)\mathbb{N}(I_j)/\mathbb{N}(I_i)},$$

where for $\alpha = x_1 + x_2 i + x_3 j + x_4 k$, $p(\alpha)$ is a harmonic homogeneous polynomial of degree s in $x_1, \ldots, x_4$. For a proof of this fact and Proposition 2.6, see [**7**, p. 353, Theorem 2.14].

3. The Vector Bundle V and its Special Points. We will now describe the quaternion algebra B and its orders and ideals in an adèlic context. This follows the description given in [**4**]. Let $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z} = \prod_p \mathbb{Z}_p$ be the profinite completion of $\mathbb{Z}$, and let $\hat{\mathbb{Q}} = \hat{\mathbb{Z}} \otimes \mathbb{Q} = \coprod_p \mathbb{Q}_p$ be the ring of finite adèles of $\mathbb{Q}$. Set $B_p = B \otimes \mathbb{Q}_p$. Then, $R_p = R \otimes \mathbb{Z}_p$ is the local component of R in B_p. Let $\hat{B} = B \otimes \hat{\mathbb{Q}} = \coprod_p B_p$ and $\hat{R} = R \otimes \hat{\mathbb{Z}} = \prod_p R_p$. Every left ideal I of R is locally principal. Therefore, for every prime p, there exists $g_p \in R_p^* \backslash B_p^*$ such that $I_p = R_p g_p$. Each left ideal I determines an element $g_I = (\ldots g_p \ldots) \in \hat{R}^* \backslash \hat{B}^*$, and conversely, every element g of $\hat{R}^* \backslash \hat{B}^*$ determines a left ideal $I = \hat{R}g \cap B$ of R. The set of left ideals of R which represent the distinct ideal classes, $I_1, I_2, \ldots, I_n$, corresponds to a choice of elements $g_1, g_2, \ldots, g_n$ in $\hat{R}^* \backslash \hat{B}^*$ such that

$$\hat{B}^* = \bigcup_{i=1}^{n} \hat{R}^* g_i B^*.$$

The right order R_i of the ideal I_i is given by

$$R_i = B \bigcap g_i^{-1} \hat{R} g_i.$$

Suppose K is a quadratic field and $f : K \longrightarrow B$ is an embedding of K into B. Let O be the subring of K whose image under f is contained in R_i. Then $f : O \longrightarrow R_i$ is called an optimal embedding of O. For any such f, $f(K) \cap g_i^{-1} \hat{R} g_i = f(O)$ in $\hat{B}$. The set of optimal embeddings of O into the orders R_i modulo conjugation by R_i^*, corresponds to the set of elements (g, f) in $(\hat{R}^* \backslash \hat{B}^* \times \mathrm{Hom}(K, B))/B^*$ such that $f(K) \cap g^{-1} \hat{R} g = f(O)$. The space $(\hat{R}^* \backslash \hat{B}^* \times \mathrm{Hom}(K, B))/B^*$ is related to algebraic curves which we will discuss next.

The quaternion algebra B over $\mathbb{Q}$ corresponds to an algebraic curve Y of genus zero over $\mathbb{Q}$. The correspondence is given by $Y(E) = \{\alpha \in B \otimes E : \alpha \neq 0, \mathrm{Tr}(\alpha) = \mathbb{N}(\alpha) = 0\}/E^*$ for any $\mathbb{Q}$-algebra E. The group B^* acts on the right of Y by conjugation $\alpha \longmapsto b^{-1}\alpha b$, and in fact, $\mathrm{Aut}_{\mathbb{Q}}(Y) = B^*/\mathbb{Q}^*$. For any quadratic field K, there exists a canonical identification $Y(K) \simeq \mathrm{Hom}(K, B)$ as follows: for any $f \in \mathrm{Hom}(K, B)$, let $y_f \in Y(K)$ be the image of the unique K-line on the quadric $\{\alpha \in B \otimes K : \mathrm{Tr}(\alpha) = \mathbb{N}(\alpha) = 0\}$ on which conjugation by $f(K^*)$ acts by multiplication by the character $k \longmapsto k/\bar{k}$. Notice that $f(K^*)$

acting on $Y(K)$ has two fixed points. One of these fixed points is y_f and the other is the image of the unique K-line on which conjugation by $f(K^*)$ acts by multiplication by the character $k \longmapsto \bar{k}/k$.

Define a curve X by

$$X = \hat{R}^* \backslash \hat{B}^* \times Y/B^*.$$

Since $\hat{B}^* = \bigcup_{i=1}^{n} \hat{R}^* g_i B^*$, we have an isomorphism

$$X \cong \coprod_{i=1}^{n} Y/\Gamma_i = \coprod_{i=1}^{n} X_i,$$

where the double coset $\hat{R}^* g_i \times y (\text{mod } B^*)$ corresponds to the coset $y (\text{mod } \Gamma_i)$ on the component $X_i = Y/\Gamma_i$.

A point $x = g \times y$ of X will be called a special point of discriminant d if it lies in the image of $\hat{R}^* \backslash \hat{B}^* \times Y(K)$ in $X(K)$ and if the embedding f corresponding to y satisfies $f(K) \cap g^{-1} \hat{R} g = f(O)$, where O is the order of discriminant d.

We can define an action of the group $\text{Pic}(O) \cong \hat{O}^* \backslash \hat{K}^* / K^*$ on the set of special points of discriminant d as follows. Let $a \in \hat{K}^*$, $x = g \times y$ a special point of discriminant d, and $f : K \longrightarrow B$ the embedding corresponding to y. Then f induces a homomorphism $\hat{f} : \hat{K}^* \longrightarrow \hat{B}^*$. Define

$$x_a = g \hat{f}(a) \times y.$$

The action does not depend upon the choice of representative for x, for if $x \equiv g' \times y'$, then $g' = gb$ and $f' = b^{-1} f b$ for some $b \in B^*$. Thus, $g' \hat{f}'(a) \times y' = gb(b^{-1} \hat{f}(a) b) \times y = g \hat{f}(a) b \times y' \equiv x_a$. To see that this is an action on the set of special points of discriminant d, we need to check that $x_a = g \hat{f}(a) \times y$ has discriminant d. But $f(K) \cap (g \hat{f}(a))^{-1} \hat{R} (g \hat{f}(a)) = (\hat{f}(a))^{-1} (f(K) \cap g^{-1} \hat{R} g) \hat{f}(a) = (\hat{f}(a))^{-1} f(O) \hat{f}(a) = f(O)$. Hence, x_a is a special point of discriminant d.

It is sometimes helpful to think of the points of discriminant d on the component X_i as corresponding to optimal embeddings $f : O \longrightarrow R_i$. In this case, the action of $\text{Pic}(O)$ on the special points of discriminant d can be described in terms of ideals. Let $\mathcal{A} = K \cap a \hat{O}$ be the ideal which is determined by $a \in \text{Pic}(O)$, and let x be a special point corresponding to the optimal embedding $f : O \longrightarrow R_i$. Let R' denote the right order of the left R_i module $R_i \mathcal{A}$. Since O acts on the right of $\mathcal{A}$, it follows that f induces an optimal embedding $O \longrightarrow R'$ which corresponds to the point x_a.

We will define a vector bundle V over X, but first we must make some preliminary definitions.

Let $B_0 = \{b \in B : \text{Tr}(b) = 0\}$, and let U be the representation of B^* on the elements $b_0 \in B_0$ with action by B^* defined by $b_0 \gamma = (1/\mathbb{N}(\gamma)) \gamma^{-1} b_0 \gamma$. Notice that the center $\mathbb{Q}^*$ of B^* acts by $b_0 x = x^{-2} b_0$ for $x \in \mathbb{Q}^*$. An inner product on U is given by $[u_1, u_2] = \frac{1}{2} \text{Tr}(u_1 \overline{u_2})$.

If W is any inner product space with inner product $[x, y]$ for $x, y \in W$, then for $k \geqq 2$, $\mathrm{Sym}^k(W)$ is also an inner product space with inner product defined by

$$(3.1)\qquad [x_1 \ldots x_k, y_1 \ldots y_k] = \sum_{\sigma \in S_k} \prod_{i=1}^{k} [x_i,\, y_{\sigma(i)}],$$

where S_k is the symmetric group on k letters. Hence $[x^k,\, y^k] = k![x,\, y]^k$.

Let $W = \mathbb{C}x \oplus \mathbb{C}y$ be the two dimensional representation of $SU(2)$ with $[x, x] = [y,\, y] = 1$ and $[x,\, y] = 0$. Then $\mathrm{Sym}^{2k-2}(W)$ is an inner product space with basis $\{x^{2k-2}, x^{2k-1}y, \ldots, y^{2k-2}\}$. The elements of this basis are orthogonal, and by (3.1) it follows that

$$[x^i y^j, x^i y^j] = i!j!.$$

Denote $W_{2k-1} = \mathrm{Sym}^{2k-2}(W)$.

The space $\mathrm{Sym}^{k-1}(U)$ is an inner product space, and it can be written as an orthogonal direct sum $\mathrm{Sym}^{k-1}(U) = \mathrm{Sym}^{2k-2}(W) \oplus M$, where W is as defined above. W_{2k-1} is the unique irreducible summand of highest weight $2k-2$, and it is a representation of B^* of dimension $2k-1$.

The vector bundle V is defined by

$$V = \hat{R}^* \backslash \hat{B}^* \times Y \times W_{2k-1}/B^*,$$

where B^* acts on the right as described earlier. The decomposition $\hat{B}^* = \bigcup_{i=1}^{n} \hat{R}^* g_i B^*$ gives an isomorphism

$$V \cong \coprod_{i=1}^{n} Y \times W_{2k-1}/\Gamma_i = \coprod_{i=1}^{n} V_i,$$

which takes $\hat{R}^* g_i \times y \times w(\mathrm{mod}\ B^*)$ to the coset $y \times w(\mathrm{mod}\Gamma_i)$ on the component $Y \times W/\Gamma_i = V_i$.

We now define $\mathrm{Pic}(V)$. For each $\Gamma_i = R_i/\langle \pm 1 \rangle$, define

$$W^{\Gamma_i} = \{w \in W : \gamma(w) = w \text{ for all } \gamma \in \Gamma_i\}.$$

Then

$$\mathrm{Pic}(V) = \bigoplus_{i=1}^{n} W^{\Gamma_i}.$$

Any $v \in V$ is equivalent to the double coset (g_i, y, w) for some i. Define the class of the point v to be the projection of w in W^{Γ_i}

$$\mathrm{class}(v) = \frac{1}{|\Gamma_i|} \sum_{\gamma \in \Gamma_i} \gamma(w).$$

To see what Pic(V) looks like in particular instances, we will consider two examples.

Suppose first that $N = 2$ and $k = 2$. By Proposition 1.2, the quaternion algebra B over $\mathbb{Q}$ ramified at 2 and ∞ is given by $B = (-1, -1)$, so that $B = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}ij$, where $i^2 = j^2 = -1$ and $ij = -ji$. By Proposition 1.3, a maximal order of B is given by

$$R = \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}ij + \mathbb{Z}\left(\frac{1+i+j+ij}{2}\right).$$

The group R^* consists of those elements $b \in R$ with $\mathbb{N}(b) = \pm 1$. In this case, R^* has order 24 and is given by

$$R^* = \left\{\pm 1, \pm i, \pm j, \pm ij, \frac{\pm 1 \pm i \pm j \pm ij}{2}\right\}.$$

Hence,

$$\begin{aligned}\Gamma = \frac{R^*}{\langle \pm 1\rangle} \cong \Bigg\{ & 1, i, j, k, \frac{1+i+j+k}{2}, \frac{1-i+j+k}{2}, \frac{1+i-j+k}{2}, \\ & \frac{1+i+j-k}{2}, \frac{-1+i+j+k}{2}, \frac{-1-i+j+k}{2}, \frac{-1+i-j+k}{2}, \\ & \frac{-1+i+j-k}{2} \Bigg\} (\text{mod } \langle \pm 1\rangle).\end{aligned}$$

In fact, it can be shown that $\Gamma \cong A_4$, the alternating group on 4 letters. Hence, $w = |\Gamma| = 12$. From Eichler's mass formula (1.4), it follows that the class number of B is $n = 1$. Thus, $\text{Pic}(V) = W^{\Gamma}$, where $W = \{b \in B : \text{Tr}(b) = 0\}$. By computing the action of the elements of Γ on W, it can be shown that in this example, $\text{Pic}(V) = 0$.

The dimension of Pic(V) is certainly not always 0. Consider the case $N = 5$ and $k = 2$. By Proposition 1.2, the quaternion algebra B over $\mathbb{Q}$,ramified at 5 and ∞, is given by $B = (-2, -5)$. Hence, $B = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}ij$, where $i^2 = -2$, $j^2 = -5$ and $ij = -ji$. By Proposition 1.3, a maximal order is given by

$$R = \mathbb{Z}j + \mathbb{Z}ij + \mathbb{Z}\left(\frac{1+j+ij}{2}\right) + \mathbb{Z}\left(\frac{i+2j+ij}{4}\right).$$

Therefore,

$$R^* = \left\{\pm 1, \pm\left(\frac{1}{2} + \frac{i}{4} - \frac{ij}{4}\right), \pm\left(\frac{1}{2} - \frac{i}{4} + \frac{ij}{4}\right)\right\},$$

and

$$\Gamma = \frac{R^*}{\langle \pm 1\rangle} \cong \left\{1, \left(\frac{-i}{2} - \frac{i}{4} + \frac{ij}{4}\right), \left(-\frac{1}{2} + \frac{i}{4} - \frac{ij}{4}\right)\right\} (\text{mod } \langle \pm 1\rangle)$$

Hence, $w = |\Gamma| = 3$, and using Eichler's mass formula (1.4), we see that $n = 1$. Therefore, $\mathrm{Pic}(V) = W^{\Gamma}$. Checking the action of Γ on W, it can be shown that $\mathrm{Pic}(V) = W^{\Gamma} = \{\alpha ij : \alpha \in \mathbb{Q}\}$, and hence, $\dim(\mathrm{Pic}(V)) = 1$.

A height pairing $\langle\ ,\ \rangle$ mapping $V \times V$ into $\mathbb{Q}$ can be defined as follows. If $v_1 = g \times \widetilde{y_1} \times \widetilde{w_1} \cong g_i \times y_1 \times w_1$ and $v_2 = h \times \widetilde{y_2} \times \widetilde{w_2} \cong g_j \times y_2 \times w_2$, then

$$\langle v_1, v_2 \rangle = \begin{cases} 0, & \text{if } i \neq j \\ \sum_{\gamma \in \Gamma_i} [w_1, \gamma(w_2)]_{W_{2k-1}} & \text{if } i = j. \end{cases}$$

We will now define a special element w_0 of $\mathrm{Sym}^{2k-2}(W)$. This w_0 will be used in the definition of a special point of V of discriminant $-D$. Fix K an imaginary quadratic field of discriminant $-D$, where the prime N is inert, and an embedding $f : K \longrightarrow B$. Let $v_0 = \sqrt{-D} \in U$. Then v_0^{k-1} lies in $\mathrm{Sym}^{k-1}(U)$. Let $w_0 \in \mathrm{Sym}^{2k-2}(W)$ be the component of v_0^{k-1} in $\mathrm{Sym}^{2k-2}(W)$.

The special points of V of discriminant $-D$ are defined to be points of the form $v = g \times y \times w_0$, where $x = g \times y$ is a special point of X of discriminant $-D$ and $w_0 \in W_{2k-1}$ is defined above. $\mathrm{Pic}(O)$ acts on the special points of V of discriminant $-D$ as follows. Let $v = x \times w_0$ be a special point of V of discriminant $-D$ and $a \in \mathrm{Pic}(O)$. Then $v_a = x_a \times w_0$.

Our main identity is an equation representing the values of certain Dirichlet series in terms of the values of the height pairings of special points on V.

4. The Correspondence t_m. A correspondence t_m can be defined on $V = \hat{R}^*\backslash\hat{B}^* \times Y \times W_{2k-1}/B^*$ as follows. For $p \neq N$, $R_p^*\backslash B_p^* \cong GL_2(\mathbb{Z}_p)\backslash GL_2(\mathbb{Q}_p)$ which index lattices in $\mathbb{Q}_p^2$, and locally, t_m maps a lattice L in $GL_2(\mathbb{Z}_p)\backslash GL_2(\mathbb{Q}_p)$ to the sum of all sublattices of index $p^{\mathrm{ord}_p(m)}$. For $p = N$, $R_p^*\backslash B_p^* \cong \mathbb{Z}$, and locally, t_m maps a coset A into $\pi^{\mathrm{ord}_p(m)}A$, where π is a uniformizing parameter in R_p. This action of t_m on $\hat{R}^*\backslash\hat{B}^*$ induces a correspondence on V

$$t_m(g \times y \times w) = \sum_{h \in t_m(g)} (h \times y \times w).$$

The correspondences t_m on V induce endomorphisms of $\mathrm{Pic}(V) = \bigoplus_{i=1}^n W^{\Gamma_i}$. With a particular choice of basis, the action of t_m on $\mathrm{Pic}(V)$ is given by $B_{2k-2}^t(m)$, the transpose of the Brandt matrix $B_{2k-2}(m)$. To study the action of t_m on $\mathrm{Pic}(V)$, first note that

$$\mathrm{End}(\mathrm{Pic}(V)) = \bigoplus_{i,j} \mathrm{Hom}(W^{\Gamma_i}, W^{\Gamma_j}),$$

and the ij^{th} block of t_m is a homomorphism from W^{Γ_i} to W^{Γ_j}. Recall that each right order R_i corresponds to a supersingular elliptic curve E_i so that $\mathrm{End}(E_i) \cong R_i$. Hence, each g_i corresponds to an elliptic curve E_i. Let $U_i = \mathrm{End}_{\mathrm{Tr}=0}(E_i) \otimes \mathbb{Q}$ be the endomorphisms of trace zero. Let $W_{2k-1}^i \subseteq \mathrm{Sym}^{k-1}(U_i)$ be the unique irreducible summand of highest weight $2k-2$ in $\mathrm{Sym}^{k-1}(U_i)$ as discussed earlier. Each isogeny $\phi \in \mathrm{Hom}(E_i, E_j)$ induces a linear map $\phi_* : U_i \longrightarrow U_j$ by $\alpha \longmapsto \phi \circ \alpha \circ \check{\phi}$, which then induces a linear map $\phi_*^{k-1} : \mathrm{Sym}^{k-1}(U_i) \longrightarrow \mathrm{Sym}^{k-1}(U_j)$ mapping W^i_{2k-1} into W^j_{2k-1}. The action of t_m on $\mathrm{Pic}(V)$ is described in the following proposition.

PROPOSITION 4.1. *The ij^{th} block of t_m is the homomorphism from $(W^i_{2k-1})^{\Gamma_i}$ to $(W^j_{2k-1})^{\Gamma_j}$*

$$\frac{1}{2w_j} \sum_{\substack{\phi \in \mathrm{Hom}(E_i,E_j) \\ \deg \phi = m}} \mathrm{Pr}_j \circ \phi_*^{k-1},$$

where Pr_j denotes the projection into $(W^j_{2k-1})^{\Gamma_j}$ given by $w \longmapsto 1/|\Gamma_j| \sum_{\gamma \in \Gamma_j} \gamma(w)$.

To prove Proposition 4.1, we need to look at how t_m acts on an element $g_i \times y \times w$, where $w \in (W^i_{2k-1})^{\Gamma_i}$. We have

$$\begin{aligned} t_m(g_i \times y \times w) &= \sum_{h \in t_m(g_i)} (h \times y \times w) \\ &= \sum_{\substack{h \in t_m(g_i) \\ h = u g_j \alpha \\ u \in \hat{R}^*, \alpha \in B^*}} (g_j \times y\alpha^{-1} \times w\alpha^{-1}). \end{aligned}$$

Therefore, we find that the $(ij)^{\text{th}}$ block of t_m takes w in $(W^i_{2k-1})^{\Gamma_i}$ to

$$\sum_{\substack{h \in t_m(g_i) \\ h = u g_j \alpha \\ u \in \hat{R}^*, \alpha \in B^*}} \mathrm{Pr}_j(w\alpha^{-1}),$$

which lies in $(W^j_{2k-1})^{\Gamma_j}$. From the definition of t_m and Tate's theorem that an isogeny is uniquely determined by its action on the Tate module T_pE_i and Dieudonné module T_NE_i, it follows that each α in the above sum corresponds to an isogeny $\phi \in \mathrm{Hom}(E_i, E_j)$ of degree m, up to multiplication by $R_j^* \cong \mathrm{Aut}(E_j)$. Further, $w\alpha^{-1} = \phi_*^{k-1}(w)$, and this completes the proof of Propositon 4.1.

5. Modular Forms of Weight $2k$. Let $f : H \longrightarrow \mathbb{C}$ be a function on the upper complex half-plane, ϵ a character of $(\mathbb{Z}/N\mathbb{Z})^*$, and $k \geqq 0$ an integer. Suppose f is holomorphic on H, at infinity, and at the cusps. Suppose, further, that

$$f\left(\frac{az+b}{cz+d}\right) = \epsilon(d)(cz+d)^k f(z),$$

for all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \quad \text{with } c \equiv 0 \bmod M.$$

Then $f(z)$ is called a modular form of weight k for $\Gamma_0(M)$ with character ϵ. If a modular form vanishes at infinity and at the cusps, then it is called a cusp form.

Every modular form f has a Fourier expansion $f(z) = \sum_{m \geqq 0} a_m q^m$, where $q = e^{2\pi i z}$. If f is a cusp form, then $a_0 = 0$, since f vanishes at infinity.

For two modular forms $f(z)$, $g(z)$ of weight k for $\Gamma_0(M)$ with trivial character, at least one of which is a cusp form, the Petersson product of f and g is defined by

$$(f,g) = 2^{k+1}\pi^k \iint_{\Gamma_0(M)\backslash H} f(z)\overline{g(z)}y^{k-2}\,dx\,dy, \qquad z = x+iy,$$

where the integral is taken over any fundamental domain for the action of $\Gamma_0(M)$ on H.

Let $S_{2k}^{\mathrm{new}}(\Gamma_0(M))$ denote the space of cusps forms of weight $2k$ for $\Gamma_0(M)$ with trivial character which are orthogonal with respect to the Petersson product to all forms $g(dz)$, where g is of level $L < M$, $dL \mid M$. For the next three sections, we will fix the following notation. Let K be an imaginary quadratic field of discriminant $-D$ where the prime N is inert. Let $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$. Set

$$\begin{aligned}
O &= \text{ring of integers in } K,\\
\epsilon &= \text{quadratic character of } (\mathbb{Z}/D\mathbb{Z})^* \text{ determined by}\\
&\quad\ \epsilon(p) = (-D/p) \text{ for } p \nmid D,\\
A &= \text{ideal class in } K,\\
h &= \text{class number of } K,\\
2u &= \text{number of units in } K,\\
R(m) &= \begin{cases} \text{number of ideals of } O \text{ of norm } m, & \text{for } m \geqq 1\\ h/(2u), & \text{for } m = 0,\end{cases}\\
r_A(m) &= \begin{cases} \text{number of integral ideals of norm } m \text{ in } A, & \text{for } m \geqq 1\\ 1/(2u), & \text{for } m = 0,\end{cases}\\
\delta(m) &= \begin{cases} 1, & \text{if } (m,D) = 0\\ 2, & \text{if } (m,D) \neq 0.\end{cases}
\end{aligned}$$

We will now define a Dirichlet series associated with f and the ideal class A. Let $\sum_{m \geqq 1} a_m q^m$ be the Fourier expansion of f. Define the Dirichlet series $L(f,A,s)$ by

$$L(f,A,s) = \sum_{\substack{m \geqq 1\\ (m,N)=1}} \left(\frac{\epsilon(m)}{m^{2s-2k+1}}\right) \sum_{m \geqq 1} \left(\frac{a_m r_A(m)}{m^s}\right).$$

$L(f,A,s)$ has an analytic continuation to an entire function of s. It also satisfies the functional equation

$$L^*(f,A,s) \stackrel{\text{def}}{=} (2\pi)^{-2s} N^s D^s \Gamma(s)^2 L(A,f,s) = -\epsilon(N) L^*(f,A,2k-s). \tag{5.1}$$

For proof of these facts, see [**5**, pp. 267, 282–283].

From (5.1), we see that if $\epsilon(N) = 1$, then $L(f, A, s)$ vanishes at $s = k$. We will study the values of $L(f, A, s)$ at $s = k$ in the case $\epsilon(N) = -1$. We will restrict ourselves to the case D prime. Our study will involve the Legendre polynomial of degree n, which is given by

$$P_n(x) = \frac{1}{2^n} \sum_{m=0}^{\left[\frac{n}{2}\right]} (-1)^m \binom{n}{m} \binom{2n-2m}{n} x^{n-2m}.$$

The following theorem is proved in [**5**, p. 291, Theorem 5.6].

THEOREM 5.2. *Let $\epsilon(N) = -1$ and let k be an integer. For $m \geqq 0$, define*

$$b_{m,A} = (mD)^{k-1} u^2 \sum_{0 \leqq n \leqq \frac{mD}{N}} \delta(n) r_A(mD - nN) R(n) P_{k-1}\left(1 - \frac{2nN}{mD}\right).$$

Then $\sum_{m \geqq 0} b_{m,A} q^m$ is a modular form of weight $2k$ and level N, and a cusp form if $k \neq 1$, and

$$L(f, A, k) = \frac{2^{2k-2}(k-1)!}{(2k-2)!} \frac{1}{D^{k-\frac{1}{2}} u^2} \left(f, \sum_{m \geqq 0} b_{m,A} q^m \right).$$

In the next chapter, an alternative representation for the values $L(f, A, k)$ will be given.

Define the series $\phi(v, w)$, $v, w \in \mathrm{Pic}(V)$, as follows

$$\text{For } k > 1, \quad \phi(v, w) = \sum_{m \geqq 1} \langle v, t_m w \rangle q^m,$$

$$\text{For } k = 1, \quad \phi(v, w) = \frac{\deg v \cdot \deg w}{2} + \sum_{m \geqq 1} \langle v, t_m w \rangle q^m.$$

We have the following result

PROPOSITION 5.3. *For any $v, w \in \mathrm{Pic}(V)$, $\phi(v, w)$ is a modular form of weight $2k$ for $\Gamma_0(N)$. If $s > 0$, it is a cusp form.*

Proof. Consider the basis $\{\alpha_1, \ldots, \alpha_n\}$ of $\mathrm{Pic}(V)$, chosen so that the action of t_m on $\mathrm{Pic}(V)$ with respect to this basis is given by $B^t_{2k-2}(m)$. If $v = \alpha_i$ and $w = \alpha_j$, then $\phi(v, w) = \theta_{ji} = \sum_{m \geqq 1} \beta_{ji} q^m$, where β_{ji} is the ji^{th} element of the Brandt matrix $B_{2k-2}(m)$. It follows from Proposition 2.6 that in this case $\phi(v, w)$ is a modular form of weight $2k$ on $\Gamma_0(N)$, and a cusp form if $s > 0$. The result now follows directly. □

6. The Main Identity. The main identity will give the value of $L(f, A, s)$ at $s = k$ in terms of the heights of special points of V of discriminant $-D$.

Let v be a fixed point of discriminant $-D$ on V. Define

$$g_A = \sum_B \phi(v_B, v_{AB}),$$

where the sum is over all classes B in $\operatorname{Pic}(O)$. Then, by Proposition 5.3, g_A is a modular form of weight $2k$ on $\Gamma_0(N)$.

The main identity is

PROPOSITION 6.1. *For $\epsilon(N) = -1$ and $k \geqq 1$ an integer*

$$L(f, A, k) = \frac{(f, g_A)}{u^2 D^{k-\frac{1}{2}}(k-1)!^2}.$$

This identity relates the central critical values of $L(f, A, s)$ with height pairings on $\operatorname{Pic}(V)$. To prove this proposition, we will make use of the result of Theorem 5.2 which states

$$L(f, A, k) = \frac{2^{2k-2}(k-1)!}{(2k-2)!}\,\frac{1}{D^{k-\frac{1}{2}}u^2}\left(f, \sum_{m\geqq 0} b_{m,A}q^m\right),$$

where

$$b_{m,A} = (mD)^{k-1}u^2 \sum_{0\leqq n\leqq \frac{mD}{N}} \delta(n) r_A(mD - nN) R(n) P_{k-1}\left(1 - \frac{2nN}{mD}\right).$$

Further, if $k \neq 1$, then $\sum_{m\geqq 0} b_{m,A}q^m$ is a cusp form. Hence, $b_{m,A} = 0$ for $k \neq 1$.

In the next section, we will compute the height pairing $\langle v_B, t_m v_{AB}\rangle$ and show that for all $m \geqq 1$

$$b_{m,A} = \frac{(2k-2)!}{(k-1)!^3 2^{2k-2}} \sum_B \langle v_B, t_m v_{AB}\rangle.$$

For $k = 1$, the constant term in g_A is given by $h/2 = u^2 b_{0,A}$. Combining these results with the identity of Theorem 5.2 will prove Proposition 6.1.

7. The Height Computation. Let v be a fixed point on V of discriminant $-D$, and let A and B lie on $\operatorname{Pic}(O)$. For $m \geqq 1$, we want to compute $\langle v_B, t_m v_{AB}\rangle$.

Write $v = x \times w_0$, $v_B = x_B \times w_0$, and $v_{AB} = x_{AB} \times w_0$, so that x_B and x_{AB} are special points of discriminant $-D$ on X. Recall that K is an imaginary quadratic field of discriminant $-D$ where the prime N is inert, and O is the ring of integers in K. K embeds in the quaternion algebra B, so that B can be written in the form $B = K + Kj$, where $j^2 = -N$ and $j\alpha = \bar{\alpha}j$ for all $\alpha \in K$. B embeds in $GL_2(K)$ by

$$\alpha + \beta j \longmapsto \begin{pmatrix} \alpha & \beta \\ -N\bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad \alpha, \beta \in K.$$

Therefore, K embeds by

$$\alpha = \alpha + 0j \longmapsto \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}, \qquad \alpha \in K.$$

Each component X_i of the curve X is indexed by a supersingular elliptic curve of characteristic N. Suppose E is the elliptic curve corresponding to x_{AB} and E' the elliptic curve corresponding to x_B. We will denote the group $\mathrm{Hom}(E, E')$ by $\mathrm{Hom}(x_{AB}, x_B)$.

Let $\mathcal{D} = (\sqrt{-D})$ be the different ideal of O. Let $\mathcal{A}$ and $\mathcal{B}$ be ideals in the classes of A and B respectively, which are relatively prime to $\mathcal{D}$. The special point $v = x \times w_0$ may be chosen so that $\mathrm{End}(x) =$ maximal order containing $O = R$. Fix one solution ϵ to the equation $\epsilon^2 \equiv -N \bmod D$. Then

$$\mathrm{End}(x) = R = \{\alpha + \beta j : \alpha, \beta \in \mathcal{D}^{-1}, \alpha \equiv \epsilon\beta \bmod O\}.$$

We have the following result

Proposition 7.1. *There exists a bijection*

$$\mathrm{Hom}(x_{AB}, x_B) \leftrightarrow \left\{\alpha + \beta j : \alpha \in \mathcal{D}^{-1}\mathcal{A}, \beta \in \mathcal{D}^{-1}\frac{\bar{\mathcal{B}}}{\mathcal{B}}\bar{\mathcal{A}}, \alpha \equiv \epsilon\beta \bmod O\right\}.$$

If $\phi \in \mathrm{Hom}(x_{AB}, x_B)$ *corresponds to* $\alpha + \beta j$, *then*

$$\deg \phi = (\mathbb{N}\alpha + N\mathbb{N}\beta)/\mathbb{N}\mathcal{A}.$$

Proof. Since $R = \mathrm{End}(x)$, the ring $\mathrm{End}(x_B)$ is the right order of the left R module $R\mathcal{B}$. In the embedding of B into $GL_2(K)$, the ideal $\mathcal{B}$ embeds

$$\mathcal{B} \longmapsto \left\{\begin{pmatrix} \gamma & 0 \\ 0 & \bar{\gamma} \end{pmatrix} : \gamma \in \mathcal{B}\right\}.$$

Therefore,

$$\begin{aligned} R\mathcal{B} &\longmapsto \left\{\begin{pmatrix} \alpha & \beta \\ -N\bar{\beta} & \bar{\alpha} \end{pmatrix}\begin{pmatrix} \gamma & 0 \\ 0 & \bar{\gamma} \end{pmatrix} : \gamma \in \mathcal{B}, \alpha, \beta \in \mathcal{D}^{-1}, \alpha \equiv \epsilon\beta \bmod O\right\} \\ &= \left\{\begin{pmatrix} \alpha\gamma & \beta\bar{\gamma} \\ -N\bar{\beta}\gamma & \bar{\alpha}\bar{\gamma} \end{pmatrix} : \gamma \in \mathcal{B}, \alpha, \beta \in \mathcal{D}^{-1}, \alpha \equiv \epsilon\beta \bmod O\right\}. \end{aligned}$$

It follows that

$$R\mathcal{B} = \{\alpha + \beta j : \alpha \in \mathcal{D}^{-1}\mathcal{B}, \beta \in \mathcal{D}^{-1}\bar{\mathcal{B}}, \alpha \equiv \epsilon\beta \bmod O\}.$$

Let $R_{\mathcal{B}} = \mathrm{End}(x_B)$ denote the right order of $R\mathcal{B}$. Then,

$$\begin{aligned} R_{\mathcal{B}} &= \{\gamma \in B : R\mathcal{B}\gamma \subset R\mathcal{B}\} \\ &= \left\{\alpha + \beta j : \alpha \in \mathcal{D}^{-1}, \beta \in \mathcal{D}^{-1}\frac{\bar{\mathcal{B}}}{\mathcal{B}}, \alpha \equiv \epsilon\beta \bmod O\right\}. \end{aligned}$$

$\mathrm{Hom}(x_{AB}, x_B)$ can be identified with the left $R_{\mathcal{B}}$ module $R_{\mathcal{B}}\mathcal{A}$. Hence, to complete the proof we need to compute $R_{\mathcal{B}}\mathcal{A}$.

$$\begin{aligned} R_{\mathcal{B}}\mathcal{A} \longmapsto & \left\{ \begin{pmatrix} \alpha & \beta \\ -N\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} \gamma & 0 \\ 0 & \bar{\gamma} \end{pmatrix} : \gamma \in \mathcal{A}, \alpha \in \mathcal{D}^{-1}, \right. \\ & \left. \beta \in \mathcal{D}^{-1}\frac{\bar{\mathcal{B}}}{\mathcal{B}}, \alpha \equiv \epsilon\beta \bmod O \right\} \\ = & \left\{ \begin{pmatrix} \alpha\gamma & \beta\bar{\gamma} \\ -N\bar{\beta}\gamma & \bar{\alpha}\bar{\gamma} \end{pmatrix} : \gamma \in \mathcal{A}, \alpha \in \mathcal{D}^{-1}, \right. \\ & \left. \beta \in \mathcal{D}^{-1}\frac{\bar{\mathcal{B}}}{\mathcal{B}}, \alpha \equiv \epsilon\beta \bmod O \right\}. \end{aligned}$$

Hence,

$$R_{\mathcal{B}}\mathcal{A} = \left\{ \alpha + \beta j : \alpha \in \mathcal{D}^{-1}\mathcal{A}, \beta \in \mathcal{D}^{-1}\frac{\bar{\mathcal{B}}}{\mathcal{B}}\bar{\mathcal{A}}, \alpha \equiv \epsilon\beta \bmod O \right\}. \qquad \square$$

Proposition 7.1 will now be used in the computation of $\langle v_B, t_m v_{AB}\rangle$. We will show

$$\begin{aligned} &\langle v_B, t_m v_{AB}\rangle \\ &= u^2 m^{k-1}[w_0, w_0] \sum_{0 \leqq n \leqq \frac{mD}{N}} \delta(m) r_{A^{-1}}(mD - nN) r_{AB^2}(n) P_{k-1}\left(1 - \frac{2nN}{mD}\right). \end{aligned}$$

Denote $v_{AB} = g \times y \times w_0$. Then

$$\langle v_B, t_m v_{AB}\rangle = \left\langle v_B, \sum_{h \in t_m(g)} (h \times y \times w_0) \right\rangle = \sum_{h \in t_m(g)} \langle v_B, (h \times y \times w_0)\rangle.$$

For each $h \in t_m(g)$, if x_B and $h \times y$ lie on the same component of X, then $R_{\mathcal{B}}\mathcal{A} = R_{\mathcal{B}}c_h$ for some $c_h \in B^*$ with $\mathbb{N}c_h = m\mathbb{N}\mathcal{A}$. We must deform by c_h before taking the inner product $\langle v_B, (h \times y \times w_0)\rangle$, and this necessitates dividing by $(\mathbb{N}\mathcal{A})^{k-1}$. We have

$$\begin{aligned} \langle v_B, t_m v_{AB}\rangle &= \sum_{h \in t_m(g)} \frac{\langle v_B, c_h(h \times y \times w_0)\rangle}{(\mathbb{N}\mathcal{A})^{k-1}} \\ &= \sum_{h \in t_m(g)} \left(\sum_{\gamma \in \Gamma_i} \frac{[w_0, \gamma c_h w_0]}{(\mathbb{N}\mathcal{A})^{k-1}} \right). \end{aligned}$$

But $\Gamma_i = R_{\mathcal{B}}^* / \langle \pm 1 \rangle$, and the elements γc_h with $\gamma \in R_{\mathcal{B}}^*$ are precisely equal to the $b \in R_{\mathcal{B}}\mathcal{A}$ with $\mathbb{N}b = m\mathbb{N}\mathcal{A}$. Furthermore, $[w_0, \gamma c_h w_0] = [w_0, -\gamma c_h w_0]$. Hence,

$$\langle v_B, t_m v_{AB}\rangle = \frac{1}{2} \sum_{\substack{b \in R_{\mathcal{B}}\mathcal{A} \\ \mathbb{N}b = m\mathbb{N}\mathcal{A}}} \frac{[w_0, bw_0]}{(\mathbb{N}\mathcal{A})^{k-1}}.$$

Applying Proposition 7.1, we have

$$(7.2)\qquad \langle v_B, t_m v_{AB}\rangle = \frac{1}{2}\sum_{\substack{\phi\in\mathrm{Hom}(x_{AB},x_B)_{\deg m}\\ \phi\leftrightarrow b,\ \mathrm{N}b=m\mathbb{N}\mathcal{A}}} \frac{[w_0, bw_0]}{(\mathbb{N}\mathcal{A})^{k-1}}$$

For the next step in the computation of the height pairing $\langle v_B, t_m v_{AB}\rangle$, we will compute the values $[w_0, bw_0]$ in the above sum. Since w_0 is the component of $\lambda(xy)^{k-1}$ in $\mathrm{Sym}^{2k-2}(W)$, where x, y, and $\lambda = \alpha^{k-1}$ are as defined in chapter three, and $[w_0, bw_0] = [\lambda(xy)^{k-1}, b(\lambda(xy)^{k-1})] = \lambda^2[(xy)^{k-1}, b(xy)^{k-1}]$, it will suffice to compute $[(xy)^{k-1}, b(xy)^{k-1}]$.

Write $b = \alpha+\beta j$, where $\alpha, \beta\in K$, $j^2 = -N$, and $j\gamma = \bar{\gamma}j$ for all $\gamma\in K$. The element b embeds into $GL_2(K)$ by

$$b \longmapsto \begin{pmatrix} \alpha & \beta\\ -N\bar{\beta} & \bar{\alpha}\end{pmatrix}.$$

Hence

$$b\begin{pmatrix} x\\ y\end{pmatrix} = \begin{pmatrix} \alpha & \beta\\ -N\bar{\beta} & \bar{\alpha}\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix} = \begin{pmatrix} \alpha x+\beta y\\ -N\bar{\beta}x+\bar{\alpha}y\end{pmatrix}.$$

Therefore, the action of b on the basis element $(xy)^{k-1}$ is given by

$$(xy)^{k-1} \longmapsto ((\alpha x+\beta y)(-N\bar{\beta}x+\bar{\alpha}y))^{k-1}.$$

When computing the inner product $[(xy)^{k-1}, b(xy)^{k-1}]$, we need only consider the inner product of $(xy)^{k-1}$ and $\eta(xy)^{k-1}$, where $\eta\in K$ is the coefficient of $(xy)^{k-1}$ in the expansion of $((\alpha x+\beta y)(-N\bar{\beta}x+\bar{\alpha}y))^{k-1}$. This is true because all of the other summands in the expansion are orthogonal to $(xy)^{k-1}$.

LEMMA 7.3. *The coefficient of* $(xy)^{k-1}$ *in the expansion* $((\alpha x+\beta y)(-N\bar{\beta}x+\bar{\alpha}y))^{k-1}$ *is given by*

$$(\alpha\bar{\alpha}+N\beta\bar{\beta})^{k-1}P_{k-1}\left(\frac{\alpha\bar{\alpha}-N\beta\bar{\beta}}{\alpha\bar{\alpha}+N\beta\bar{\beta}}\right).$$

Proof.

$$((\alpha x+\beta y)(-N\bar{\beta}x+\bar{\alpha}y))^{k-1} = ((\alpha\bar{\alpha}-N\beta\bar{\beta})xy-\alpha\bar{\beta}Nx^2+\bar{\alpha}\beta y^2)^{k-1}$$
$$= \sum_{r=0}^{k-1}\binom{k-1}{r}(\alpha\bar{\alpha}-N\beta\bar{\beta})^{k-1-r}(xy)^{k-1-r}(-\alpha\bar{\beta}Nx^2+\bar{\alpha}\beta y^2)^r,$$

and

$$(-\alpha\bar{\beta}Nx^2+\bar{\alpha}\beta y^2)^r = \sum_{j=0}^{r}(-\alpha\bar{\beta}N)^{r-j}(\bar{\alpha}\beta)^j\binom{r}{j}x^{2r-2j}y^{2j}.$$

The powers of x and y in the last sum are equal only when $j = r/2$, so the coefficient of $(xy)^{k-1}$ is given by

$$\sum_{r=0}^{\left[\frac{k-1}{2}\right]} \binom{k-1}{2r}\binom{2r}{r}(\alpha\bar{\alpha} - N\beta\bar{\beta})^{k-1-2r}(-\alpha\bar{\beta}N)^{2r-r}(\bar{\alpha}\beta)^r$$
$$= \sum_{r=0}^{\left[\frac{k-1}{2}\right]} \binom{k-1}{2r}\binom{2r}{r}(\alpha\bar{\alpha} - N\beta\bar{\beta})^{k-1-2r}((\alpha\bar{\alpha})(-N\beta\bar{\beta}))^r.$$

In order to simplify the notation, let $a = \alpha\bar{\alpha}$ and $b = N\beta\bar{\beta}$. We can write the quantity $\binom{k-1}{2r}\binom{2r}{r}$ in the form

$$\begin{aligned}\binom{k-1}{2r}\binom{2r}{r} &= \frac{(k-1)!}{(2r)!(k-1-2r)!}\cdot\frac{(2r)!}{r!r!}\\ &= \frac{(k-1)!}{r!(k-1-r)!}\cdot\frac{(k-1-r)!}{r!(k-1-2r)!}\\ &= \binom{k-1}{r}\binom{k-1-r}{r}.\end{aligned}$$

The coefficient of $(xy)^{k-1}$ is given by

$$\sum_{r=0}^{\left[\frac{k-1}{2}\right]} \binom{k-1}{r}\binom{k-1-r}{r}(-1)^r(a-b)^{k-1-2r}(ab)^r.$$

We want to show

$$\text{(7.4)}\qquad \frac{1}{(a+b)^{k-1}}\sum_{r=0}^{\left[\frac{k-1}{2}\right]} \binom{k-1}{r}\binom{k-1-r}{r}(-1)^r(a-b)^{k-1-2r}(ab)^r = P_{k-1}\left(\frac{a-b}{a+b}\right).$$

The Legendre polynomials satisfy the recurrence relation [**3**, p. 1026]

$$\text{(7.5)}\qquad P_{n+1}(x) = \frac{1}{n+1}\left[(2n+1)xP_n(x) - nP_{n-1}(x)\right].$$

Hence, to show that (7.4) holds, we need only show that the left hand side is equal to $P_{k-1}\left(\frac{a-b}{a+b}\right)$ for $k = 1$ and $k = 2$ and that it satisfies the recurrence relation of (7.5) for $n = k-2$, $k \geqq 3$, and $x = \left(\frac{a-b}{a+b}\right)$. This is carried out using an algebraic argument, and it completes the proof of Lemma 7.3.

Applying Lemma 7.3 to the computation of $[(xy)^{k-1}, b(xy)^{k-1}]$, where $b = \alpha + \beta j$, we find that

$$[(xy)^{k-1}, b(xy)^{k-1}] = (\alpha\bar{\alpha} + N\beta\bar{\beta})^{k-1} P_{k-1}\left(\frac{\alpha\bar{\alpha} - N\beta\bar{\beta}}{\alpha\bar{\alpha} + N\beta\bar{\beta}}\right) \times [(xy)^{k-1}, (xy)^{k-1}].$$

As explained earlier, we have $[w_0, bw_0] = \gamma^2[(xy)^{k-1}, b(xy)^{k-1}]$ and $[w_0, w_0] = \gamma^2[(xy)^{k-1}, (xy)^{k-1}]$. Therefore, by substituting into the above equation we obtain

$$[w_0, bw_0] = (\alpha\bar{\alpha} + N\beta\bar{\beta})^{k-1} P_{k-1}\left(\frac{\alpha\bar{\alpha} - N\beta\bar{\beta}}{\alpha\bar{\alpha} + N\beta\bar{\beta}}\right)[w_0, w_0].$$

Note that $\alpha\bar{\alpha} + N\beta\bar{\beta} = \mathbb{N}b = m\mathbb{N}\mathcal{A}$. Hence,

$$(7.6) \qquad [w_0, bw_0] = m^{k-1}(\mathbb{N}\mathcal{A})^{k-1} P_{k-1}\left(\frac{\alpha\bar{\alpha} - N\beta\bar{\beta}}{\alpha\bar{\alpha} + N\beta\bar{\beta}}\right)[w_0, w_0].$$

Combining the results of (7.2) and (7.6), we have

$$\langle v_B, t_m v_{AB}\rangle = \frac{1}{2} \sum_{\substack{\phi \in \mathrm{Hom}(x_{AB}, x_B)_{\deg m} \\ \phi \leftrightarrow b = \alpha + \beta j, \mathbb{N}b = m\mathbb{N}\mathcal{A}}} m^{k-1} P_{k-1}\left(\frac{\alpha\bar{\alpha} - N\beta\bar{\beta}}{\alpha\bar{\alpha} + N\beta\bar{\beta}}\right)[w_0, w_0].$$

Applying the correspondence of Proposition 7.1, it follows that

$$(7.7) \qquad \langle v_B, t_m v_{AB}\rangle = \frac{1}{2} \sum_{\substack{\alpha + \beta j \\ \alpha \in \mathcal{D}^{-1}\mathcal{A}, \beta \in \mathcal{D}^{-1}(\bar{\mathcal{B}}/\mathcal{B})\bar{\mathcal{A}} \\ \alpha \equiv \epsilon\beta \bmod O}} m^{k-1} P_{k-1}\left(\frac{\alpha\bar{\alpha} - N\beta\bar{\beta}}{\alpha\bar{\alpha} + N\beta\bar{\beta}}\right)[w_0, w_0].$$

Consider the ideals

$$\mathcal{C} = (\alpha)\mathcal{D}\mathcal{A}^{-1} \qquad \mathcal{C}' = (\beta)\mathcal{D}\mathcal{B}\bar{\mathcal{B}}^{-1}\bar{\mathcal{A}}^{-1},$$

which satisfy the identity

$$(7.8) \qquad \mathbb{N}\mathcal{C} + N\mathbb{N}\mathcal{C}' = mD.$$

Suppose $n = \mathbb{N}\mathcal{C}'$. Then, $\mathbb{N}\mathcal{C} = md - nN$. Since $\mathbb{N}\mathcal{C}' = (\mathbb{N}\beta)D(1/\mathbb{N}\mathcal{A}) = n$, it follows that $\mathbb{N}\beta = (n/D)\mathbb{N}\mathcal{A}$. Since $\mathbb{N}\mathcal{C} = (\mathbb{N}\alpha)D(1/\mathbb{N}\mathcal{A}) = mD - nN$, it follows that $\mathbb{N}\alpha = ((mD - nN)/D)\mathbb{N}\mathcal{A}$. Hence, when $n = \mathbb{N}\mathcal{C}'$,

$$\frac{\alpha\bar{\alpha} - N\beta\bar{\beta}}{\alpha\bar{\alpha} + N\beta\bar{\beta}} = \frac{\mathbb{N}\alpha - N\mathbb{N}\beta}{\mathbb{N}\alpha + N\mathbb{N}\beta} = \frac{((mD - nN)/D)\mathbb{N}\mathcal{A} - N(n/D)\mathbb{N}\mathcal{A}}{((mD - nN)/D)\mathbb{N}\mathcal{A} + N(n/D)\mathbb{N}\mathcal{A}}$$
$$= \frac{mD - 2nN}{mD} = 1 - \frac{2nN}{mD}.$$

The ideal C lies in the class of A^{-1}, and the ideal C' lies in the class of AB^2. Hence, the number of solutions to (7.8) with $\mathbb{N}C' = n$ is given by

$$\begin{array}{ll} r_{A^{-1}}(mD - nN)r_{AB^2}(n) & \text{if } n \neq 0, \dfrac{mD}{N} \\ r_{A^{-1}}(mD) & \text{if } n = 0 \\ r_{AB^2}\left(\dfrac{mD}{N}\right) & \text{if } n = \dfrac{mD}{N} \end{array}$$

Notice that each solution C, C' to (7.8) corresponds to a solution to the identity

$$\mathbb{N}\alpha + N\mathbb{N}\beta = m\mathbb{N}\mathcal{A}$$

$$\text{(7.9)} \qquad \alpha \in \mathcal{D}^{-1}\mathcal{A}, \qquad \beta \in \mathcal{D}^{-1}\frac{\bar{\mathcal{B}}}{\mathcal{B}}\bar{\mathcal{A}}$$

$$\alpha \equiv \epsilon\beta \bmod O.$$

Each pair C, C' with $\mathbb{N}C' = n \neq 0, (mD)/N$, so that $\mathbb{N}C \neq 0$, gives $(2u)^2$ candidates (α, β) for solutions to (7.9). If $n \equiv 0 \bmod D$, then all of these candidates turn out to be solutions since the condition $\alpha \equiv \epsilon\beta \bmod O$ is satisfied. If $n \not\equiv 0 \bmod D$, then either $\alpha \equiv \epsilon\beta \bmod O$ or $\alpha \equiv -\epsilon\beta \bmod O$, but not both. Hence, only one half of the candidates are actually solutions to (7.9). Summarizing this information, we see that the number of solutions to the identity (7.9) is given by

$$\begin{array}{ll} (2u)^2\left(\dfrac{1}{2}\right)\delta(n)r_{A^{-1}}(mD - nN)r_{AB^2}(n) & \text{if } n \neq 0, \dfrac{mD}{N} \\ (2u)r_{A^{-1}}(mD) & \text{if } n = 0 \\ (2u)r_{AB^2}\left(\dfrac{mD}{N}\right) & \text{if } n = \dfrac{mD}{N} \end{array}$$

Using the convention $r_A(0) = 1/2u$ for any ideal class A, the number of solutions to the identity (7.9) is given by

$$2u^2\delta(n)r_{A^{-1}}(mD - nN)r_{AB^2}(n).$$

Therefore, using this result, the fact that $(\alpha\bar{\alpha} - N\beta\bar{\beta})/(\alpha\bar{\alpha} + N\beta\bar{\beta}) = 1 - (2nN)/(mD)$ when $\mathbb{N}C' = n$, and (7.7) we have

$$\begin{aligned} \text{(7.10)} \quad & \langle v_B, t_m v_{AB} \rangle \\ &= \frac{1}{2}\sum_{0 \leqq n \leqq \frac{mD}{N}} 2u^2\delta(n)r_{A^{-1}}(mD - nN)r_{AB^2}(n)m^{k-1}P_{k-1}\left(1 - \frac{2nN}{mD}\right)[w_0, w_0] \\ &= u^2 m^{k-1}[w_0, w_0]\sum_{0 \leqq n \leqq \frac{mD}{N}} \delta(n)r_{A^{-1}}(nD - nN)r_{AB^2}(n)P_{k-1}\left(1 - \frac{2nN}{mD}\right). \end{aligned}$$

To complete the computation of $\langle v_B, t_m v_{AB}\rangle$, we will next compute the value of $[w_0, w_0]$.

Let U and W be as defined in section three. If v is any vector in $U \otimes \mathbb{R}$ of length 1, then v^{k-1} lies in $\mathrm{Sym}^{k-1}(U)$. Let w be the component of v^{k-1} in the subspace $\mathrm{Sym}^{2k-2}(W)$ with inner product coming from $\mathrm{Sym}^{k-1}(U)$. In order to compute $[w, w]$, we will make use of the operator

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

in the Lie Algebra $SL_2(\mathbb{C})$, which acts by taking $x^i y^j \longmapsto x^{i+1}y^{j-1}$. Since the component M of the decomposition $\mathrm{Sym}^{k-1}(U) = \mathrm{Sym}^{2k-2}(W) \oplus M$ has no vector of weight $2k-2$, it follows that $(E_+)^{k-1}(v^{k-1}) = (E_+)^k w$. Further, $(E_+)^{k-1}(v^{k-1}) = (E_+v)^{k-1}$ in $\mathrm{Sym}^{k-1}(U)$. Hence, $w = {E_+}^{-(k-1)}(E_+v)^{k-1}$.

Assume now that $v = xy$ in the representation $\mathrm{Sym}^2(W) = U$. Then $(E_+v) = x^2$ and $[E_+v, E_+v] = 2!0! = 2$. The vector $(E_+v)^{k-1}$ lies in $\mathrm{Sym}^{2k-2}(W) \subseteq \mathrm{Sym}^{k-1}(U)$ and $[(E_+v)^{k-1}, (E_+v)^{k-1}] = 2^{k-1}(k-1)!$. On the representation $\mathrm{Sym}^{k-1}(U)$, ${E_+}^{-(k-1)}(x^{2k-2}) = x^{k-1}y^{k-1}$ alters the inner product by the factor

$$\frac{[x^{k-1}y^{k-1}, x^{k-1}y^{k-1}]}{[x^{2k-2}, x^{2k-2}]} = \frac{(k-1)!^2}{(2k-2)!}.$$

Hence,

$$[w, w] = \frac{(k-1)!^2}{(2k-2)!}\,[(E_+v)^{k-1}, (E_+v)^{k-1}] = \frac{2^{k-1}(k-1)!^3}{(2k-2)!}.$$

Recall that w_0 is the component of v_0^{k-1} in $\mathrm{Sym}^{2k-2}(W)$, where $v_0 = \sqrt{-D}$. Since $[v_0, v_0] = 2D$, we can write $v_0 = \alpha v$, where v has length 1 and $\alpha = \sqrt{2D}$. If w is the component of v^{k-1} in $\mathrm{Sym}^{2k-2}(W)$, then $[w_0, w_0] = \alpha^{2k-2}[w, w]$. Therefore,

$$(7.11) \quad [w_0, w_0] = \frac{2^{2k-2}D^{k-1}(k-1)!^3}{(2k-2)!}.$$

We can now use the height computation to prove the following result.

PROPOSITION 7.12. *For all $m \geqq 1$*

$$\sum_B \langle v_B, t_m v_{AB}\rangle = \frac{u^2(mD)^{k-1}2^{2k-2}(k-1)!^3}{(2k-2)!} \sum_{0 \leqq n \leqq \frac{mD}{N}} \delta(n) r_A(mD - nN)R(n) \times P_{k-1}\left(1 - \frac{2nN}{D}\right),$$

where the sum is over all classes B in $\mathrm{Pic}(O)$.

Proof. Applying (7.10), we have

$$\begin{aligned}\sum_B \langle v_B, t_m v_{AB}\rangle &= \sum_B u^2 m^{k-1}[w_0, w_0]\\ &\times \sum_{0\leqq n\leqq \frac{mD}{N}} \delta(n) r_{A^{-1}}(mD-nN) r_{AB^2}(n) P_{k-1}\left(1-\frac{2nN}{mD}\right)\\ &= u^2 m^{k-1}[w_0, w_0] \sum_{0\leqq n\leqq \frac{mD}{N}} \delta(n) r_{A^{-1}}(mD-nN)\\ &\times \left(\sum_B r_{AB^2}(n)\right) P_{k-1}\left(1-\frac{2nN}{mD}\right).\end{aligned}$$

Since $\sum_B r_{AB^2}(n) = R(n)$, $r_{A^{-1}}(k) = r_A(k)$, and by (7.11), we have

$$[w_0, w_0] = \frac{2^{2k-2} D^{k-1}(k-1)!^3}{(2k-2)!},$$

and the result follows. □

As explained in the last section, to complete the proof of Proposition 6.1, we need only show that for all $m \geqq 1$,

$$\frac{(2k-2)!}{(k-1)!^3 2^{2k-2}} \sum_B \langle v_B, t_m v_{AB}\rangle = b_{m,A}.$$

By Proposition 7.12, it follows that

$$\begin{aligned}&\frac{(2k-2)!}{(k-1)!^3 2^{2k-2}} \sum_B \langle v_B, t_m v_{AB}\rangle\\ &= u^2 (mD)^{k-1} \sum_{0\leqq n\leqq \frac{mD}{N}} \delta(n) r_A(mD-nN) R(n) P_{k-1}\left(1-\frac{2nN}{mD}\right)\\ &= b_{m,A}.\end{aligned}$$

This completes the proof of Proposition 6.1, the main identity.

8. Special Values of L-series. Let χ be a complex character of the group $\mathrm{Pic}(O)$, and let $f = \sum_{m\geqq 1} a_m q^m$ be a normalized eigenform for the Hecke algebra $\mathbb{T}$. Define

(8.1) $$L(f, \chi, s) = \sum_A \chi(A) L(f, A, s),$$

where the sum is over all A in $\mathrm{Pic}(O)$. We are interested in the value of $L(f,\chi,s)$ at the special value $s=k$.

Define $e_\chi = \sum_A \chi^{-1}(A)v_A$, and let $e_{f,\chi}$ be the projection of the divisor e_χ in $\mathrm{Pic}(V)\otimes\mathbb{C}$ to the f-isotypical eigenspace for $\mathbb{T}$.

We now prove the following result.

PROPOSITION 8.2.

$$L(f,\chi,k) = \frac{(f,f)}{u^2D^{k-\frac{1}{2}}(k-1)!^2}\langle e_{f,\chi}, e_{f,\chi}\rangle.$$

Proof. We will extend the $\mathbb{R}$-bilinear pairings $\langle\ ,\ \rangle$ and $\phi(\ ,\ )$ to the complex pairings which are linear in the first argument and anti-linear in the second argument. Then

$$\begin{aligned}\langle e_\chi, e_\chi\rangle &= \left\langle \sum_A \chi^{-1}(A)v_A, \sum_B \chi^{-1}(B)v_B\right\rangle \\ &= \sum_{A,B}\chi(A^{-1}B)\langle v_A, v_B\rangle.\end{aligned}$$

Recall that the main identity states that

$$L(f,A,k) = \frac{(f,g_A)}{u^2D^{k-\frac{1}{2}}(k-1)!^2}.$$

Substituting this equality into (8.1) we have

$$\begin{aligned}(8.3)\qquad L(f,\chi,k) &= \sum_A \chi(A)\frac{(f,g_A)}{u^2D^{k-\frac{1}{2}}(k-1)!^2} \\ &= \frac{(f,\sum_A\chi(A)g_A)}{u^2D^{k-\frac{1}{2}}(k-1)!^2}.\end{aligned}$$

Expanding $\sum_A \chi(A)g_A$, we see that

$$\begin{aligned}\sum_A \chi(A)g_A &= \sum_A \chi(A)\sum_B \phi(v_B, v_{AB}) \\ &= \sum_{A,B}\chi(A)\phi(v_B, v_{AB}).\end{aligned}$$

Let $A' = B$ and $B' = AB$. Then

$$\sum_A \chi(A)g_A = \sum_{A',B'}\chi((A')^{-1}B')\phi(v_{A'}, v_{B'}) = \phi(e_\chi, e_\chi).$$

Note that in the special case $k=1$, deg $e_{f,\chi}=0$ since f is a cusp form. Also, ϕ is $\mathbb{T}$-bilinear, and it follows that the f-eigencomponent of the modular form $\phi(e_\chi, e_\chi)$ is given by

$$\begin{aligned}\phi(e_{f,\chi}, e_{f,\chi}) &= \sum_{m\geqq 1}\langle e_{f,\chi}, t_m e_{f,\chi}\rangle q^m \\ &= \sum_{m\geqq 1}\langle e_{f,\chi}, e_{f,\chi}\rangle a_m q^m \\ &= \langle e_{f,\chi}, e_{f,\chi}\rangle \cdot \sum_{m\geqq 1} a_m q^m \\ &= \langle e_{f,\chi}, e_{f,\chi}\rangle \cdot f.\end{aligned}$$

Hence,

$$\left(f, \sum_A \chi(A) g_A\right) = \langle e_{f,\chi}, e_{f,\chi}\rangle \cdot (f,f)$$

Substituting this equality into (8.3) completes the proof. □

The following corollary is a direct result of Proposition 8.2 and the fact that the height pairing $\langle\ ,\ \rangle$ induces a positive definite Hermitian pairing on $\mathrm{Pic}(V)\otimes\mathbb{C}$.

COROLLARY 8.4. *$L(f,\chi,k)\geqq 0$ with equality if and only if $e_{f,\chi}=0$.*

For any automorphism α of $\mathbb{C}$,

$$\langle e_{f,\chi}, e_{f,\chi}\rangle^\alpha = \langle e_{f^\alpha,\chi^\alpha}, e_{f^\alpha,\chi^\alpha}\rangle.$$

Therefore, we have the following result.

COROLLARY 8.5. *If α is an automorphism of $\mathbb{C}$, then*

$$\left(\frac{L(f,\chi,k)D^{k-\frac{1}{2}}}{(f,f)}\right)^\alpha = \frac{L(f^\alpha,\chi^\alpha,k)D^{k-\frac{1}{2}}}{(f^\alpha,f^\alpha)},$$

and the ratio lies in $\mathbb{Q}(f,\chi)$, the numberfield generated by the values of χ and the Fourier coefficients of the eigenform f. Hence, $L(f,\chi,k)=0$ if and only if $L(f^\alpha,\chi^\alpha,k)=0$.

Consider now the special case $\chi=1$. Then, $L(f,\chi,k)$ can be written in the form

$$L(f,\chi,k) = L(f,k)L(f\otimes\epsilon,k),$$

where $f\otimes\epsilon=\sum_{m\geqq 1} a_m\epsilon(m)q^m$ is the twist of f, $L(f,s)=\sum_{m\geqq 1} a_m m^{-s}$, and $L(f\otimes\epsilon,s)=\sum_{m\geqq 1} a_m\epsilon(m)m^{-s}$.

Define e_D to be the class in $\mathrm{Pic}(V)$ of the rational divisor

$$\frac{1}{2u} \sum_{\mathrm{disc}(v)=-D} (v).$$

Then,

$$e_\chi = \sum_A v_A \equiv u e_D \qquad \text{in } \mathrm{Pic}(V).$$

Since $\mathrm{Pic}(O)\times\mathrm{Gal}(K/\mathbb{Q})$ acts simply transitively on the special points, it follows that the points v_A and $\overline{v_A}$ lie on the same component of V. Hence, in the special case $\chi = 1$, Proposition 8.2 becomes

COROLLARY 8.6.

$$L(f,k)L(f\otimes\epsilon,k) = \frac{(f,f)}{D^{k-\frac{1}{2}}(k-1)!^2}\langle e_{f,D}, e_{f,D}\rangle.$$

Notice that changing the discriminant D changes the value of $\langle e_{f,D}, e_{f,D}\rangle$ by a square in the field $\mathbb{Q}(f)$ generated by the Fourier coefficients of f. Therefore, we have the following, which is a refinement of a result of Waldspurger [**9**].

COROLLARY 8.7. *If $L(f\otimes\epsilon_D,k)\neq 0$, then the ratio*

$$\frac{L(f\otimes\epsilon_D,k)D^{k-\frac{1}{2}}}{L(f\otimes\epsilon_{D'},k)(D')^{k-\frac{1}{2}}}$$

lies in $\mathbb{Q}(f)^2$.

REFERENCES

1. M. Eichler, *Zur Zahlentheorie der Quaternionen-Algebren*, Crelle J. *195* (1955), 127–151.

2. ———, *Lecture Notes in Mathematics 320, The Basis Problem for Modular Forms and the Traces of the Hecke Operators*, (Springer-Verlag, New York, 1973), 75–151.

3. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, (Academic Press, New York, 1980).

4. B. H. Gross, *Heights and the special values of L-series*, CMS Conf. Proc. *7* (1986).

5. B. H. Gross and D. Zagier, *Heegner points and derivatives of L-series*, Invent. Math. *84* (1986), 225–320.

6. J. Milnor and D. Husemoller, *Symmetric Bilinear Forms*, (Springer-Verlag, New York, 1973).

7. A. Pizer, *An Algorithm for Computing Modular Forms on $\Gamma_0(N)$*, J. Algebra *64* (1980), 340–390.

8. A. Pizer, *Theta Series and Modular Forms of Level p^2M*, Compositio Math. *40* (1980), 177–241

9. J-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier*, J. Math. pures et appl. *60* (1981), 375–484.

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