Canad. Math. Bull. Vol. **24** (2), 1981

A PROBLEM OF R. H. FOX

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The purpose of this expository article is to familiarize the reader with one of the fundamental problems in the theory of infinite groups. We give an up-to-date account of the so-called Fox problem which concerns the identification of certain normal subgroups of free groups arising out of certain ideals in the free group rings. We assume that the reader is familiar with the elementary concepts of algebra.

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1. **Fox Problem.** Let F be a non-cyclic free group freely generated by a finite set X and let $\mathbf{Z}F$ denote the integral group ring of F. The elements of $\mathbf{Z}F$ are finite sums of the form $\sum_i n_if_i$, $n_i \in \mathbf{Z}$, $f_i \in F$ (with $1 = e$, the identity element of F). The addition and multiplication in $\mathbf{Z}F$ are defined by the rules:

$$\sum_i n_if_i + \sum_i m_if_i = \sum_i (n_i + m_i)f_i$$

and

$$\left(\sum_i n_if_i\right) \cdot \left(\sum_j m_jf_j\right) = \sum_{i,j} n_im_jf_if_j$$

The trivial map $F \to \mathbf{Z}$ yields the ring homomorphism $\tau : ZF \to \mathbf{Z}$ defined by $\tau(\sum_i n_if_i) = \sum_i n_i$ whose kernel $\mathfrak{f}$ is an ideal of $\mathbf{Z}F$, called the augmentation ideal of $\mathbf{Z}F$. Since $f_i - f_j = f_j(f_j^{-1}f_i - 1)$, $\mathfrak{f}$ consists of elements of the form $\sum_i u_i(f_i - 1)$, $u_i \in \mathbf{Z}F$, $f_i \in F$. Further, since $(f_if_j - 1) = f_i(f_j - 1) + (f_i - 1)$ and $(f_i^{-1} - 1) = -f_i^{-1}(f_i - 1)$, it follows that, for $f \in F$, $(f - 1)$ is of the form $\sum_x a_x(x - 1)$, $a_x \in \mathbf{Z}F$,

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Received by the editors April 7, 1980. This paper is one of a series of survey papers written at the invitation of the editors.

$x \in X$. Thus

$$\mathfrak{f} = \text{ideal}_{\mathbf{Z}F}\{(x-1),\ x \in X\}.$$

More generally, for $n \geq 1$,

$$\mathfrak{f}^n = \text{ideal}_{\mathbf{Z}F}\{(x_1-1)\cdots(x_n-1),\ x_i \in X\}$$

For each $n \geq 1$, the set

$$D_n(F) = F \cap (1+\mathfrak{f}^n) = \{f \in F \mid f-1 \in \mathfrak{f}^n\}$$

is easily seen to be a normal subgroup of F. Since $[f_1, f_2]-1 = f_1^{-1}f_2^{-1}f_1f_2-1 = f_1^{-1}f_2^{-1}(f_1f_2-f_2f_1) = f_1^{-1}f_2^{-1}((f_1-1)(f_2-1)-(f_2-1)(f_1-1)) \in \mathfrak{f}^2$, it follows that F', the commutator subgroup of F, is contained in $D_2(F)$; and a simple induction on n, using the above argument, shows that $D_n(F) \geq \gamma_n(F)$, the n-th term of the lower central series of F.(1) The equality $D_n(F) = \gamma_n(F)$ for $n \geq 1$ is a classical result due to Wilhelm Magnus [8].

Now, let R be a normal subgroup of F. Since subgroups of free groups are free, R is itself a free group freely generated by a set Y, say (the elements of Y are certain reduced words of the form $x_1^{\varepsilon_1}\cdots x_l^{\varepsilon_l}$ $(l \geq 1)$, $\varepsilon_i \in \{1, -1\}$, $x_i \in X$). The linear extension of the natural homomorphism $\theta: F \to F/R$ yields the ring homomorphism $\theta: \mathbf{Z}F \to \mathbf{Z}(F/R)$ defined by $\theta(\sum_i n_i f_i) = \sum_i n_i\theta(f_i)$, whose kernel $\mathfrak{r}$ is an ideal of $\mathbf{Z}F$ (contained in $\mathfrak{f}$) given by

$$\mathfrak{r} = \text{ideal}_{\mathbf{Z}F}\{(y-1),\ y \in Y\}$$

For each $n \geq 1$, we consider the ideal

$$\mathfrak{r}\mathfrak{f}^n = \text{ideal}_{\mathbf{Z}F}\{(y-1)(x_1-1)\cdots(x_n-1),\ y \in Y,\ x_i \in X\}.$$

Then, the set

$$F(n, R) = F \cap (1+\mathfrak{r}\mathfrak{f}^n) = \{f \in F \mid f-1 \in \mathfrak{r}\mathfrak{f}^n\}$$

is a normal subgroup of F. We call $F(n, R)$ the n-th Fox subgroup of F relative to R. The identification of $F(n, R)$ is known as the Fox problem. It is a generalization of the case $R = F$, where we know that $F(n, F) = D_{n+1}(F) = \gamma_{n+1}(F)$.

For $r_1, r_2 \in R$, $[r_1, r_2]-1 = r_1^{-1}r_2^{-1}((r_1-1)(r_2-1)-(r_2-1)(r_1-1)) \in \mathfrak{r}^2 \subseteq \mathfrak{r}\mathfrak{f}$ and it follows that $F(1, R) \geq R'$. The equality $F(1, R) = R'$ is once again a classical result due to Wilhelm Magnus [9].(2) For $n \geq 2$, the identification of $F(n, R)$ turns out to be a difficult and challenging problem in group theory.(3)

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(1) For a group G, $\gamma_1(G) = G$, $\gamma_{i+1}(G) = [\gamma_i(G), G]$, $i \geq 1$, where for subgroups A, B of G, $[A, B] = sgp\{[a, b],\ a \in A,\ b \in B\}$, $[a, b] = a^{-1}b^{-1}ab$ is the commutator of a and b.

(2) This result is also attributed to Schuman [10] (see Fox [3] for a proof).

(3) This problem also appears as Problem 13 in Birman [1].

2. **Free differential calculus.** For a detailed account of Fox's free differential calculus and its various applications we refer the reader to Chapter 3 of Birman [1]. Here we simply recall the basic definitions and a connection with the Fox subgroups $F(n, R)$.

Given $w \in \mathbf{Z}F$, for each $x \in X$, the (left) partial derivative $\partial w/\partial x$ of w is defined by the mapping

$$\frac{\partial}{\partial x}: \mathbf{Z}F \to \mathbf{Z}F$$

given by

(i) $$\frac{\partial(u+v)}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}, \qquad u, v \in \mathbf{Z}F;$$

(ii) $$\frac{\partial(uv)}{\partial x} = u\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\tau(v), \qquad u, v \in \mathbf{Z}F;$$

(iii) $$\frac{\partial x'}{\partial x} = \begin{cases} 1 & \text{if } x' = x,\ x' \in X \\ 0 & \text{if } x' \neq x,\ x' \in X; \end{cases}$$

(iv) $$\frac{\partial(1)}{\partial x} = 0.$$

It follows, for instance, that if $f = f_1 x^{\varepsilon_1} f_2 x^{\varepsilon_2} \cdots f_l x^{\varepsilon_l} f_{l+1} \in F$, $l \geq 1$, $\varepsilon_i \in \{1, -1\}$, f_i are elements of F not involving x, then

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{l} \varepsilon_i f_i x^{\varepsilon_1} \cdots f_i x^{(\varepsilon_i - 1)/2}.$$

Further

$$f_1 x^{\varepsilon_1} \cdots f_i x^{(\varepsilon_i - 1)/2}(x-1) = \varepsilon_i (f_1 x^{\varepsilon_1} \cdots f_i x^{\varepsilon_i} - f_1 x^{\varepsilon_i} \cdots f_i)$$

implies that

$$\sum_x \sum_{i=1}^{l} \varepsilon_i f_1 x^{\varepsilon_1} \cdots f_i x^{(\varepsilon_i - 1)/2}(x-1) = f - 1.$$

This yields the fundamental formula

$$f - 1 = \sum_x \frac{\partial f}{\partial x}(x-1),$$

where x ranges over all elements of X which occur in f. The higher order Fox derivatives are defined inductively by

$$\frac{\partial^k u}{\partial x_1 \cdots \partial x_k} = \frac{\partial}{\partial x_k}\left(\frac{\partial^{k-1}(u)}{\partial x_1 \cdots \partial x_{k-1}}\right).$$

THEOREM 2.1 (Fox). *Let R be a normal subgroup of F and let $\theta:\mathbf{Z}F\to\mathbf{Z}(F/R)$ be the linear extension of the group homomorphism $F\to F/R$. Then*

$$F(n,R)=\left\{f\in F \mid \theta\left(\frac{\partial^k f}{\partial x_1\cdots\partial x_k}\right)=0, \quad \text{for all } 0\le k\le n \text{ and all } x_i\in X\right\}.$$

[*The computation with the free differential calculus for $n\ge 2$ is quite involved and is unlikely to yield the required identification of $F(n,R)$*].

3. **A matrix representation.** The group $F/F(n,R)$ admits a faithful matrix representation of degree $n+1$ over a suitable ring. This was first observed by Enright [2]. Here we give a faithful representation of $\mathbf{Z}F/\mathfrak{r}\mathfrak{f}^n$ which, in turn, yields a faithful representation of $F/F(n,R)$ similar to the one obtained by Enright.

Let F be freely generated by X. For each $n\ge 1$, let

$$\Lambda_n=\{\lambda_{i,i+1}^{(x)};\ 1\le i<n+1;\ x\in X\}$$

be a set of independent and commuting indeterminates. Let $A_n=\mathbf{Z}(F/R)[\Lambda_n]$ be the ring of polynomials in $\lambda_{i,i+1}^{(x)}$'s over the group ring $\mathbf{Z}(F/R)$. We set $A_0=\mathbf{Z}(F/R)$ and denote by $T(A_n)$, $n\ge 0$, the ring of all $n+1\times n+1$ upper triangular matrices over A_n. For each $x\in X$, the matrix

$$\varphi_n(x)=\begin{bmatrix} \mathbf{x} & \lambda_{1,2}^{(x)} & 0 & - & - & - & 0 & 0\\ 0 & \mathbf{1} & \lambda_{2,3}^{(x)} & - & - & - & 0 & 0\\ - & - & - & - & - & - & - & -\\ - & - & - & - & - & - & - & -\\ - & - & - & - & - & - & - & -\\ 0 & 0 & 0 & - & - & - & \mathbf{1} & \lambda_{n,n+1}^{(x)}\\ 0 & 0 & 0 & - & - & - & 0 & \mathbf{1} \end{bmatrix}$$

is an invertible element of $T(A_n)$, where for $f\in F$, $\mathbf{f}=\theta(f)=fR\in F/R$. It follows that the map

$$\varphi_n: X\to T(A_n)$$

given by $x\to\varphi_n(x)$ defines a homomorphism of F into $T(A_n)$. Extending φ_n by linearity yields the ring homomorphism

$$\varphi_n^*:\mathbf{Z}F\to T(A_n)$$

THEOREM 3.1 (Gupta and Passi [6]). *For all $n\ge 0$, $\mathfrak{r}\mathfrak{f}^n$ is the kernel of φ_n^*.*

Since $\operatorname{Ker}\varphi_n=F\cap(1+\operatorname{Ker}\varphi_n^*)=F\cap(1+\mathfrak{r}\mathfrak{f}^n)$, the above theorem yields the following corollary.

COROLLARY 3.2. *$F(n, R) = \operatorname{Ker} \varphi_n$; or equivalently, $F/F(n, R)$ is isomorphic to the group of $n+1 \times n+1$ matrices generated by all $\varphi_n(x)$, $x \in X$.*

REMARK 1. In Gupta and Passi [6] the matrices used are lower triangular and the representation obtained is that of $\mathbf{Z}F/\mathfrak{f}^n\mathfrak{r}$. However, the proof in our case is essentially the same.

REMARK 2. The case $n = 1$ is the well-known Magnus representation of F/R' (since $F(1, R) = R'$); and the case $R = F$ gives a faithful representation of the free nilpotent group $F/\gamma_{n+1}(F)$ (since $F(n, F) = \gamma_{n+1}(F)$).

REMARK 3. If $R = F'$, A_n is a commutative integral domain and hence can be embedded in its field of fractions. Thus in this case $F/F(n, F')$ is isomorphic to a linear group. Further, since $F(n, F')$ is a fully invariant subgroup of F, we have a sequence of relatively free torsion free linear groups.

REMARK 4. A connection with the free differential calculus is provided by the following observation. For $w \in \mathbf{Z}F$, let $\alpha_{i,j}(w)$, $1 \le i < j \le n+1$, denote the ij entry of the matrix $\varphi_n^*(w)$. Then $\alpha_{i,j}(w)$ is a sum of terms of the form

$$\mu(u(x_i, \ldots, x_{j-1}))\lambda_{i,i+1}^{(x_i)} \cdots \lambda_{j-1,j}^{(x_{j-1})},$$

where

$$\mu(u) = \begin{cases} \theta(u) & \text{if} \quad i = 1 \\ \tau(u) & \text{if} \quad i > 1 \end{cases}$$

and

$$u(x_i, \ldots, x_{j-1}) = \frac{\partial^{j-i}(w)}{\partial x_i \cdots \partial x_{j-1}}$$

REMARK 5. The matrix representation provides a useful tool for studying the behaviour of the upper and the lower central series of $F/F(n, R)$ (see Gupta and Passi [6]).

4. **Identification of Fox subgroups.** It is easy to identify some of the subgroups of $F(n, R)$. For $t \ge 1$, let $R_t = R \cap \gamma_t(F)$. Then $R_t \le F \cap (1 + \mathfrak{r} \wedge \mathfrak{f}^t)$, and it follows that

$$[R_{t_1}, R_{t_2}] \le F \cap (1 + \mathfrak{r}\mathfrak{f}^t),$$

where $t = \min\{t_1, t_2\}$. More generally, for $n \ge 1$, let

$$\mathbf{t}(m) = (t_1, \ldots, t_m), \qquad 2 \le m \le n+1,$$

be an m-tuple of positive integers satisfying

$$t_1 + \cdots + \hat{t_i} + \cdots + t_m \ge n, \ \forall i,$$

https://doi.org/10.4153/CMB-1981-023-6 Published online by Cambridge University Press

($\hat{t}_i$ indicates t_i missing). Then it is quite straightforward to verify that

$$[R_{t_1}, \ldots, R_{t_m}] \le F \cap (1 + \mathfrak{r}^n\mathfrak{f}) = F(n, R).^{(4)}$$

In particular, if

$$G(n, R) = \prod_{m=2}^{n+1} \prod_{\mathbf{t}(m)} [R_{t_1}, \ldots, R_{t_m}],$$

where the product is taken over all m-tuples $(t_1, \ldots, t_m)$ satisfying $t_1 + \cdots + \hat{t}_i + \cdots + t_m \ge n, \forall i$, then $G(n, R) \le F(n, R)$. As written, $G(n, R)$ may contain several redundant factors which can be eliminated by using P. Hall's three subgroup Lemma ($[A, B, C] \le [B, C, A][C, A, B]$, $A, B, C \trianglelefteq F$). Thus for example

$$\begin{aligned}
G(1, R) &= [R_1, R_1](= R')\\
G(2, R) &= [R_2, R_2][R_1, R_1, R_1]\\
G(3, R) &= [R_3, R_3][R_2, R_1, R_2][R_1, R_1, R_1, R_1]\\
G(4, R) &= [R_4, R_4][R_3, R_1, R_3][R_2, R_2, R_2]\\
&\quad \times [R_2, R_1, R_1, R_2][R_2, R_1, R_2, R_1][R_1, R_1, R_1, R_1, R_1].
\end{aligned}$$

Enright [2] and Hurley [7] proved that $F(2, R) = G(2, R)$ for all $R \le F$. Also, Gupta and Gupta [5] proved that $F(n, F') = G(n, F')$ for all $n \ge 1$. These observations leads us to ask:

$$\text{Is } F(n, R) = G(n, R)?$$

It follows from the matrix representation of $F/F(n, R)$ that $F/F(n, R)$ is always a torsion free group. Since $F/G(n, R)$ may have periodic elements we must replace $G(n, R)$ by its isolator

$$G^*(n, R) = I_R(G(n, R)) = \{w \in R \mid w^k \in G(n, R), \quad \text{for some} \quad \mathrm{k} = \mathrm{k}(\mathrm{w}) \ge 1\}^{(5)}$$

The above question now takes the form:

$$\text{Is } F(n, R) = G^*(n, R)?$$

It has been shown in Gupta [4], that if F/RF' is finite then $F(n, R) = G^*(n, R)$ for all n. In particular, the Fox problem is solved when F/R is a finitely generated periodic group. Moreover, if F/RF' is finite then $G^*(n, R) = I_R([R_n, R_n]\gamma_{n+1}(R))$. Apart from certain small values of n, the solution of the general Fox problem remains open.

(4) If $G_1, \ldots, G_m$ are subgroups of a group G, then $[G_1, \ldots, G_m] = sgp\{[g_1, \ldots, g_m], g_i \in G_i\}$ where $[g_1, \ldots, g_m] = [[\cdots[g_1, g_2], \ldots], g_m]$ is a left-normed commutator of weight m.

(5) For any $S \trianglelefteq R \trianglelefteq F$, with R/S nilpotent, $S^* = I_R(S)$ is again a normal subgroup of F.

5. **A reduction theorem.** The difficulty in the solution of the general Fox problem is mainly due to the arbitrary nature of the normal subgroup R of F. It seems to be hinged on another equally important problem. Since $[R_i, R_j] \leq R' \cap \gamma_{i+j}(F)$, it follows that if $H(n, R) = \Pi_{i+j=n}[R_i, R_j]$, then $H(n, R) \leq R' \cap \gamma_n(F)$. Here we ask:

$$\text{Is } H(n, R) = R' \cap \gamma_n(F)?$$

Once again $F/R' \cap \gamma_n(F)$ is torsion free and we replace $H(n, R)$ by its isolator

$$H^*(n, R) = I_R(H(n, R)).$$

It seems reasonable to expect that $H^*(n, R) = R' \cap \gamma_n(F)$. In anticipation we make the following definition.

DEFINITION. Let R be a normal subgroup of a finitely generated free group F. Set $H(1, R) = R'$. For $n \geq 1$, R is said to be n-separable if $R' \cap \gamma_n(F) = H^*(n, R)$. If R is n-separable for all $n \geq 1$, then we say that R is separable.

If $R \leq \gamma_c(F)$, then R is trivially n-separable for $n \in \{1, \ldots, 2c\}$. If F/R is a free polynilpotent group then R is separable (Smel'kin [11], Ward [12]). If F/RF' is finite then R is separable (Gupta [4]). Beyond these we do not know any general result concerning separability. We conclude this article with the statement of the following reduction theorem (the details will be published elsewhere).

THEOREM 5.1. *Let R be a normal subgroup of a finitely generated free group F. If for some $n \geq 2$, R is $(n-1)$-separable then $F(n, R) = G^*(n, R)$.*

REMARK. A uniform structure theorem for R (too complicated to be included here) enables us to establish n-separability of R for small values of n (e.g. $n \leq 8$).

REFERENCES

1. Joan S. Birman, *Braids, links and mapping class groups*, Ann. Math. Studies **82,** 1974.
2. Dennis E. Enright, *Triangular matrices over group rings*, Doctoral thesis, New York University, 1968.
3. R. H. Fox, *Free differential Calculus I—Derivations in free group rings*, Ann. of Math. **57** (1953), 547–560.
4. Narain Gupta, *Fox subgroups of free groups*, J. Pure and Appl. Algebra **11** (1977), 1–7.
5. C. K. Gupta and N. D. Gupta, *Power series and matrix representations of certain relatively free groups*, Lecture Notes in Mathematics, No 372 (Springer-Verlag, 1974), 318–329.
6. N. D. Gupta, and I. B. S. Passi, *Some properties of Fox subgroups of free groups*, J. Algebra **43** (1976), 198–211.
7. T. C. Hurley, *On a problem of Fox*, Invent. Math. **21** (1973), 139–141.
8. W. Magnus, *Über Beziehungen Zwischen Höheren Kommutatoren*, J. Reine Angew. Math. **177** (1937), 105–115.
9. W. Magnus, *On a theorem of Marshall Hall.* Ann. of Math. Ser II **40** (1939), 764–768.

10. H. G. Schumann, *Über Modulun und Gruppenbilder*, Math. Ann. **114** (1935), 385–413.

11. A. L. Smel'kin, *Free polynilpotent groups*, Izv. Akad. Nauk. SSSR Ser. Mat. **28** (1964), 91–122 (English transl. Amer. Math. Soc. Transl. (2) **55** (1966), 270–304).

12. M. A. Ward, *Basic commutators*, Phil. Trans. Royal Soc. of London Ser. A, **264** (1969), 343–412.

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