Can. J. Math. Vol. 47 (6), 1995 pp. 1290–1316

THE TOPOLOGY OF QUASIBUNDLES

H. MOVAHEDI-LANKARANI AND R. WELLS

ABSTRACT. Let $\mathcal{M}(N,N)$ be the space of all $N \times N$ real matrices and let $\mathcal{G}(N)$ be the set of all linear subspaces of $\mathbb{R}^N$. The maps ker and coker from $\mathcal{M}(N,N)$ onto $\mathcal{G}(N)$ induce two quotient topologies, the right and left respectively. A quasibundle over a space X is defined as a continuous map from X into $\mathcal{G}(N)$; it is a right quasibundle if $\mathcal{G}(N) = \mathcal{M}(N,N)/\ker$ and a left quasibundle if $\mathcal{G}(N) = \mathcal{M}(N,N)/\operatorname{coker}$. The following is established. Theorem: Let ξ be a left quasibundle over a closed subset of some Euclidean space. Then the following statements are equivalent: (i) ξ has enough sections pointwise. (ii) Sections zero at infinity over closed subsets may be extended globally. (iii) A vector subbundle over a closed subset extends to a vector subbundle over a neighborhood. (iv) ξ is a fibrewise sum of local vector subbundles. (v) There exist finitely many global sections spanning ξ. (vi) ξ is an image quasibundle. (vii) ξ results from a Swan construction. These results are used to prove a version of the Hirsch-Smale immersion theorem for locally compact subsets of Euclidean space.

1. **Introduction.** Real, complex, and algebraic varieties as well as a closed invariant set of a diffeomorphism of a manifold are among the examples of nonmanifold objects which arise naturally in differential topology. In order to study the differential topology of these objects, the varieties may be decomposed into manifolds via Whitney stratification, but the invariant sets (*e.g.* the Henon attractor [7]) are too wild for such a decomposition. Since the notion of the tangent bundle plays a primary role in the classical differential topology, any attempt to study these more general objects must begin with the definition of a tangent space. For instance, if X is a Whitney stratified set with k-dimensional manifold stratum X_k, then we may define the Whitney tangent space at $x \in X$ to be that of the unique stratum containing x. The most notable properties of this definition are the following two:

i) The dimension of the tangent space may vary from point to point.

ii) Nonetheless, the tangent space varies continuously (in a certain sense) from point to point. (This is indeed condition A for a Whitney stratification.)

This Whitney tangent space is superseded by the generalized tangent spaces of Goresky and MacPherson investigated in [4]. In this set up the correspondence $x \longmapsto T_xX$ is not single valued, but satisfies properties (i) and (ii) in a suitable sense.

The invariant sets above are not so easily handled because there are many ways to define what is meant by a vector tangent to a subset X of Euclidean space [2, 3, 11, 12,

The second author was supported through grant N00014-90-J-4012 of the Office for Naval Research.

Received by the editors April 5, 1994.

AMS subject classification: Primary: 55R55, 55R65; secondary: 54C20.

Key words and phrases: quasibundles, image bundles, kernel bundles, fibre, classifying space, monotopy, immersion.

16]. If we use the definition presented in [3, 11], then the set of vectors tangent to X, labeled by source, $TX = \{(x, v) \mid v \in \mathbb{R}^N \text{ is tangent to } X \text{ at } x\} \subset X \times \mathbb{R}^N$ becomes a natural generalization of the tangent bundle of a smooth manifold and possesses the same two properties above. However, the sense of continuity in (ii) for TX is essentially dual to that for the Whitney tangent bundle. One advantage of this definition is that it allows us to prove a version of the Inverse Function Theorem [3, 11]: If X is locally compact and T_xX is the fibre of TX over x, then the orthogonal projection $\pi_x: \mathbb{R}^N \longrightarrow T_xX$ carries a neighborhood U of x in X *diffeomorphically* onto $\pi_x(U)$. That is, there is a C^1 map $\sigma_x: \pi_x(U) \longrightarrow U$ such that $\sigma_x = (\pi_x|_U)^{-1}$. Moreover, no vector subspace Q of $\mathbb{R}^N$ with dimension smaller than the dimension of T_xX has this orthogonal projection property.

With the above facts as motivation, we proceed to study this general type of fibration. We note that our example TX comes equipped with a classifying map, which we denote by the same symbol $TX: X \longrightarrow \mathcal{G}(N)$, where $\mathcal{G}(N)$ is the *set* of *all* vector subspaces of $\mathbb{R}^N$. The map TX is defined by requiring that $TX: x \longmapsto T_xX$. Still another natural example of a generalization of a vector bundle is the normal bundle of X defined by $NX = \{(x, v) \mid v \perp T_xX\}$ and the classifying map $NX: X \longrightarrow \mathcal{G}(N)$. There are two natural (and in a sense dual) topologies on $\mathcal{G}(N)$ producing topological spaces $\mathcal{G}_r(N)$ and $\mathcal{G}_\ell(N)$ so that the classifying maps $TX: X \longrightarrow \mathcal{G}_r(N)$ and $NX: X \longrightarrow \mathcal{G}_\ell(N)$ are continuous. (See Section 2.) These spaces $\mathcal{G}_r(N)$ and $\mathcal{G}_\ell(N)$ are not Hausdorff, but they are compact. Indeed, each is a finite disjoint union of Grassmannian manifolds topologized as usual. With the two spaces $\mathcal{G}_r(N)$ and $\mathcal{G}_\ell(N)$ available, we define a *right* (respectively, *left*) *quasibundle* over X to be a continuous map from X into $\mathcal{G}_r(N)$ (respectively, $\mathcal{G}_\ell(N)$). Using [11] it is a routine task to show that for X a locally compact subset of $\mathbb{R}^N$, the tangent quasibundle $TX: X \longrightarrow \mathcal{G}_r(N)$ is indeed a right quasibundle. Then, because the canonical maps given by the orthogonal complement operation $\perp: \mathcal{G}_r(N) \longrightarrow \mathcal{G}_\ell(N)$ and $\perp: \mathcal{G}_\ell(N) \longrightarrow \mathcal{G}_r(N)$ are homeomorphisms, the normal quasibundle $NX: X \longrightarrow \mathcal{G}_\ell(N)$ is a left quasibundle. More generally, a quasibundle ξ is a left quasibundle if and only if $\perp \circ \xi$ is a right one.

The *total space* $|\xi|$ of a quasibundle ξ is defined as $|\xi| = \{(x, v) \mid x \in X, v \in \xi(x)\}$ with the topology it inherits from the product topology of $X \times \mathbb{R}^N$. This space is equipped with a natural map $\rho: |\xi| \longrightarrow X$ given by $\rho(x, v) = x$, which corresponds to the ordinary vector bundle projection. For a connected space X a map $\xi: X \longrightarrow \mathcal{G}(N)$ is continuous with respect to both the right and left topologies if and only if there is some Grassmannian manifold $G_{k,N} \subset \mathcal{G}(N)$ with $\xi: X \longrightarrow G_{k,N} \subset \mathcal{G}(N)$, *i.e.*, $\rho: |\xi| \longrightarrow X$ is an ordinary vector bundle. It is an interesting fact that the total space $|\xi|$ is closed in $X \times \mathbb{R}^N$ if and only if $\xi: X \longrightarrow \mathcal{G}_r(N)$ is continuous.

One of the most useful elementary properties of a vector bundle is the fact that any vector over a point is the value at that point of a global section. Here, we turn this property into a definition and say that a quasibundle ξ over a space X has *enough sections pointwise* if, for any point $x \in X$ there exist finitely many global sections $X \longrightarrow |\xi|$ whose values at x form a basis for $\rho^{-1}(x)$. We observe that if ξ is a left quasibundle and y is sufficiently near x, these sections remain linearly independent at y. On the other hand, if ξ is a right quasibundle and y is sufficiently close to x, these sections remain a spanning set over y.

These observations are related to the fact if ξ is a left quasibundle, then $\dim \xi(x)$ is locally a minimum at x, while if ξ is a right quasibundle, then $\dim \xi(x)$ is locally a maximum at x. In either case, we stratify the base space X by the dimension of the fibre. Specifically, for $\xi: X \longrightarrow \mathcal{G}(N)$ we set $X_i = \{x \mid \dim \xi(x) = i\}$. This provides a filtration $\phi = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^N = X$, where $X^j = \bigcup_{i \leq j} X_i$. Note that each filtration set X^j is open when ξ is a right quasibundle and closed when ξ is a left quasibundle.

In [15], a quasibundle is defined as a triplet $p: E \longrightarrow X$, where p is continuous and every fibre $p^{-1}(x)$ has the structure of a vector space. Thus, our two definitions are more restrictive because, in the corresponding triplet $\rho: |\xi| \longrightarrow X$ for both the right and the left cases, the vector space operations in the fibre $\rho^{-1}(x)$ are related by our condition that $\xi: X \longrightarrow \mathcal{G}(N)$ be continuous with respect to either the right or the left topology. The class of quasibundles then considered in [15], for which the dimension function is lower semi-continuous, includes our left quasibundles. In order to prove the main theorem in [15], Samsonovicz imposes the additional condition that the base space is a finite CW complex and that the filter sets are subcomplexes. Under these hypotheses, it is shown in [15] that the existence of enough sections pointwise is equivalent to any of the following three conditions:

(i) The quasibundle is the image of an ordinary vector bundle morphism.

(ii) The quasibundle admits a tight para-atlas. (See [15] for definition.)

(iii) The quasibundle admits a decomposition. (See [15] for definition.)

In this paper we develop the theory of quasibundles $\xi: X \longrightarrow \mathcal{G}(N)$ with X homeomorphic to a closed subset of a Euclidean space, with unrestricted strata. We show that for a left quasibundle $\xi: X \longrightarrow \mathcal{G}_\ell(N)$ with X such a space, the existence of enough sections pointwise is equivalent to any of the following six statements.

(i) ξ has the section extension property.

That is, any section defined over a closed subset of X and vanishing at infinity extends to a global section vanishing at infinity.

(ii) ξ has the bundle extension property.

That is, a true vector subbundle of ξ over a closed subset of X extends to a true vector subbundle over a neighborhood of that closed set.

(iii) ξ is pieced.

That is, ξ is the fibrewise sum of vector subbundles defined over open subsets of X. This statement extends the existence of a decomposition in [15].

(iv) There exist finitely many sections spanning ξ.

That is, there exist sections $s_1, \ldots, s_q: X \longrightarrow |\xi|$ so that for every $x \in X$, the vectors $s_1(x), \ldots, s_1(x)$ span $\rho^{-1}(x)$.

We note a subtlety at this point: It does not appear to follow that the sections $s_1, \ldots, s_q$ above span the module $\Gamma(\xi)$ of (continuous) sections over the ring $C_0(X)$ of continuous functions on X vanishing at infinity.

(v) ξ is an image quasibundle.

That is, there exists an ordinary vector bundle morphism $\Phi: X \times \mathbb{R}^q \longrightarrow X \times \mathbb{R}^q$ with image $|\xi|$. This statement extends the image property in [15].

(vi) ξ results from a Swan construction.

That is, there exists a finitely generated topological module M over $C(X^+)$ and closed submodules J_x of M, for $x \in X$, so that the map $x \longmapsto M/J_x$ is naturally equivalent to ξ and M becomes a submodule of $\Gamma(\xi)$.

It is also shown that if ξ is a left quasibundle with enough sections pointwise, then $\perp \circ \xi$ is isomorphic to $\ker \Psi$, where Ψ is an ordinary vector bundle morphism of the trivial bundle. Some rudimentary homotopy properties in the general spirit of [10] also follow from the above results.

We note at this point that the most crucial open problem appears to be the following one:

PROBLEM. Does every left quasibundle over a closed subset of a Euclidean space have enough sections pointwise?

However, it is easily shown that if ξ is a left quasibundle over a locally compact space X, then for each $x \in X$ and $v \in \xi(x)$ there is a map $s_x : X \longrightarrow |\xi|$ with $s_x(x) = v$ which is continuous *at* x, but not necessarily at nearby points.

Our second main result (Theorem 3.4) is a refinement of an earlier bundle extension theorem for right quasibundles. This theorem is used heavily in the proof of our last main result (Theorem 3.5) which, partially addressing the homotopy classification problem, deals with extensions of homotopies via quasibundles monomorphisms. We remark here that this work does not address the issues relating to K-Theory, or characteristic classes for quasibundles. For current work in these directions see [14], [15], and [16].

As an application of our main results we prove a generalized version of the Hirsch-Smale Immersion Theorem [5, 8, 13] for differential immersions into a Euclidean space of certain locally compact subsets of another Euclidean space.

We thank L. Vaserstein for the idea of ordering the cover $\mathcal{V}$ in Lemma 4.4.

2. **Quasibundles.** In this section we define quasibundles and prove some preliminary results about them. As in the case with vector bundles, we would like each quasibundle to be induced from a universal quasibundle by a map into a universal base space. Our candidate for such a base space is the set $\mathcal{G}(N)$ of *all* linear subspaces of $\mathbb{R}^N$. There are two natural topologies that we wish to define on this set. To this end, let $\mathcal{M}(p,q)$ denote the set of all real $p \times q$ matrices with the obvious topology. For $A \in \mathcal{M}(p,q)$ let $\ker(A)$ and $\operatorname{coker}(A)$ denote the null space and the column space of A, respectively. Then both functions $\ker: \mathcal{M}(p,q) \longrightarrow \mathcal{G}(q)$ and $\operatorname{coker}: \mathcal{M}(p,q) \longrightarrow \mathcal{G}(p)$ are surjective. The two topologies on $\mathcal{G}(N)$ are then defined as the corresponding quotient topologies from $\mathcal{M}(N,N)$. We use the notation $\mathcal{G}_r(N) = \mathcal{M}(N,N)/\ker$ and $\mathcal{G}_\ell(N) = \mathcal{M}(N,N)/\operatorname{coker}$ to denote $\mathcal{G}(N)$ with these topologies. We will show that both topologies are non-Hausdorff and induce the standard (metric) topology on each Grassmannian manifold $G_{k,N}$, $0 \leq k \leq N$. (Here, $G_{k,N}$ is the subset of $\mathcal{G}(N)$ consisting of the k-dimensional subspaces.) This is accomplished by showing that both topologies are defined by the unsymmetric Hausdorff distance.

Let $\theta(x,y)$ be the standard Riemannian metric on the unit sphere S^{N-1} of $\mathbb{R}^N$. That is, $\theta(x,y)$ is the angle between x and y. As usual, for $x \in S^{N-1}$ and $Q \in \mathcal{G}(N)$, the distance between x and Q is defined as

$$\begin{aligned}\theta(x,Q) &= \inf\{\theta(x,y) \mid y \in Q \cap S^{N-1}\}\\ &= \cos^{-1} \|\pi_Q x\|,\end{aligned}$$

where π_Q denotes the orthogonal projection on Q and $\|\cdot\|$ is the Euclidean norm. Then for $P, Q \in \mathcal{G}(N)$, the *distance from P to Q* is defined by

$$d(P,Q) = \sup\{\theta(x,Q) \mid x \in P \cap S^{N-1}\}.$$

This distance function has the following properties which are used (either explicitly or implicitly) throughout this paper:

i) $0 \le d(P,Q) \le \pi/2$ for all $P, Q \in \mathcal{G}(N)$.
ii) $d(P,Q) = 0$ if and only if $P \subset Q$.
iii) If $d(P,Q) = 0$, then
  a) $d(Q,P) = 0$ if and only if $P = Q$;
  b) $d(Q,P) = \pi/2$ if and only if $P \subsetneq Q$.
iv) $d(P,Q) \le d(P,R) + d(R,Q)$ for all $P, Q, R \in \mathcal{G}(N)$.
v) If $P \cap Q^\perp \ne \{0\}$, then $d(P,Q) = \pi/2$.
vi) If $\dim P = 1$, then $d(P,Q) + d(P,Q^\perp) = \pi/2$.
vii) $d(P,Q) + d(P,Q^\perp) \ge \pi/2$ for all $P, Q \in \mathcal{G}(N)$.
viii) For any $P, Q \in \mathcal{G}(N)$ we have $d(P,Q) = d(Q^\perp, P^\perp)$.
ix) Let $P, Q \in \mathcal{G}(N)$ and let π_Q denote the orthogonal projection on Q. Then for all $y \in P$ we have $\|\pi_Q y\| \ge \cos\big(d(P,Q)\big)\|y\|$.
x) For $P, Q \in \mathcal{G}(N)$ we have $d(P,Q) = d(P, \pi_Q P)$.

The two topologies on $\mathcal{G}(N)$ are now defined as follows. In the first topology (*the right topology*) a neighborhood base for Q is defined by the sets $\mathcal{N}_\epsilon^r(Q) = \{P \mid d(P,Q) < \epsilon\}$. In the second topology (*the left topology*) we use the base of sets $\mathcal{N}_\epsilon^\ell(P) = \{Q \mid d(P,Q) < \epsilon\}$. Let $\tilde{\mathcal{G}}_r(N)$ and $\tilde{\mathcal{G}}_\ell(N)$ denote $\mathcal{G}(N)$ with the right and the left topology, respectively. Observe that the map $P \longmapsto P^\perp$ is an anti-isometry from $\tilde{\mathcal{G}}_r(N)$ to $\tilde{\mathcal{G}}_\ell(N)$ and from $\tilde{\mathcal{G}}_\ell(N)$ to $\tilde{\mathcal{G}}_r(N)$. It is a homeomorphism from $\tilde{\mathcal{G}}_r(N)$ to $\tilde{\mathcal{G}}_\ell(N)$ and from $\tilde{\mathcal{G}}_\ell(N)$ to $\tilde{\mathcal{G}}_r(N)$.

Our aim is to show that $\mathcal{G}_r(N) \cong \tilde{\mathcal{G}}_r(N)$ and $\mathcal{G}_\ell(N) \cong \tilde{\mathcal{G}}_\ell(N)$. But first, let us record the following criterion for convergence in $\tilde{\mathcal{G}}_r(N)$; the proof is trivial.

Let $\{P_n\}_{n\ge1}$ and P be in $\tilde{\mathcal{G}}_r(N)$. The $P_n \longrightarrow P$ in $\tilde{\mathcal{G}}_r(N)$ if and only if for some n_0 there exists a constant dimensional sequence $\{V_n\}_{n\ge n_0}$ with

i) $\{V_n\}_{n\ge n_0}$ converging to some V in the appropriate Grassmannian;
ii) $P_n \subset V_n$ for $n_0 \le n$;
iii) $V \subset P$.

The following criterion for convergence in $\tilde{\mathcal{G}}_\ell(N)$ is analogous to the one above:

Let $\{P_n\}_{n\geq 1}$ and P be in $\tilde{\mathcal{G}}_\ell(N)$. Then $P_n \longrightarrow P$ in $\tilde{\mathcal{G}}_\ell(N)$ if and only if for some n_0 there exists a constant dimensional sequence $\{V_n\}_{n\geq n_0}$ with

i) $\{V_n\}_{n\geq n_0}$ converging to some V in the appropriate Grassmannian;
ii) $V_n \subset P_n$ for $n_0 \leq n$;
iii) $P \subset V$.

LEMMA 2.1. *We have $\mathcal{G}_r(N) \cong \tilde{\mathcal{G}}_r(N)$ and $\mathcal{G}_\ell(N) \cong \tilde{\mathcal{G}}_\ell(N)$.*

PROOF. It suffices to show that the maps coker: $\mathcal{M}(N,N) \longrightarrow \tilde{\mathcal{G}}_\ell(N)$ and ker: $\mathcal{M}(N,N) \longrightarrow \tilde{\mathcal{G}}_r(N)$ are quotient maps. That coker: $\mathcal{M}(N,N) \longrightarrow \tilde{\mathcal{G}}_\ell(N)$ is continuous follows directly from the criterion for convergence in $\tilde{\mathcal{G}}_\ell(N)$. Also, since $\perp$: $\tilde{\mathcal{G}}_\ell(N) \longrightarrow \tilde{\mathcal{G}}_r(N)$ is a homeomorphism and $\big(\mathrm{coker}(A^T)\big)^\perp = \ker(A)$, we have that ker: $\mathcal{M}(N,N) \longrightarrow \tilde{\mathcal{G}}_r(N)$ is continuous. Here, A^T denotes the transpose of A.

Next we show that coker (and hence ker) is an open map. To this end, let $\epsilon > 0$ be given and let $A \in \mathcal{M}(N,N)$. We must show that there is $\delta > 0$ such that if $d\big(\mathrm{coker}(A), P\big) < \delta$, then $P = \mathrm{coker}(B)$ with $\|A-B\| < \epsilon$, where $\|\ \|$ denotes the *Euclidean* norm on $\mathcal{M}(N,N)$. For $0 < K < 1$, there is $\delta_1 > 0$ such that $\|\pi_P y\| \geq K\|y\|$ for $y \in Q$ provided that $d(Q,P) < \delta_1$. Let $P \in \tilde{\mathcal{G}}_\ell(N)$ with $d\big(\mathrm{coker}(A), P\big) < \delta_1$. Write $A = [A_1, \ldots, A_N]$ and set $B = [B_1, \ldots, B_N]$, where $B_i = \pi_P A_i$, $1 \leq i \leq N$. Then there is $\delta_2 > 0$ such that $\|A_i - B_i\| < \epsilon/\sqrt{N}$, for $d\big(\mathrm{coker}(A), P\big) < \delta_2$. In this case we get

$$\|A-B\| = \Big(\sum_{i=1}^{N} \|A_i - B_i\|^2\Big)^{1/2} < \epsilon.$$

Using property (x) above, we obtain $d\big(\mathrm{coker}(A), P\big) = d\big(\mathrm{coker}(A), \pi_P\,\mathrm{coker}(A)\big) = d\big(\mathrm{coker}(A), \mathrm{coker}(B)\big)$.

Let $\delta = \min\{\delta_1, \delta_2\}$ and suppose that $d\big(\mathrm{coker}(A), P\big) < \delta$. Then we may find B as above so that $\|A - B\| < \epsilon$ and $d\big(\mathrm{coker}(A), \mathrm{coker}(B)\big) < \delta$. If $\mathrm{coker}(B) = P$, we are done. If not, let R be the orthogonal complement of coker(B) in P. We may suppose that the columns $A_1, \ldots, A_m$ of A form a basis for coker(A). Then the columns $B_1, \ldots, B_m$ form a basis for coker(B). We may find vectors $B'_{m+1}, \ldots, B'_N$ so that $\|B_i - B'_i\|$ are small for $m+1 \leq i \leq N$, and so that $B_{m+1} - B'_{m+1}, \ldots, B - B'_N$ span R. Then the matrix $\tilde{B} = [B_1, \ldots, B_m, B'_{m+1}, \ldots, B'_N]$ has the two properties $\|A - \tilde{B}\| < \epsilon$ and $\mathrm{coker}(\tilde{B}) = P$. ∎

Because of the above lemma, we simplify our notation and use $\mathcal{G}_r(N)$ and $\mathcal{G}_\ell(N)$ to denote $\mathcal{G}(N)$ with the right or the left topology.

Observe that if $\{P_n\}_{n\geq 1}$ and $\{Q_n\}_{n\geq 1}$ are constant dimensional sequences in $\mathcal{G}(N)$ with $P_n \subset Q_n$ for all n and $P_n \longrightarrow P$, $Q_n \longrightarrow Q$ in the appropriate Grassmannians, then $P \subset Q$.

LEMMA 2.2. *The map span: $\mathcal{G}_\ell(N) \times \mathcal{G}_\ell(N) \longrightarrow \mathcal{G}_\ell(N)$ is continuous. Thus $\mathcal{G}_\ell(N)$ is an abelian monoid.*

PROOF. Let $\{P_n\}_{n\geq 1}$ and $\{Q_n\}_{n\geq 1}$ converge to P and Q respectively. We must show that span$\{P_n, Q_n\} \longrightarrow$ span$\{P, Q\}$ in $\mathcal{G}_\ell(N)$. We may assume that the sequences $\{P_n\}_{n\geq 1}$,

$\{Q_n\}_{n\geq 1}$ and $\{\operatorname{span}\{P_n, Q_n\}\}_{n\geq 1}$ are all constant dimensional and that in the appropriate Grassmannians $P_n \longrightarrow \bar{P} \supset P$, $Q_N \longrightarrow \bar{Q} \supset Q$, and $\operatorname{span}\{P_n, Q_n\} \longrightarrow R$. Then $\bar{P} \subset R$ and $\bar{Q} \subset R$ because both P_n and Q_n are contained in $\operatorname{span}\{P_n, Q_n\}$. Hence, $\operatorname{span}\{P, Q\} \subset \operatorname{span}\{\bar{P}, \bar{Q}\} \subset R$ as desired. ∎

We are now ready to define a quasibundle.

DEFINITION 2.3. Let X be a topological space. A *right* (respectively, *left*) (N, L)-*quasibundle* over X is a continuous map ξ from X into $\mathcal{G}_r(N)$ (respectively, $\mathcal{G}_\ell(N)$) such that $\max\{\dim \xi(x) \mid x \in X\} = L$.

When there is no danger of confusion, *i.e.*, N and/or L are implicitly understood or are unimportant, we use the term right (respectively, left) quasibundle. For simple examples of quasibundles which are not vector bundles, we give the following two constructions. Let U be an open subset of a topological space X and let $f: U \longrightarrow G_{k,N}$ be a vector bundle. (Here, $G_{k,N}$ denotes a Grassmannian manifold.) Then the map $\eta: X \longrightarrow \mathcal{G}_r(N)$ given by

$$\eta(x) = \begin{cases} f(x) & \text{for } x \in U \\ \mathbb{R}^N & \text{otherwise} \end{cases}$$

is a right quasibundle. Similarly, the map $\xi: X \longrightarrow \mathcal{G}_\ell(N)$ defined by

$$\xi(x) = \begin{cases} f(x) & \text{for } x \in U \\ 0 & \text{otherwise} \end{cases}$$

is a left quasibundle over X.

Observe that if ξ is a right (respectively, left) quasibundle over X, then $\dim(\xi(x))$ is locally a maximum (respectively, minimum) at x. Hence, if ξ is both a right and a left quasibundle, then $\dim(\xi(x))$ is locally a constant. Therefore, each component of X maps into a single Grassmannian $G_{k,N}$ for some $0 \leq k \leq N$.

The following lemma is just a rephrasing of the definition of a quasibundle on a first countable space.

LEMMA 2.4. *Let X be a first countable space.*

i) A map $\xi: X \longrightarrow \mathcal{G}_r(N)$ is a right quasibundle if and only if $d(\xi(x_n), \xi(x)) \longrightarrow 0$ whenever $x_n \longrightarrow x$.

ii) A map $\xi: X \longrightarrow \mathcal{G}_\ell(N)$ is a left quasibundle if and only if $d(\xi(x), \xi(x_n)) \longrightarrow 0$ whenever $x_n \longrightarrow x$.

For a quasibundle ξ, set

$$|\xi| = \{(x, v) \mid x \in X, v \in \xi(x)\}$$

and

$$\Sigma(\xi) = \{(x, v) \mid x \in X, v \in \xi(x), \|v\| = 1\} = |\xi| \cap (X \times S^{N-1})$$

with the topology they inherit from $X \times \mathbb{R}^N$. In addition, we define a map $\rho: |\xi| \longrightarrow X$ by setting $\rho(x, v) = x$. We also denote by ρ the restriction of this map to $\Sigma(\xi)$. Clearly, $\Sigma(\xi)$ is closed in $X \times \mathbb{R}^N$ if and only if $|\xi|$ is closed in $X \times \mathbb{R}^N$. We call $|\xi|$ the *total space of* ξ and the map $\Sigma(\xi) \longrightarrow X$ the *sphere quasibundle associated with* ξ. The following lemma (whose proof is straightforward) will be needed for the proof of Theorem 3.4.

LEMMA 2.5. *ξ is a right quasibundle over a first countable space X if and only if $\Sigma(\xi)$ is closed in $X \times \mathbb{R}^N$. Consequently, ξ is a right quasibundle over a first countable space X if and only if $|\xi|$ is closed in $X \times \mathbb{R}^N$.*

Observe that the analogous result for left quasibundles is false. Indeed, let $\xi\colon [0,1] \longrightarrow \mathcal{G}_\ell(2)$ be the left quasibundle with total space $|\xi| = \big([0,1) \times \mathbb{R}\big) \cup (\{1\} \times \{0\})$ which is not closed (nor open) in $[0,1] \times \mathbb{R}^2$.

Now consider the following two natural examples:

i) id: $\mathcal{G}(N) \longrightarrow \mathcal{G}(N)$ is a right quasibundle over $\mathcal{G}_r(N)$ and a left quasibundle over $\mathcal{G}_\ell(N)$.

ii) $\perp\colon \mathcal{G}_r(N) \longrightarrow \mathcal{G}_\ell(N)$ is a left quasibundle and $\perp\colon \mathcal{G}_\ell(N) \longrightarrow \mathcal{G}_r(N)$ is a right quasibundle.

Therefore, to each right quasibundle $\eta\colon X \longrightarrow \mathcal{G}_r(N)$ there corresponds a left quasibundle, which we call *the perp* of η and denote by $\eta^\perp$, by setting

$$\eta^\perp = \perp \circ \eta\colon X \longrightarrow \mathcal{G}_\ell(N).$$

Similarly, if ξ is a left quasibundle over X, then its perp $\xi^\perp = \perp \circ \xi\colon X \longrightarrow \mathcal{G}_r(N)$ is a right quasibundle. Furthermore, for any quasibundle ζ, we have $(\zeta^\perp)^\perp = \zeta$.

Let ξ and η be quasibundles over the same space X. If both ξ and η are right (respectively, left) quasibundles, then their Whitney sum $\xi \oplus \eta$ is defined as usual and is a right (respectively, left) quasibundle. In general, however, the Whitney sum of a left and a right quasibundle may not be defined as a right or left quasibundle. Nevertheless, for any quasibundle ξ over a space X we have $\xi \oplus \xi^\perp = \epsilon^N$, the unique (N,N)-bundle over X.

A quasibundle map from one quasibundle to another over the same base space is defined exactly as a bundle map. Specifically, $\varphi\colon \xi \longrightarrow \eta$ is a quasibundle map if $\varphi = \{\varphi_x\}_{x \in X}$ so that, for each $x \in X$, the map $\varphi_x\colon \xi(x) \longrightarrow \eta(x)$ is linear, and the following diagram commutes:

$$\begin{array}{ccc} |\xi| & \xrightarrow{\Phi} & |\eta| \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ X & \xrightarrow{\text{id}} & X \end{array}$$

where Φ is a continuous map, defined by setting $\Phi(x,v) = (x, \varphi_x v)$. In particular, ξ and η are *equivalent* if the map Φ is a homeomorphism. In this case we use the notation $\xi \cong \eta$. Of course, it is possible for two quasibundles ξ and η to be equivalent while $\xi^\perp$ is *not* equivalent to $\eta^\perp$.

DEFINITION 2.6. Let X be a topological space and let ξ and η be quasibundles over X. If for some natural number q the sequence

$$0 \longrightarrow |\eta| \longrightarrow X \times \mathbb{R}^q \longrightarrow |\xi| \longrightarrow 0$$

is exact and continuous, then we say that ξ is an *image quasibundle* and η is a *kernel quasibundle.*

It is clear that every image quasibundle is a left quasibundle and every kernel quasibundle is a right quasibundle. One of our major goals is to determine when the converse of this statement is true.

Now let X be a locally compact space. We say that a map $f\colon X \longrightarrow \mathbb{R}^N$ *vanishes at infinity* if for each $\epsilon > 0$ there is a compact set $K(\epsilon) \subset X$ such that $\|f(x)\| < \epsilon$ for $x \in X \setminus K(\epsilon)$. As usual, by a *section* of a quasibundle ξ over X we mean a continuous map $s\colon X \longrightarrow |\xi|$ such that $\rho\big(s(x)\big) = x$ for every $x \in X$. We say a section $s\colon X \longrightarrow |\xi| \subset X \times \mathbb{R}^N$ vanishes at infinity if and only if the map $\pi_2 \circ s\colon X \longrightarrow \mathbb{R}^N$ vanishes at infinity. Here, π_2 denotes the projection onto the second factor.

For a topological space X, we use the standard notation $C(X)$ to denote the continuous real valued functions on X and $C_0(X)$ to denote the set of functions in $C(X)$ which vanish at infinity. We denote the one-point compactification of X by X^+ and we set $C(X^+) = \mathbb{R} + C_0(X)$. If $\xi\colon X \longrightarrow \mathcal{G}(N)$ is a quasibundle, we may extend it to $\xi^+\colon X^+ \longrightarrow \mathcal{G}(N)$ by setting $\xi^+(\infty) = \{0\}$ if ξ is a left quasibundle and $\xi^+(\infty) = \mathbb{R}^N$ if ξ is a right quasibundle.

Here is a way to piece together vector bundles in order to form left quasibundles. Let $\mathcal{U}$ be a locally finite open cover of a topological space X and for each $U \in \mathcal{U}$ let $\xi_U\colon U \longrightarrow G_{k(U),N}$ be a vector bundle. Then define $\xi\colon X \longrightarrow \mathcal{G}_\ell(N)$ be setting

$$\xi(x) = \operatorname{span}\{\xi_U(x) \mid x \in U \in \mathcal{U}\}.$$

To check that ξ is a left quasibundle, let $x \in X$ and let $U_1, \ldots, U_m$ be the members of $\mathcal{U}$ containing the point x. It follows from Lemma 2.2 that span: $\big(\mathcal{G}_\ell(N)\big)^m \longrightarrow \mathcal{G}_\ell(N)$ is continuous. Hence, $\xi|_{U_1 \cap \cdots \cap U_m} = \operatorname{span}\{\xi_{U_1}, \ldots, \xi_{U_m}\}$ is continuous as desired since it is a composition of continuous maps

$$U_1 \cap \cdots \cap U_m \xrightarrow{\Delta_m} (U_1 \cap \cdots \cap U_m)^m \xrightarrow{\xi_{U_1} \times \cdots \times \xi_{U_m}} \big(\mathcal{G}_\ell(N)\big)^m \xrightarrow{\text{span}} \mathcal{G}_\ell(N).$$

Here, Δ_m denotes the diagonal in $(U_1 \cap \cdots \cap U_m)^m$.

We summarize the above remarks as the following lemma.

LEMMA 2.7. *Let $\mathcal{U}$ be a locally finite open cover of a topological space X and suppose that for each $U \in \mathcal{U}$ there is a continuous map (a vector bundle) $\xi_U\colon U \longrightarrow G_{k(U),N}$. Then the map $\xi\colon X \longrightarrow \mathcal{G}_\ell(N)$ defined by setting*

$$\xi(x) = \operatorname{span}\{\xi_U(x) \mid x \in U \in \mathcal{U}\}$$

is a left quasibundle over X.

DEFINITION 2.8. We say that a left quasibundle is *pieced* if it is constructed as above. We call the vector bundles $\xi_U\colon U \longrightarrow G_{k(U),N}$ the *pieces* (of the construction).

As was mentioned earlier, the existence of sections remains as one of the fundamental open problems. In the remainder of this section we give an example and construct two classes of left quasibundles which do have sections. But first we record the following easy result for the sake of completeness.

PROPOSITION 2.9. *Let ξ be a quasibundle over a locally compact space X and let $x_0 \in X$. Suppose that there exist a neighborhood U of x_0 and local sections $\sigma_1, \ldots, \sigma_k$: $U \longrightarrow |\xi|$, $k = \dim \xi(x_0)$, such that $\sigma_1(x), \ldots, \sigma_k(x)$ are linearly independent for each $x \in U$ and $\sigma_1(x_0), \ldots, \sigma_k(x_0)$ form a basis for $\xi(x_0)$. Then there exist global sections $s_1, \ldots, s_k: X \longrightarrow |\xi|$ such that $s_1(x), \ldots, s_k(x)$ are linearly independent for each $x \in U$ and $s_1(x_0), \ldots, s_k(x_0)$ form a basis for $\xi(x_0)$.*

DEFINITION 2.10. Let ξ be a quasibundle over a topological space X. We say that ξ has *enough sections pointwise* if for each $x \in X$ there exist sections $s_1, \ldots, s_{k(x)}$, $k(x) = \dim \xi(x)$, such that $s_1(x), \ldots, s_{k(x)}(x)$ form a basis for $\xi(x)$.

Of course, if ξ is a left quasibundle with enough sections pointwise, then for each $x \in X$ there exist a neighborhood U of x and sections $s_1, \ldots, s_{k(x)}$ such that $s_1(x), \ldots, s_{k(x)}(x)$ form a basis for $\xi(x)$ and $s_1(y), \ldots, s_{k(x)}(y)$ are linearly independent for each $y \in U$.

Clearly, there is no hope for a general right quasibundle to have enough sections pointwise. However, the situation is much more promising for left quasibundles. Indeed, it is essentially trivial to show that the Whitney tangent bundle of a Whitney stratified set in $\mathbb{R}^N$ is a left quasibundle with enough sections pointwise. Furthermore, if X is a locally compact subset of $\mathbb{R}^N$, then the normal bundle (in the sense of [3, 11]) of X is a left quasibundle with enough sections pointwise. This follows from the fact that for each $x \in X$ there is, by the Inverse Function Theorem (see the Introduction), a neighborhood U of x in X and a C^1 manifold M such that $U \subset M$ and $T_xX = T_xM$. Hence, any vector $v \in (T_xX)^\perp = (T_xM)^\perp$ extends (as a C^1 section) to a neighborhood of x in M and so to a neighborhood of x in X.

The next result provides another class of examples (albeit obvious) of left quasibundles with enough sections pointwise.

THEOREM 2.11. *Let ξ be a left quasibundle over a normal space X. If ξ is pieced, then it has enough sections pointwise.*

PROOF. Let $\mathcal{U}$ be a locally finite open cover of X and let $x_0 \in X$. Let $U_1, \ldots, U_r$ be the members of $\mathcal{U}$ containing x_0. Then there are open neighborhoods $V_1, \ldots, V_r$ of x_0 so that $\bar{V}_i \subset U_i$, $i = 1, \ldots, r$. We may find a basis $e_1, \ldots, e_{k(x_0)}$ for $\xi(x_0)$ such that each vector e_j lies in some $\xi_{U_{i(j)}}(x)$. Also, we may find sections σ_j of $\xi_{U_{i(j)}}$ with $\sigma_j(x_0) = e_j$. Let $f_1, \ldots, f_{k(x_0)}$ be Urysohn functions so that $f_i(x) = 1$ for $x \in \bar{V}_i$ and $f_i(x) = 0$ for $x \in X \setminus U_i$. We may then define sections $s_j: X \longrightarrow |\xi|, j = 1, \ldots, k(x_0)$, by setting $s_j(x) = f_{i(j)}(x)\sigma_j(x)$for $x \in U_{i(j)}$ and $s_j(x) = 0$ for $x \in X \setminus U_{i(j)}$. Clearly $s_1, \ldots, s_{k(x_0)}$ have the desired properties. ∎

In Section 4 we will prove the converse of the above theorem when X is homeomorphic to a closed subset of a Euclidean space.

Next, we construct yet another class of left quasibundles with enough sections pointwise. We will refer to the following construction as the *Swan Construction*. (See [17].) Let X be a locally compact Hausdorff space and let M be a finitely generated topological module over $C(X^+)$. Let $\varphi_1, \ldots, \varphi_N$ generate M. Suppose that we have a family $\{J_x\}_{x \in X}$ of submodules of M with the following properties:

i) For each $x \in X$, $I_x M \subset J_x$, where I_x is the ideal $I_x = \{f \in C_0(X) \mid f(x) = 0\}$.
ii) The set $J = \bigcup\{(x \times J_x) \mid x \in X\}$ is *closed* in $X \times M$.

Observe that condition (ii) implies that J_x is *closed* in M because $x \times J_x = J \cap (x \times M)$. Moreover, that J is closed is analogous to the closure condition in Lemma 2.5. For $x \in X$ define

$$\Lambda_J(x) = \left\{(a_1, \ldots, a_N) \in \mathbb{R}^N \mid \sum_{i=1}^{N} f_i \varphi_i \in J_x \text{ for } f_i(x) = a_i\right\}.$$

This is well defined since $a_i = f_i(x) = \tilde{f}_i(x)$ implies that $(f_i - \tilde{f}_i) \in I_x$ and so $\sum_{i=1}^{N}(f_i - \tilde{f}_i)\varphi_i \in I_x M \subset J_x$. Because J_x is closed in M, the quotient $V_J(x) = M/J_x$ is a topological quotient for each $x \in X$. We may then define, for each $x \in X$, a map $\alpha(x)\colon \mathbb{R}^N \longrightarrow V_J(x)$ by setting

$$\alpha(x)b = \sum_{i=1}^{N} g_i \varphi_i + J_x = \sum_{i=1}^{N} b_i \varphi_i + J_x,$$

where $g_i(x) = b_i$, $1 \leq i \leq N$. Again this is independent of the choice of g_i for b_i. Note that for each $x \in X$, $\alpha(x)$ is surjective because $\varphi_1, \ldots, \varphi_N$ generate M and that $\ker\big(\alpha(x)\big) = \Lambda_J(x)$ because $\alpha(x)b = 0$ implies $\sum_{i=1}^{N} b_i\varphi_i + J_x = 0$ giving $\sum_{i=1}^{N} b_i \varphi_i \in J_x$. This of course means that $b \in \Lambda_J(x)$ by definition of $\Lambda_J(x)$. Consequently, for each $x \in X$ the map $\beta(x) = \alpha(x)|_{\Lambda_J^{\perp}(x)}\colon \Lambda_J^{\perp}(x) \longrightarrow V_J(x)$ is an algebraic isomorphism and hence a homeomorphism.

LEMMA 2.12. *The map $x \longmapsto \Lambda_J(x)$ is a right quasibundle over X.*

PROOF. Let $\{x_n\}_{n\geq 1} \subset X$ converge to $x \in X$. We may assume (by choosing subsequences) that $\dim \Lambda_J(x_n)$ is constant and that $\{\Lambda_J(x_n)\}_{n\geq 1}$ converges in the appropriate Grassmannian to some $\tilde{\Lambda}_J$. It suffices to show that $\tilde{\Lambda}_J \subset \Lambda_J(x)$. Let $b \in \tilde{\Lambda}_J$. Then there is a sequence $\{b_n\}_{n\geq 1} \subset \mathbb{R}^N$ with $b_n \in \Lambda_J(x_n)$ and converging (in $\mathbb{R}^N$) to b. Denote $b_n = (b_{n,1}, \ldots, b_{n,N})$. By the continuity of multiplication $\mu_n = \sum_{i=1}^{N} b_{n,i}\varphi_i$ converges to $\mu = \sum_{i=1}^{N} b_i \varphi_i$. But $\mu_n \in J_{x_n}$ because $b_n \in \Lambda_J(x_n)$. Hence, $\{(x_n, \mu_n)\}_{n\geq 1}$ converges to (x, μ) and $(x, \mu) \in J$. Therefore, $\mu \in J_x$ and so $b \in \Lambda_J(x)$. ∎

THEOREM 2.13. *Let X be a locally compact Hausdorff space and let M be a finitely generated topological module over $C(X^+)$. Then, in the above notation, the map $\xi\colon X \longrightarrow \mathcal{G}_\ell(N)$ defined by setting $\xi(x) = \Lambda_J^{\perp}(x)$ is a left quasibundle over X with enough sections pointwise.*

PROOF. That ξ is a left quasibundle over X follows immediately from Lemma 2.12. It remains to show that ξ has enough sections pointwise. Consider the following diagram

$$\begin{array}{ccc}
X \times M & & \\
{\scriptstyle p}\big\downarrow & \overset{\gamma}{\nwarrow} & \\
(X \times M)/J = V_J & \overset{\beta}{\longleftarrow} & |\xi| \\
{\scriptstyle \pi_J}\big\downarrow & & \big\downarrow{\scriptstyle \rho} \\
X & \overset{\text{id}}{\longrightarrow} & X
\end{array}$$

where $(x,\mu) \sim (y,\nu) \bmod J$ if and only if $x = y$ and $\mu - \nu \in J_x$; $(X \times M)/J = (X \times M)/\sim$ and $p: X \times M \longrightarrow (X \times M)/J$ is the quotient map; $\beta(x,b) = \beta(x)b = \sum_{i=1}^{N} b_i\varphi_i + J_x$; and $\gamma(x,c) = (x, \sum_{i=1}^{N} c_i\varphi_i)$ for $c = (c_1,\ldots,c_N)$. Clearly γ is continuous. Note that the diagram is commutative because

$$\begin{array}{ccc} (x, \sum_{i=1}^{N} c_i\varphi_i) & \xleftarrow{\ \gamma(X)\ } & \\ \downarrow p & & \\ (x, \sum_{i=1}^{N} c_i\varphi_i + J_x) & \xleftarrow{\ \beta(x)\ } & (x,c) \end{array}$$

and the square is clearly commutative. The commutativity of the triangle implies that the map $\beta = p \circ \gamma$ is continuous because it is the composition of two continuous maps. Furthermore, β is a bijection because it is so at the fibre level.

Now let $\varphi \in M$ and define sections s_J (with respect to π_J) and s (with respect to ρ) by setting $s_J(x) = (x, \varphi + J_x)$ and $s(x) = \beta^{-1}s_J(x) = \beta^{-1}(x, \varphi + J_x)$. We claim that $p^{-1}\big(s_J(X)\big)$ is closed in $X \times M$. To see this, let $(x_n, \mu_n) \in p^{-1}\big(s_J(X)\big)$ converge to (x,μ). Then $\mu_n \in \varphi + J_{x_n}$ and so $(\mu_n - \varphi) \in J_{x_n}$ converges to $(\mu - \varphi) \in J_x$ by the assumption that the set J is closed. Therefore $\mu \in \varphi + J_x$ and so $(x,\mu) \in p^{-1}\big(s_j(X)\big)$.

Finally, the continuity of γ implies that $\gamma^{-1}p^{-1}\big(s_J(X)\big)$ is closed. But $\gamma^{-1}p^{-1}\big(s_J(X)\big) = \beta^{-1}\big(s_J(X)\big) = s(X)$. Hence, s is continuous. ∎

Again, in Section 4 we will prove the converse of this theorem when X is homeomorphic to a closed subset of a Euclidean space.

3. **Main results.** In this section we list our main results. But first let us introduce the following terminology.

DEFINITION 3.1. We say that a topological space X is *Euclidean closed* if and only if X is homeomorphic to a closed subset of a Euclidean space.

It follows from the classical embedding theorem of Menger-Nöbeling [9] that a space X is Euclidean closed if and only if X is a locally compact separable metric space of finite covering dimension.

THEOREM 3.2. *Let X be a Euclidean closed space and let $\xi: X \longrightarrow \mathcal{G}_\ell(N)$ be a left quasibundle. Then the following statements are equivalent.*

i) ξ has enough sections pointwise.

ii) ξ has the section extension property. That is, if $Y \subset X$ is closed and $s_Y: Y \longrightarrow |\xi|$ is a section over Y vanishing at infinity, then there is a global section $S: X \longrightarrow |\xi|$, vanishing at infinity, which extends s_Y.

iii) ξ has the bundle extension property. That is, if $Y \subset X$ is closed and $\psi: Y \longrightarrow G_{k,N}$ is a vector bundle with $\psi(y) \subset \xi(y)$ for all $y \in Y$, then there is an open neighborhood U of Y and a vector bundle $\zeta: U \longrightarrow G_{k,N}$ such that $\zeta|_Y = \psi$ and $\zeta(x) \subset \xi(x)$ for all $x \in U$.

iv) ξ is pieced.

v) There exist finitely many (global) sections spanning ξ. That is, there exist $q \geq N$ and sections $s_1, \ldots, s_q$ such that for every $x \in X$ we have $\xi(x) = \operatorname{span}\{s_1(x), \ldots, s_q(x)\}$.
vi) ξ is an image quasibundle.
vii) ξ results from a Swan Construction.

COROLLARY 3.3. *Let X be a Euclidean closed space and let $\eta: X \longrightarrow \mathcal{G}_r(N)$ be a right quasibundle such that $\xi = \eta^{\perp}: X \longrightarrow \mathcal{G}_\ell(N)$ has enough sections pointwise. Then there exists a bundle morphism $\Psi: X \times \mathbb{R}^N \longrightarrow X \times \mathbb{R}^N$ such that $|\eta| = \ker \Psi$.*

Next, we recall the following notation. If ζ is a left (respectively, right) (N, L)-quasibundle over a space X, we set

$$X_i = \{x \in X \mid \dim \zeta(x) = i;\ 0 \leq i \leq L\}$$

and

$$X^i = \{x \in X \mid \dim \zeta(x) \leq i;\ 0 \leq i \leq L\} = \bigcup_{j \leq i} X_j.$$

This provides us with a filtration of X by sets closed (respectively, open) in X:

$$\phi = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^L = X.$$

Clearly, if X is locally compact, then so are the sets X_i and X^i, $0 \leq i \leq L$.

Our second main result is a *refined bundle extension theorem* for right quasibundles (*cf.* Theorem 4.5, Corollary 4.6, and [3; Chapter 2, Proposition III]).

THEOREM 3.4. *Let X be a locally compact subset of a Euclidean space and let $\eta: X \longrightarrow \mathcal{G}_r(N)$ be a right (N, L)-quasibundle such that $\eta^{\perp}: X \longrightarrow \mathcal{G}_\ell(N)$ has enough sections pointwise. Then there exist neighborhoods $V_0, V_1, \ldots, V_L$ of $X_0, X_1, \ldots, X_L$ respectively and vector bundles $\gamma_i: V_i \longrightarrow G_{i,N}$ for $i = 0, 1, \ldots, L$ such that*
i) $V_i \cap (X \setminus X^i) = \phi$;
ii) $\gamma_i|_{X_i} = \eta|_{X_i}$;
iii) $\eta(x) \subset \gamma_i(x)$ for all $x \in V_i$; and
iv) $\gamma_i(x) \subset \gamma_{i+1}(x)$ for all $x \in V_i \cap V_{i+1}$.

Before stating the next result, we introduce some notation and terminology. If $\zeta: X \longrightarrow \mathcal{G}_*(N)$ is a quasibundle, right or left, and $I = [0, 1]$, we write $\zeta \times I$ for the composition $X \times I \xrightarrow{\text{proj}} X \xrightarrow{\zeta} \mathcal{G}_*(N)$. Clearly, $|\zeta \times I|$ may be identified with $|\zeta| \times I$ and contains $|\zeta| \times 0$ and $|\zeta| \times 1$ canonically. We say that a quasibundle map

$$\begin{array}{ccc} |\zeta| & \xrightarrow{\varphi} & |\beta| \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is a *quasibundle monomap* if it is a monomorphism on each fiber; the base map $X \longrightarrow Y$ may be many-to-one. Then a *monotopy* is a quasibundle monomap from $\zeta \times I$ to β; it is a monotopy *from* its restriction to $|\zeta| \times 0$ *to* its restriction to $|\zeta| \times 1$. We may define quasibundle epimaps and epitopies in the same way.

If $\varphi: |\zeta| \longrightarrow |\beta|$ is a quasibundle map and C is a subspace of the base space X of ζ, we write $\varphi \parallel C$ for the restriction of φ to $\big||\zeta|_C\big|$.

THEOREM 3.5 (THE MONOTOPY EXTENSION THEOREM). *Let X be a locally compact subset of Euclidean space and let $\eta: X \longrightarrow \mathcal{G}_r(N)$ be a right (N, L)-quasibundle such that $\eta^{\perp}: X \longrightarrow \mathcal{G}_\ell(N)$ has enough sections pointwise. Suppose that for some n*

$$\begin{array}{ccc} |\eta| & \xrightarrow{\varphi} & \mathbb{R}^n \\ \downarrow & & \downarrow \\ X & \longrightarrow & * \end{array}$$

is a quasibundle monomap and that

$$\begin{array}{ccc} \big|\eta|_{X_L}\big| \times [0,1] & \xrightarrow{\Phi} & \mathbb{R}^n \\ \downarrow & & \downarrow \\ X \times [0,1] & \longrightarrow & * \end{array}$$

is a monotopy with start $\varphi \parallel X_L$. Then there exists a monotopy

$$\begin{array}{ccc} |\eta| \times [0,1] & \xrightarrow{\Psi} & \mathbb{R}^n \\ \downarrow & & \downarrow \\ X \times [0,1] & \longrightarrow & * \end{array}$$

extending Φ and starting at φ.

In Section 6, as an application of our main results, we state and prove a generalized version (Theorem 6.1) of the Hirsch-Smale Immersion Theorem [5, 8, 13] for differential immersions into a Euclidean space of locally compact subsets of another Euclidean space.

4. **Proof of Theorem 3.2.**

The outline of the proof of Theorem 3.2 is as follows:

$$\begin{array}{ccccccccccc} \text{(i)} & \Longrightarrow & \text{(ii)} & \Longrightarrow & \text{(iii)} & \Longrightarrow & \text{(iv)} & \Longrightarrow & \text{(v)} & \Longrightarrow & \text{(vi)} \\ & & & & & & & & \Downarrow & & \Downarrow \\ & & & & & & & & \text{(vii)} & \Longrightarrow & \text{(i)} \end{array}$$

More specifically, we have
(i) $\Longrightarrow$ (ii): Theorem 4.1
(ii) $\Longrightarrow$ (iii): Theorem 4.5
(iii) $\Longrightarrow$ (iv): Theorem 4.7
(iv) $\Longrightarrow$ (v): Lemma 4.9
(v) $\Longrightarrow$ (vi): Corollary 4.10
(vi) $\Longrightarrow$ (i): A trivial consequence of the definition of an image quasibundle.
(v) $\Longrightarrow$ (vii): Theorem 4.11

(vii) $\Longrightarrow$ (i): Theorem 2.13

Let X be a set and let $\mathcal{F}$ be a family of subsets of X. By the *order* of the family $\mathcal{F}$ we mean the largest integer n such that the family $\mathcal{F}$ contains $n+1$ sets with nonempty intersection. If no such integer exists, then $\mathcal{F}$ has order ∞. This, of course, means that if $\mathcal{F}$ is of order n, then every point of X is in at most $n+1$ members of $\mathcal{F}$. It follows from a straight forward argument that if X is a Euclidean closed space with covering dimension of $X \leq D$, then every open covering of X has a *locally finite* open refinement of order $\leq 4(D+1)$. This implies that the integer q in (v) and (vi) of Theorem 3.2 may be chosen so that $N \leq q \leq N(N+1)(D+1)/2$. (See Lemma 4.8 and Lemma 4.9.)

THEOREM 4.1 (THE SECTION EXTENSION THEOREM). *Let X be a Euclidean closed space and let $\xi: X \longrightarrow \mathcal{G}_\ell(N)$ be a left quasibundle with enough sections pointwise. Let Y be a closed subset of X and suppose that $s_Y: Y \longrightarrow |\xi|$ is a section over Y vanishing at infinity. Then there is a section $s: X \longrightarrow |\xi|$, vanishing at infinity, which extends s_Y.*

PROOF. Replace X with its one point compactification X^+ and replace ξ with its canonical extension $\xi^+: X^+ \longrightarrow \mathcal{G}_\ell(N)$. So we may assume that $X^0 \neq \phi$ and that X is compact. Then the theorem is an immediate consequence of the following lemma.

LEMMA 4.2. *Let X be a compact Euclidean closed space of covering dimension $\leq D$ and let ξ be a left quasibundle over X with enough sections pointwise. Let Y be a closed subset of X and set $Y^n = Y \cap X^n$, $n \geq -1$. If $s_n: Y^n \longrightarrow |\xi|$ is a section, then there is a section $s: X \longrightarrow |\xi|$ such that $s|Y^n = s_n$.*

PROOF. We will prove the lemma by induction on n. Clearly, $s_{-1} = \phi$ and $s_0 = \{0\} = s|Y^0$.

Assume that the lemma is true for $n-1$. Let $s_n: Y^n \longrightarrow |\xi|$ be a section. By the induction hypothesis, there is a section $s'_n: X \longrightarrow |\xi|$ such that $s'_n|_{Y^{n-1}} = s_n|_{Y^{n-1}}$. Set $S = s_n - s'_n$ and observe that $S|_{Y^{n-1}} = 0$. Let $x_0 \in Y^n \setminus Y^{n-1}$. Then (by hypothesis) there is a neighborhood U of x_0 in $X \setminus Y^{n-1}$ and n (global) sections $\sigma_1, \sigma_2, \ldots, \sigma_n$ such that $\sigma_1(x_0), \ldots, \sigma_n(x_0)$ form a basis for $\xi(x_0)$ and $\sigma_1(x), \ldots, \sigma_n(x)$ are linearly independent for each $x \in U$. Hence, if $x \in U \cap X^n$, the vectors $\sigma_1(x), \ldots, \sigma_n(x)$ form a basis for $\xi(x)$. Let V be an open neighborhood of x_0 in $X \setminus Y^{n-1}$ with the property that $V \subset \bar{V} \subset U$. Then for each $x \in \bar{V} \cap Y^n$ we have $S(x) = \sum_{i=1}^n c_i(x)\sigma_i(x)$, where each $c_i: \bar{V} \cap Y^n \longrightarrow \mathbb{R}$, $1 \leq i \leq n$, is continuous. By the Tietze Extension Theorem, each c_i extends to a continuous function $c'_i: X \longrightarrow \mathbb{R}$. Therefore, we may define a section $S_{x_0}: X \longrightarrow |\xi|$ by setting

$$S_{x_0}(y) = \sum_{i=1}^{n} c'_i(y)\sigma_i(y), \quad y \in X. \tag{4.2.1}$$

Next, we use the family of sections $\{S_x\}_{x \in Y^n \setminus Y^{n-1}}$ (defined as in (4.2.1)) to find a family of open pairs $\{(W_j, O_j)\}_{j \in J}$ in $X \setminus Y^{n-1}$ and sections $S_{n,j}: X \longrightarrow |\xi|$ with the following properties:

1) $O_j \subset \bar{O}_j$ compact $\subset W_j \subset X \setminus Y^{n-1}$;
2) the family $\{W_j\}_{j \in J}$ is locally finite;

3) $\mathrm{diam}(W_j)$ (and hence $\mathrm{diam}(O_j)$) approaches zero as j tends to infinity;
4) $Y^n \setminus Y^{n-1} \subset \bigcup_{j\in J} O_j$;
5) $(S_{n,j} - S)|_{\bar{O}_j \cap (Y^n \setminus Y^{n-1})} = 0$;
6) $S_{n,j}|_{(X\setminus W_j)} = 0$.

Let $O = \bigcup_{j\in J} O_j$ which is open in $X \setminus Y^{n-1}$. Then O is a Euclidean closed space of covering dimension $\leq D$. Therefore, there is a locally finite refinement $\{U_i\}_{i\in I}$ of $\{O_j\}_{j\in J}$ of order $\leq 4(D+1)$. Furthermore, we may assume that $\mathrm{diam}(U_i)$ goes to zero as i tends to infinity. Let $\{U_i'\}_{i\in I}$ be a shrinking of $\{U_i\}_{i\in I}$; of course the order of $\{U_i'\}_{i\in I}$ is at most $4(D+1)$. We let $\mu_i = \max\{\|S(x)\| \mid x \in \bar{U}_i' \cap (Y^n \setminus (Y^{n-1})\}$, $i \in I$. Then for each i, we let O_j be a member of O containing U_i. Let $\varphi_i: X \longrightarrow [0,1]$ be a continuous function which is identically 1 on $\bar{U}_i' \cap (Y^n \setminus Y^{n-1})$ and has support in a sufficiently small neighborhood of $\bar{U}_i' \cap (Y^n \setminus Y^{n-1})$ contained in U_i. Writing $S'_{n,i} = \varphi_i S_{n,j}$ we may then assume that

1) support of $S'_{n,i}$ is in U_i;
2) $S'_{n,i}|_{\bar{U}_i' \cap (Y^n \setminus Y^{n-1})} = S|_{\bar{U}_i' \cap (Y^n \setminus Y^{n-1})}$;
3) $\max\{\|S'_{n,i}(x)\| \mid x \in U_i\} < 2\mu_i$.

CLAIM 1. $\lim_{i\to\infty} \mu_i = 0$.

If not, then there exist $\epsilon > 0$, a sequence of indices $i_1 < i_2 < i_3 < \cdots$, and a sequence of points $\{x_{i_k}\}_{k\geq 1}$ with $x_{i_k} \in \bar{U}_{i_k}' \cap (Y^n \setminus Y^{n-1})$ such that $\mu_{i_k} \geq \epsilon$ and $\|S(x_{i_k})\| \geq \epsilon$. Because X (and hence Y^n) is compact, we may assume that the sequence $\{x_{i_k}\}_{k\geq 1}$ converges to some $x \in Y^n$. There are two possibilities:

a) If $x \in Y^n \setminus Y^{n-1}$, then the cover $\{\bar{U}_i'\}_{i\in I}$ is not locally finite at x and we have a contradiction.

b) If $x \in Y^{n-1}$, then $\|S(x_{i_k})\| \longrightarrow \|S(x)\| = 0$ by continuity. This contradicts the assertion that $\|S(x_{i_k})\| \geq \epsilon > 0$ for all k and the claim is established.

Now there is a partition of unity $\{f_i\}_{i\in I}$ subordinate to the cover $\{U_i'\}_{i\in I}$. Since the cover $\{U_i\}_{i\in I}$ is locally finite, we may define a section $\tilde{S}: O \longrightarrow |\xi|$ by setting $\tilde{S}(x) = \sum_i f_i(x) S'_{n,i}(x)$. (That $\tilde{S}$ is continuous follows from the finite order of $\{U_i'\}_{i\in I}$.) Observe that if $x \in Y^n \setminus Y^{n-1}$, then x is only in some $\bar{U}_{i_1}', \ldots, \bar{U}_{i_\ell}'$, $\ell \leq 4(D+1)$, and so

$$\tilde{S}(x) = \sum_r f_{i_r}(x) S'_{n,i_r}(x) = \sum_r f_{i_r}(x) S(x) = S(x).$$

Therefore, $\tilde{S}|_{Y^n\setminus Y^{n-1}} = S|_{Y^n \setminus Y^{n-1}}$. (Recall that S is defined only on Y^n and that $\{f_i\}_{i\in I}$ are subordinate to $\{U_i'\}_{i\in I}$.)

CLAIM 2. *$\tilde{S}$ vanishes at infinity with respect to O.*

For $x \in O$ define $I(x) = \{i \in I \mid f_i(x) > 0\}$ and note that this is a set of cardinality $\leq 4(D+1)$. Let $\{x_m\}_{m\geq 1}$ be a sequence in O which goes to infinity with respect to O. Then the minimum element of $I(x_m)$ tends to infinity because at most $4(D+1)$ terms are non-zero. Set $\mu(x) = \max\{\mu_j \mid j \in I(x)\}$ for $x \in O$ and note that $\lim_{m\to\infty} \mu(x_m) = 0$. But $\|\tilde{S}(x_m)\| < 2\mu(x_m)$; thus $\lim_{m\to\infty} \|\tilde{S}(x_m)\| = 0$. This proves the claim.

Finally, define a section $\tilde{S}': X \longrightarrow |\xi|$ by setting $\tilde{S}'(x) = \tilde{S}(x)$ for $x \in O$ and $\tilde{S}'(x) = 0$ for $x \in X \setminus O$. That $\tilde{S}'$ is continuous follows from Claim 2. Then $s = \tilde{S} + s_n' : X \longrightarrow |\xi|$ is a section with the property that $s|_{Y^n} = s_n$. ■

We note that the following is an easy consequence of Theorem 4.1.

COROLLARY 4.3 (RUDIMENTARY COVERING HOMOTOPY THEOREM). *Let ξ be a left quasibundle over a space X which has enough sections pointwise. Let $\rho: |\xi| \longrightarrow X$ denote the projection. Let Y be a Euclidean closed space and suppose that there is a continuous map $f: Y \longrightarrow |\xi|$. If there is a homotopy $g: Y \times [0,1] \longrightarrow X$ such that $\rho \circ f = g(\,, 0)$, then there is a homotopy $F: Y \times [0,1] \longrightarrow |\xi|$ with $\rho \circ F = g$ and $F(\,, 0) = f$.*

In other words, there exists $F: Y \times [0,1] \longrightarrow |\xi|$ so that the following diagram commutes.

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{f} & |\xi| \\ \cap & \nearrow^{F} & \downarrow \rho \\ Y \times [0,1] & \xrightarrow{g} & X \end{array}$$

This result, of course, implies that if $\xi: X \longrightarrow \mathcal{G}_\ell(N)$ is a left quasibundle which has enough sections pointwise, then the projection $\rho: |\xi| \longrightarrow X$ is a fibration (with respect to Euclidean closed spaces). It is interesting to note, however, that if $\rho: \Sigma(\xi) \longrightarrow X$ is the sphere quasibundle associated with ξ, then there is a Covering Homotopy Theorem (with respect to finite CW complexes) for $\Sigma(\xi)$ if and only if ξ is a true vector bundle. Indeed, since the Covering Homotopy Theorem implies that $\rho: \Sigma(\xi) \longrightarrow X$ is a fibration (with respect to finite CW complexes), then all fibres must be (weakly) homotopically equivalent and hence are spheres of the same dimension. But then the fibres of $\rho: |\xi| \longrightarrow X$ also all have the same dimension.

Our goal now is to prove a Bundle Extension Theorem which states that a *vector* subbundle of a left quasibundle, lying over a closed subset of the base space, may be extended to a vector subbundle of the left quasibundle, lying over a neighborhood of the closed subset. Our Bundle Extension Theorem also implies a dual statement for a right quasibundle. But first we have to establish a crucial lemma.

We say that a metric space Y is *uniformly locally convex* if there is $\epsilon > 0$ such that for any $y \in Y$ there is a map $h_y: B(y,\epsilon) \times B(y,\epsilon) \times [0,1] \longrightarrow B(y,\epsilon)$ such that $h_y(y_1,y_2,0) = y_1$ and $h_y(y_1,y_2,1) = y_2$ and $h_y(y_1,y_2,t)$ depends continuously on all four variables. Here, $B(y,\epsilon)$ denotes the *closed* ball of radius ϵ centered at y. We call this number ϵ a *modulus of convexity* for Y.

Observe that for Y compact we have an easily proved uniformity property:

UNIFORMITY PROPERTY. Suppose that $\{y_n\}_{n\geq 1}$ and $\{z_n\}_{n\geq 1}$ are sequences in Y such that $y_n \longrightarrow y \in Y$ and $z_n \in B(y_n,\epsilon)$, $n \geq 1$. Suppose further that $\{t_n\}_{n\geq 1} \subset [0,1]$ is a sequence converging to zero. Then $\lim_{n\to\infty} h_{y_n}(y_n,z_n,t_n) = y$.

It follows from a theorem of J. H. C. Whitehead [6] that a compact Riemannian manifold M is uniformly locally convex. Indeed, the function $h_y(y_1,y_2,t)$ is given by $h_y(y_1,y_2,t) = \gamma(t)$, where $\gamma: [0,1],0,1 \longrightarrow M,y_1,y_2$ is the geodesic from y_1 to y_2.

LEMMA 4.4. *Let X be a Euclidean closed space and let $\mathcal{V} = \{V_0, V_1, \ldots\}$ be an ordered locally finite family of closed subsets of X such that the interiors $\{V_0^\circ, V_1^\circ, \ldots\}$ cover X. Let $\xi: X \longrightarrow \mathcal{G}_\ell(N)$ be a left quasibundle and let Y be a closed subset of X. Suppose that for each $i \geq 0$ there exists a k-dimensional vector bundle $g_i: V_i \longrightarrow G_{k,N}$ with the following three properties:*

i) $g_i(x) \subset \xi(x)$ for all $x \in V_i$;

ii) $d\big(g_i(x), g_j(x)\big) < \epsilon$ for all $x \in V_i \cap V_j$;

iii) $g_i(x) = g_j(x)$ for all $x \in Y \cap V_i \cap V_j$.

Here, $\epsilon > 0$ is a modulus of convexity for $G_{k,N}$ with the standard Riemannian metric. Then there is a k-dimensional vector bundle $g: X \longrightarrow G_{k,N}$ such that

i) $g(x) \subset \xi(x)$ for all $x \in X$;

ii) $d\big(g(x), g_i(x)\big) < 2\epsilon$ for all $x \in V_i$, $i \geq 0$;

iii) $g(x) = g_i(x)$ for all $x \in Y \cap V_i$, $i \geq 0$.

PROOF. For $x \in X$ define $\mathcal{V}(x) = \{V_i \mid x \in V_i\}$ and set $\tilde{X}_1 = \{(x, V_i) \mid x \in V_i\}$ with the obvious topology so that the projection $\rho: \tilde{X}_1 \longrightarrow X$ defined by setting $\rho(x, V_i) = x$ restricts to a homeomorphism $V_i \times \{V_i\} \longrightarrow V_i$. Next, set

$$\Delta_\infty = \Big\{f \in [0,1]^{\mathcal{V}} \mid f(V_i) = 0 \text{ for all but finitely many } i \text{ and } \sum_{V_i \in \mathcal{V}(x)} f(V_i) = 1\Big\}.$$

Then Δ_∞ inherits the product topology from $[0,1]^{\mathcal{V}}$. Furthermore, this topology coincides with the weak topology with respect to $\Delta_0 \subset \Delta_1 \subset \Delta_2 \cdots \subset \Delta_\infty$, where $\Delta_r = \{f \in \Delta_\infty \mid f(V_i) = 0 \text{ for } i > r\}$ with the obvious topology. Therefore, if K is a compact subset of Δ_∞, then $K \subset \Delta_r$ for some r. Indeed, it follows that Δ_∞ is a k-space; *i.e.*, it has the weak topology determined by the family of its compact subspaces. Since X is locally compact and Δ_∞ is a k-space, we have that $X \times \Delta_\infty$ is also a k-space. (See [1].) Moreover, if $C \subset X \times \Delta_\infty$ is compact, then $C \subset X \times \Delta_r$ for some r and so the product topology of $X \times \Delta_\infty$ coincides with the weak topology with respect to the family $X \times \Delta_0 \subset X \times \Delta_1 \subset \cdots \subset X \times \Delta_\infty$.

Now define $\tilde{X}_2 = \{(x,f) \mid f \in [0,1]^{\mathcal{V}(x)} \text{ and } \sum_{V_i \in \mathcal{V}(x)} f(V_i) = 1\}$. We give $\tilde{X}_2$ the topology determined by the injection $i: \tilde{X}_2 \hookrightarrow X \times \Delta_\infty$ defined by setting $i(x,f) = (x, \tilde{f})$, where $\tilde{f}(V_i) = f(V_i)$ for $V_i \in \mathcal{V}(x)$ and $f(V_i) = 0$ for $V_i \notin \mathcal{V}(x)$. We wish to define a continuous map $\tilde{g}: \tilde{X}_2 \longrightarrow G_{k,N}$ such that

$$\tilde{g}(x, \hat{V}_i) = g_i(x), \tag{4.4.1}$$

where $\hat{V}_i(V_j) = 0$ for $i \neq j$ and $\hat{V}_i(V_j) = 1$ for $i = j$. To this end, let $\tilde{X}_{2,r} = i^{-1}(X \times \Delta_r)$. We begin by setting $\tilde{g}_0(x, \hat{V}_0) = g_0(x)$. Suppose inductively that we have defined maps $\tilde{g}_n: \tilde{X}_{2,n} \longrightarrow G_{k,N}$ for $0 \leq n \leq r$, so that $\tilde{g}_n$ extends $\tilde{g}_{n-1}$ and that $\tilde{g}_r(x, \hat{V}_i) = g_i(x)$ for $0 \leq i \leq r$. Now let $(x,f) \in \tilde{X}_{2,r+1} \backslash \tilde{X}_{2,r}$; then the support of f is a set $\{V_{i_1}, V_{i_2}, \ldots, V_{i_s}\} \subset \mathcal{V}(x)$ with $i_1 < i_2 < \cdots < i_s = r+1$. Because $G_{k,N}$ is uniformly locally convex with modulus of convexity $\epsilon > 0$, for each $p \in G_{k,N}$ there is a map $h_p: B(p,\epsilon)\times B(p,\epsilon)\times[0,1] \longrightarrow B(p,\epsilon)$ such that $h_p(p_1,p_2,0) = p_1$ and $h_p(p_1,p_2,1) = p_2$. Set

$$f_r(V_i) = \frac{f(V_i)}{1-f(V_{r+1})} \quad \text{for } 0 \le i \le r$$

and define

$$\tilde{g}_{r+1}(x,f) = h_{g_{r+1}(x)}\big(g_{r+1}(x), \tilde{g}_r(x,f_r), 1-f(V_{r+1})\big) \tag{4.4.2}$$

for $(x,f) \in \tilde{X}_{2,r+1} \setminus X \times \{\hat{V}_{r+1}\}$ and $\tilde{g}_{r+1}(x,\hat{V}_{r+1}) = g_{r+1}(x)$. This map is clearly continuous on $\tilde{X}_{2,r+1} \setminus X \times \{\hat{V}_{r+1}\}$ and satisfies $\tilde{g}_{r+1}|_{\tilde{X}_{2,r}} = \tilde{g}_r$. (Note that $\tilde{X}_{2,r} \subset \tilde{X}_{2,r+1} \setminus X \times \{\hat{V}_{r+1}\}$.) That $\tilde{g}_{r+1}$ is continuous at any point of $V_{r+1} \times \{\hat{V}_{r+1}\}$, follows from the Uniformity Property of h. Thus $\tilde{g}_{r+1}$ is continuous and satisfies the equation $\tilde{g}_{r+1}(x,\hat{V}_i) = g_i(x)$ for $0 \le i \le r+1$.

Having completed the induction step, we set $\tilde{g} = \bigcup\{\tilde{g}_r \mid r = 0,1,2,\ldots\}$. Then, because $\tilde{X}_2$ has the weak topology with respect to the family of subspaces $\tilde{X}_{2,r}$, we see that $\tilde{g}$ is continuous. Of course, $\tilde{g}$ satisfies (4.4.1) for all i because each $\tilde{g}_r$ does so for all $i \le r$.

CLAIM. *There is a continuous map $\varphi: X \longrightarrow \tilde{X}_2$ such that $\rho \circ \varphi = id_X$.*

To see this, let $\{\varphi_i\}_{i\ge 0}$ be a partition of unity subordinate to the cover $\mathcal{V}^0 = \{V_i^0\}_{i\ge 0}$. For each $x \in X$, define $\varphi_x: \mathcal{V}(x) \longrightarrow [0,1]$ by setting $\varphi_x(V_i) = \varphi_i(x)$. Then $(x,\varphi_x) \in \tilde{X}_2$ and $\rho(x,\varphi_x) = x$. Define φ by setting $\varphi(x) = (x,\varphi_x)$.

Next, set $g = \tilde{g} \circ \varphi: X \longrightarrow G_{k,N}$. Let $x \in X$ and let V_{r+1} be the set of highest index in the support of φ_x. Then $g(x) = \tilde{g}(x,\varphi_x) \in B(\tilde{g}_{r+1}(x,\hat{V}_{r+1}),\epsilon)$. But $\tilde{g}_{r+1}(x,\hat{V}_{r+1}) = g_{r+1}(x)$ and so $d\big(g(x), g_{r+1}(x)\big) < \epsilon$. If $x \in V_i$ for some $0 \le i \le r$, then $d\big(g(x), g_i(x)\big) < 2\epsilon$ by the triangle inequality. Furthermore, if $x \in Y \cap V_i$, then by induction $\tilde{g}(x,\varphi_x) = g_i(x)$ because they all agree and so $g(x) = g_i(x)$ for $x \in Y \cap V_i$, $i \ge 0$.

Finally, in the construction, equation (4.4.2), of $\tilde{g}_{r+1}|_{\tilde{X}_{2,r+1}\setminus\tilde{X}_{2,r}}$, we may assume inductively that $\tilde{g}_r(x,f_r) \subset \xi(x)$ for any possible f_r. In addition, we have $g_{r+1}(x) \subset \xi(x)$ by hypothesis. Let $G_k\big(\xi(x)\big)$ be the subspace of $G_{k,N}$ consisting of k-dimensional subspaces of $\xi(x)$. Then $G_k\big(\xi(x)\big)$ is totally geodesic in $G_{k,N}$ with respect to the standard Riemannian metric. (If M is a symmetric space and N is a submanifold of M invariant under any symmetry about a point of N, then N is totally geodesic in M.) Consequently, we have the relation

$$h_{g_{r+1}(x)}(g_{r+1}(x), \tilde{g}_r(x,f_r), 1-f\big(V_{r+1}\big)\big) \in G_k\big(\xi(x)\big)$$

which implies that we have the inclusion $\tilde{g}_{r+1}(x,f) \subset \xi(x)$. ∎

THEOREM 4.5 (BUNDLE EXTENSION THEOREM). *Let Y be a closed subset of a Euclidean closed space X and let $\xi: X \longrightarrow \mathcal{G}_\ell(N)$ be a left quasibundle with enough sections pointwise. Suppose that there is a vector bundle $\gamma: Y \longrightarrow G_{k,N}$ such that $\gamma(y) \subset \xi(y)$ for all $y \in Y$. Then there exist an open neighborhood U of Y and a vector bundle $\zeta: U \longrightarrow G_{k,N}$ such that $\zeta|_Y = \gamma$ and $\zeta(x) \subset \xi(x)$ for all $x \in U$.*

PROOF. Since X is a Euclidean closed space and since $G_{k,N}$ is a compact Riemannian manifold, we may cover Y by interiors of a locally finite family $\{V_i\}_{i\ge 0}$ of closed subsets of X such that for each $i \geq 0$ there is a continuous map $g_i: V_i \longrightarrow G_{k,N}$ with the following three properties:

i) $g_i(y) = \gamma(y)$ for all $y \in Y$;

ii) $d\big(g_i(x), g_j(x)\big) < \epsilon$ for $x \in V_i \cap V_j$;

iii) $g_i(x) \subset \xi(x)$ for all $x \in V_i$.

Here, $\epsilon > 0$ is a modulus of convexity for $G_{k,N}$. To be more specific, for each $y \in Y$ there exist sections $s_1, \ldots, s_k$ of γ such that $s_1(y), \ldots, s_k(y)$ form a basis for $\gamma(y) \subset \xi(y)$ and $s_1(x), \ldots, s_k(x)$ are linearly independent for $x \in V_i$. By the Section Extension Theorem, each s_j, $1 \leq j \leq k$, extends to a section $\sigma_j: X \longrightarrow |\xi|$. For $x \in V_i$, let $g_i(x) = \operatorname{span}\{\sigma_1(x), \ldots, \sigma_k(x)\}$. Set $U = \bigcup_{i\geq 0} V_i$. Then by Lemma 4.4 there is a continuous map (a vector bundle) $\zeta: U \longrightarrow G_{k,N}$ such that $\zeta|_Y = \gamma$, $d\big(\zeta(x), g_i(x)\big) < \epsilon$ for $x \in V_i$, and $\zeta(x) \subset \xi(x)$ for all $x \in U$. ■

We note that the above theorem implies a dual statement for a right quasibundle.

COROLLARY 4.6 (THE SECOND BUNDLE EXTENSION THEOREM). *Let Y be a closed subset of a Euclidean closed space X and let $\eta: X \longrightarrow \mathcal{G}_r(N)$ be a right quasibundle. Suppose that there is a vector bundle $\gamma: Y \longrightarrow G_{k,N}$ such that $\gamma(y) \supset \eta(y)$ for all $y \in Y$. Then there exist an open neighborhood U of Y and a vector bundle $\zeta: U \longrightarrow G_{k,N}$ such that $\zeta|_Y = \gamma$ and $\zeta(x) \supset \eta(x)$ for all $x \in U$.*

PROOF. Let $\xi = \eta^\perp$. Then there exist a neighborhood U of Y and a vector bundle $\beta: U \longrightarrow G_{N-k,N}$ such that $\beta(x) \subset \xi(x)$ for all $x \in U$ and $\beta|_Y = \gamma^\perp$. Set $\zeta = \beta^\perp: U \longrightarrow G_{k,N}$. ■

Now we are ready to prove the converse of Theorem 2.11 when the base space is Euclidean closed.

THEOREM 4.7 (THE PIECING THEOREM). *Let X be a Euclidean closed space and let $\xi: X \longrightarrow \mathcal{G}_\ell(N)$ be a left quasibundle. Then there exists an open cover $\{U_1, \ldots, U_r\}$ of X and vector bundles $\xi_i: U_i \longrightarrow G_{k_i,N}$ for $i = 1, \ldots, r$ such that $\xi(x) = \operatorname{span}\{\xi_i(x) \mid x \in U_i\}$.*

PROOF. We prove the theorem by induction on ℓ, the number of nonempty strata. When $\ell = 1$, then $X = U$, and $\xi = \xi_1$. Assume by induction that the theorem is proved for the case of ℓ strata and consider the case of $\ell + 1$ strata $X = X_{k_1} \cup \cdots \cup X_{k_{\ell+1}}$ with $0 \leq k_1 < \cdots < k_{\ell+1}$. Then $X \backslash X_{k_1}$ is open in X and $\xi|_{X \backslash X_{k_1}}$ is an ℓ-strata left quasibundle. Consequently, there is an open cover $\{U_1, \ldots, U_r\}$ of $X \backslash X_{k_1}$ and vector bundles $\xi_i: U_i \longrightarrow G_{k_i,N}$ so that $\xi(x) = \operatorname{span}\{\xi_i(x) \mid x \in U_i\}$ for $x \in X \setminus X_{k_1}$. Because X_{k_1} is closed and because $\xi|_{X_{k_1}}$ is a vector bundle, by the Bundle Extension Theorem (Theorem 4.5) there is an open neighborhood U_{r+1} of X_{k_1} in X and a vector bundle $\xi_{r+1}: U_{r+1} \longrightarrow G_{k_1,N}$ extending $\xi|_{X_{k_1}}$ with $\xi_{r+1}(x) \subset \xi(x)$ for all $x \in U_{r+1}$. Then $\{U_1, \ldots, U_{r+1}\}$ is the desired open cover of X and $\xi_i: U_i \longrightarrow G_{k_i,N}$, $i = 1, \ldots, r+1$, are the desired vector bundles. ■

A glance at the above proof shows that we have in fact proved somewhat more, as stated in the following addendum to Theorem 4.7.

ADDENDUM. i) If $X = X_{k_1} \cup \cdots \cup X_{k_r}$, then we may assume that each U_i is an open neighborhood of X_{k_i} and that ξ_i is a vector bundle extension to U_i of the vector bundle $\xi|_{X_{k_i}}$.

ii) We may assume that there is an open cover $\{V_1, \ldots, V_r\}$ of X with $U_i \subset \bar{U}_i \subset V_i$ and vector bundles $\xi_i \colon V_i \longrightarrow G_{k_i,N}$ such that $\xi_i(x) \subset \xi(x)$ for $x \in V_i$ and $\xi(x) = \operatorname{span}\{\xi_i(x) \mid x \in V_i\}$.

In order to prove our next result we need the following lemma whose proof may be found in [14].

LEMMA 4.8. *Let X be a metric space of covering dimension $\leq D$ and let ζ be an n-dimensional vector bundle over X. Then there exist $n(D+1)$ sections $s_{k,j}$ for $1 \leq k \leq n$ and $1 \leq j \leq D+1$ such that for each $x \in X$ we have $\zeta(x) = \operatorname{span}\{s_{k,j}(x) \mid 1 \leq k \leq n$, $1 \leq j \leq D+1\}$.*

LEMMA 4.9. *Let X be a Euclidean closed space and let $\xi \colon X \longrightarrow \mathcal{G}_\ell(N)$ be a left quasibundle with enough sections pointwise. Then there exist an integer $q \geq N$ and global sections $s_1, \ldots, s_q$ such that for each $x \in X$ we have $\xi(x) = \operatorname{span}\{s_1(x), \ldots, s_q(x)\}$.*

PROOF. By the addendum to Theorem 4.7 there are covers $\{U_1, \ldots, U_r\}$ and $\{V_1, \ldots, V_r\}$ of X with $U_i \subset \bar{U}_i \subset V_i$ for $1 \leq i \leq r$ and vector bundles $\xi_i \colon V_i \longrightarrow G_{k_i,N}$ such that $\xi(x) = +\{\xi_i(x) \mid x \in V_i\}$. For each $i = 1, \ldots, r$ there exist, by Lemma 4.8, q_i sections $\sigma_1, \ldots, \sigma_{q_i}$ spanning ξ_i. Extend these to global sections $s_1, \ldots, s_{q_i}$ which agree with the σ_i on U_i (Proposition 2.9). Then the sections $s_1, \ldots, s_q$, where $N \leq q = \sum_{i=1}^r q_i$ are global sections spanning ξ. ∎

We note that in the above lemma if X has covering dimension $\leq D$, then $q_i = k_i(D+1)$, $1 \leq i \leq r$. Hence, we may take $q \leq N(N+1)(D+1)/2$. (See the remarks at the beginning of this section.)

COROLLARY 4.10 (THE IMAGE BUNDLE THEOREM). *Let X be a Euclidean closed space and let $\xi \colon X \longrightarrow \mathcal{G}_\ell(N)$ be a left quasibundle with enough sections pointwise. Then there exist an integer $q \geq N$ and a bundle morphism $\Phi \colon X \times \mathbb{R}^q \longrightarrow X \times \mathbb{R}^q$ such that $\xi = \Phi(X \times \mathbb{R}^q)$.*

PROOF. We have global sections $s_1, \ldots, s_q \colon X \longrightarrow |\xi|$ with $\xi(x) = \operatorname{span}\{s_1(x), \ldots, s_q(x)\}$ for all $x \in X$. Define the map $\Phi \colon X \times \mathbb{R}^q \longrightarrow |\xi| \subset X \times \mathbb{R}^n \subset X \times \mathbb{R}^q$ by setting

$$\Phi(x, t_1, \ldots, t_q) = \big(x, t_1 s_1(x) + \cdots + t_q s_q(x)\big). \qquad ∎$$

A similar result to this corollary is proved in [15] for the special case where X is a finite CW-complex and the open sets $X \setminus X^k$ are subcomplexes.

We are now ready to prove the final implication in Theorem 3.2.

THEOREM 4.11 (THE MODULE THEOREM). *A left quasibundle ξ over a Euclidean closed space X has enough sections pointwise if and only if ξ results from a Swan construction.*

PROOF. The if part is Theorem 2.13.

https://doi.org/10.4153/CJM-1995-066-8 Published online by Cambridge University Press

Conversely, suppose that ξ has enough sections pointwise. Then there exist finitely many sections $s_1, \ldots, s_q$ spanning ξ. If X is not compact, we may assume that the s_i, $1 \le i \le q$, vanish at infinity so that they span the canonical extension ξ^+ of ξ to X^+. Thus it suffices to prove the assertion for X compact. To this end, let $M = \{\sum_{i=1}^q v_i s_i \mid v = (v_1, \ldots, v_q): X \longrightarrow \mathbb{R}^q$ is continuous$\}$. Then M is a finitely generated topological module over $C(X)$ with the sup metric. For each $x \in X$ let $I_x = \{f \in C(X) \mid f(x) = 0\}$ and set $J_x = \{\mu \in M \mid \mu(x) = 0\}$. Then $I_x M = \{\sum_{i=1}^q v_i s_i \mid v_i \in I_x\}$ and hence $I_x M \subset J_x$. Set $J = \{(x, \mu) \mid \mu \in J_x\}$ and observe that J is closed in $X \times M$ because if $(x_n, \mu_n) \longrightarrow (x, \mu)$, then $\mu_n(x_n) = 0 \longrightarrow \mu(x) = 0$. Let ζ be the result of the Swan construction with respect to $s_1, \ldots, s_q$. For $x \in X$ let $\alpha_x: \mathbb{R}^q \longrightarrow M/J_x$ by setting

$$\alpha_x(b) = \sum_{i=1}^{q} b_i s_i + J_x,$$

where $b = (b_1, \ldots, b_q) \in \mathbb{R}^q$. Then we have

$$\mathbb{R}^q \xrightarrow{\alpha_x} M/J_x \xrightarrow{\cong} \xi(x).$$

The isomorphism follows from the following commutative diagram.

$$\begin{array}{ccc} J_x & \longrightarrow & 0 \\ \cap & & \cap \\ M & \xrightarrow{\text{ev}} & \xi(x) \\ \downarrow & \nearrow \cong & \\ M/J_x & & \end{array}$$

where ev denotes evaluation. Let $\eta(x) = \{b \in \mathbb{R}^q \mid \sum_{i=1}^q b_i s_i \in J_x\}$ and note that $\zeta(x) = \eta(x)^\perp$ by definition. Then the following diagram commutes

$$\begin{array}{ccccc} \mathbb{R}^q & \xrightarrow{\alpha_x} & M/J_x & \xrightarrow{\cong} & \xi(x) \\ \cup & \cong \nearrow \alpha_x & & \nearrow \text{ev} & \\ \zeta(x) & & \cong & & \end{array}$$

and so $\zeta \cong \xi$. ∎

Note that the result of a Swan construction is independent of the choice of sections because M/J_x depends only on the choice of J but not sections.

We finish this section by giving a proof of Corollary 3.3.

PROOF OF COROLLARY 3.3. For q sufficiently large, there exists (Lemma 4.9) a continuous map $S: X \longrightarrow \mathcal{M}(N, q)$ so that $\xi = \operatorname{coker} \circ S$. Define a vector bundle morphism $\Psi_1: X \times \mathbb{R}^N \longrightarrow X \times \mathbb{R}^q$ by setting $\Psi_1(x, t) = \big(x, S^T(x)t\big)$ and observe that

$$\begin{aligned} \ker \Psi_1 &= \{(x, t) \mid t \perp \operatorname{coker} S(x)\} \\ &= \{(x, t) \mid t \in \eta(x)\} \\ &= |\eta|. \end{aligned}$$

We define $\Psi_2: X \times \mathbb{R}^q \longrightarrow X \times \mathbb{R}^N$ by setting $\Psi_2(x, u) = \big(x, S(x)u\big)$. Finally, we set $\Psi = \Psi_2 \circ \Psi_1$ and note that we have $\ker S^T(x) = \ker S(x) S^T(x)$ implying $\ker \Psi = \ker \Psi_1 = |\eta|$. ∎

5. Proof of Theorem 3.4 and Theorem 3.5.

We begin with the following lemma.

LEMMA 5.1. *Let $\eta: X \longrightarrow \mathcal{G}_r(N)$ be a right quasibundle. Then the partition of $\Sigma(\eta) = |\eta| \cap (X \times S^{N-1})$ into fibers of the projection $\Sigma(\eta) \longrightarrow X$ is upper semi-continuous.*

PROOF. For $x \in X$, let $S(x) = \{v \in \mathbb{R}^N \mid v \in \eta(x) \text{ and } \|v\| = 1\}$. Let $\epsilon > 0$ and let U be the ϵ-neighborhood of $S(x)$ in $\mathbb{R}^N$. Choose $0 < \delta < \pi/4$ so that $2\sin(\delta/2) < \epsilon$. Then $V = \{y \in X \mid d(\eta(y), \eta(x) < \delta\}$ in an open neighborhood of x in X such that $y \in V$ implies $S(y) \subset U$. The lemma is proved. ∎

PROOF OF THEOREM 3.4. Recall that $L = \max\{\dim \eta(x) \mid x \in X\}$. By the Second Bundle Extension Theorem (Corollary 4.6) there exist an open neighborhood V_L of X_L in X and a vector bundle $\gamma_L: V_L \longrightarrow G_{L,N}$ such that $\gamma_L|_{X_L} = \eta|_{X_L}$ and $\eta(x) \subset \gamma_L(x)$ for all $x \in V_L$. Similarly, there is an open neighborhood V'_{L-1} of X_{L-1} in X and a vector bundle $\gamma'_{L-1}: V'_{L-1} \longrightarrow G_{L-1,N}$ such that $\gamma'_{L-1}|_{L-1} = \eta|X_{L-1}$ and $\eta(x) \subset \gamma'_{L-1}(x)$ for all $x \in V'_{L-1}$. We may assume also that $V'_{L-1} \cap X_L = \phi$.

Now for $\epsilon > 0$ set

$$U = \left\{x \in V_L \cap V'_{L-1} \mid d\left(\gamma'_{L-1}(x), \gamma_L(x)\right) < \epsilon\right\}.$$

We claim that U is open. To see this, consider

$$d\left(\gamma'_{L-1}(y), \gamma_L(y)\right) \leq d\left(\gamma'_{L-1}(y), \gamma'_{L-1}(x)\right) + d\left(\gamma'_{L-1}(x), \gamma_L(x)\right) + d\left(\gamma_L(x), \gamma_L(y)\right).$$

If $x \in U$, then the middle term is smaller than ϵ and, by taking y sufficiently near x, each of the two end terms may be made as small as possible. (This is because $d(P, Q) = d(Q, P)$ when $\dim P = \dim Q$.) Thus U is open. Then there is a neighborhood V_{L-1} of X_{L-1} in $X \setminus X_L$ such that $V_{L-1} \cap V_L = U$. This implies that $d\left(\gamma'_{L-1}(x), \gamma_L(x)\right) < \epsilon < \pi/2$ for $x \in V_{L-1} \cap V_L$.

Next, we let $\pi(x)$ denote the orthogonal projection of $\mathbb{R}^N$ onto $\gamma_L(x)$. Then the function $x \longmapsto \pi(x)$ is continuous on V_L because γ_L is a vector bundle. We may assume $\bar{V}_L$ is contained in a larger neighborhood W_L of X_L to which the vector bundle γ_L extends while preserving the property that $\eta(x) \subset \gamma_L(x)$. Let $h: V_L \longrightarrow [0, 1]$ be a continuous function which is identically equal to 1 on $V_{L-1} \cap \bar{V}_L$ and identically equal to zero on $V_{L-1} \setminus W_L$. Define a vector bundle isotopy Φ of $\gamma'_{L-1}|V_{L-1}$ by setting

$$\Phi(x, v, t) = \Big(x, h(x)[(1-t)v + t\pi(x)v] + \big(1 - h(x)\big)v\Big).$$

Note that for $x \in W_L$ the isotopy Φ is fixed on $\eta(x)$ because $\eta(x) \subset \gamma_L(x)$, $x \in W_L$. We let γ_{L-1} be the vector bundle which is the final image of γ'_{L-1} under Φ. That is, we set $\gamma_{L-1}(x) = \{u \mid (x, u) = \Phi(x, v, 1) \text{ for } v \in \gamma'_{L-1}(x)\}$. Then γ_{L-1} is a vector bundle over V_{L-1} with the properties

a) $\eta(x) \subset \gamma_{L-1}(x)$ for $x \in V_{L-1}$;
b) $\gamma_{L-1}(x) \subset \gamma_L(x)$ for $x \in V_{L-1} \cap V_L$; and
c) $\gamma_{L-1}|_{X_{L-1}} = \eta|_{X_{L-1}}$.

We observe that in the above argument, once V_L and γ_L were chosen, we altered neither. Thus, we may complete the proof by induction on L: Assume that the theorem is true for $\eta|_{(X\setminus X_L)}$ with γ_{L-1} and V_{L-1} prescribed as above. Then we have vector bundles $\gamma_0, \gamma_1, \ldots, \gamma_{L-1}$ over open subsets $V_0, V_1, \ldots, V_{L_1}$ of $X \setminus X_L$ satisfying the conclusions of the theorem. Because X_L is closed in X, (η is a right quasibundle) we may use these for η over all X appending γ_L and V_L to our list. ■

With Theorem 3.4 available, we give a proof of Theorem 3.5. Observe that this theorem may be viewed as a very restricted version for right quasibundles of the Covering Homotopy Extension Property for vector bundles.

PROOF OF THEOREM 3.5. In the proof of Theorem 3.4, we may shrink the set V_k and assume that V_k is a closed neighborhood of X_k in $X^k = \bigcup_{j\leq k} X_j$. The isotopy Φ of $\eta|_{X_L} = \gamma|_{X_L}$ extends to an isotopy Φ_L of γ_L with start $\varphi \parallel V_L$. Next, the isotopy Φ_L restricted to η over $X_{L-1} \cap V_L$ extends to an isotopy Φ'_{L-1} of $\eta|_{X_{L-1}} = \gamma_{L-1}|_{X_{L-1}}$. Then Φ'_{L-1} extends to an isotopy Φ_{L-1} of γ_{L-1}, which extends the restriction of Φ_L to γ_{L-1} over $V_L \cap V_{L-1}$, with start $\varphi \parallel V_{L-1}$. But the restrictions of Φ_L and Φ_{L-1} to η agree over $V_L \cup V_{L-1}$ and so we may define an isotopy $\Phi_L|_\eta \cup \Phi_{L-1}|_\eta$ over $V_L \cup V_{L-1}$. This isotopy extends Φ and has start $\varphi \parallel (V_L \cup V_{L-1})$. By repeating this procedure L steps, we are done. ■

6. **An application.** Our principal goal in this section is to prove an extension of the Hirsch-Smale Immersion Theorem [5, 8, 13] to locally compact subsets of Euclidean space. The actual work of proving this extension is done in the proof of the Hirsch-Smale Immersion Theorem [8]. The material here on quasibundle and in [11] constitute the organizational work necessary to transfer the effect of this theorem from the category of smooth manifolds to the C^1 category of locally compact subsets of Euclidean space. We first recall that the normal bundle (in the sense of [3, 11]) of an object in this category is a left quasibundle with enough sections pointwise; see the remarks following Definition 2.10.

THEOREM 6.1. *Let X be a locally compact subset of $\mathbb{R}^N$ and suppose that for some n there exists a quasibundle monomorphism*

$$\begin{array}{ccc} TX & \xrightarrow{\varphi} & \mathbb{R}^n \\ \downarrow & & \downarrow \\ X & \longrightarrow & * \end{array}$$

Then there exists an immersion $f: X \longrightarrow \mathbb{R}^n$ with df monotopic to φ.

For the proof of this theorem we need the following result from [11].

THEOREM 6.2 ([11], THEOREM 4.3). *Let X be a locally compact subset of $\mathbb{R}^N$. Then there exist C^1 submanifolds $\{M_i\}_{i\geq 1}$ of $\mathbb{R}^N$ with the following properties:*

i) $\dim M_i = i$

ii) $X_i \subset M_i$

https://doi.org/10.4153/CJM-1995-066-8 Published online by Cambridge University Press

iii) $T_xX = T_xM_i$ *for* $x \in X_i$

iv) $M_i \cap (X \setminus X^i) = \phi$.

v) For any metric defining the topology of $G_{k,N}$, *if* $\{x_n\}_{n\geq 1}$ *is a sequence from* M_k *with* $\lim_{n\to\infty} x_n = x \in (X \setminus X^k)$, *then there is a sequence* $\{x'_n\}_{n\geq 1}$ *in* X_k *also converging to x with* $\lim_{n\to\infty} d(T_{x_n}M_k, T_{x'_n}X) = 0$.

PROOF OF THEOREM 6.1. Let $L = \max\{\dim T_xX \mid x \in X\}$ and let M_L be the C^1 manifold of dimension L containing X_L, as provided by Theorem 6.2. Because X_L is closed in M_L, we may extend $\varphi \parallel X_L$ to a quasibundle monomorphism $\hat{\varphi}_L: TM_L \longrightarrow \mathbb{R}^n$. Then the Hirsch-Smale Immersion Theorem [8] yields a C^1 immersion $\hat{f}_L: M_L \longrightarrow \mathbb{R}^n$ with $\hat{\varphi}_L$ monotopic (actually isotopic) to $d\hat{f}_L$. Because X_L is closed in X, we may extend $\hat{f}_L|X_L$ to a C^1 map $f_L: X \longrightarrow \mathbb{R}^n$. Then for $x \in X_L$ we have $d\hat{f}_L(x) = df_L(x)$ so that f_L is an immersion on a neighborhood U_L of X_L in X. The bundle monotopy from $\hat{\varphi}_L$ to $d\hat{f}_L$ restricted to X_L yields a monotopy Φ from $\varphi \parallel X_L$ to $df_L \parallel X_L$. Then, by Theorem 3.5, the monotopy Φ may be extended over all of X. The end of the extended monotopy is a quasibundle monomorphism φ_L, isotopic to φ, such that $\varphi_L \parallel X_L = df_L \parallel X_L$.

Let M_{L-1} be the structure manifold (Theorem 6.2) for X_{L-1}. We may extend f_L to an open neighborhood of $X \cup M_{L-1}$ in $\mathbb{R}^N$. By shrinking M_{L-1} about X_{L-1}, we may assume that g, the restriction to M_{L-1} of the extend f_L, is an immersion on an open subset U'_{L-1} of M_{L-1} with $U_L \cap X_{L-1} = U'_L \cap X_{L-1}$. Then we may extend $\varphi_L \parallel X_{L-1}$ to a bundle monomorphism $\hat{\varphi}_{L-1}: TM_{L-1} \longrightarrow \mathbb{R}^n$ such that $\hat{\varphi}_{L-1} \parallel U'_{L-1} = dg \parallel U'_{L-1}$. By applying the Hirsch-Smale Immersion Theorem [8], we obtain an immersion $\hat{f}_{L-1}: M_{L-1} \longrightarrow \mathbb{R}^n$ so that $\hat{\varphi}_{L-1}$ is monotopic to $d\hat{f}_{L-1} \bmod U'_{L-1}$. We restrict the monotopy from $\hat{\varphi}_{L-1}$ to $d\hat{f}_{L-1}$ over X_{L-1} to get a monotopy from $\hat{\varphi}_{L-1} \parallel X_{L-1}$ to $d\hat{f}_{L-1} \parallel X_{L-1}$. This monotopy may be pasted to the fixed monotopy over U_L in order to obtain a monotopy from $\varphi_L \parallel (U_L \cup X_{L-1})$ to $(\varphi_L \parallel U_L) \cup (d\hat{f}_{L-1} \parallel X_{L-1})$. Again, by Theorem 3.5, this monotopy may be extended to one from φ_L to a bundle monomorphism φ_{L-1} such that $\varphi_{L-1} \parallel (X_{L-1} \cup X_L) = df_{L-1} \parallel (X_{L-1} \cup X_L)$, where $f_{L-1}: X \longrightarrow \mathbb{R}^n$ is a C^1 map which is an immersion on a neighborhood U_{L-1} of $X_{L-1} \cup X_L$ in X.

In the same way, we construct further monotopies φ_k to φ_{k-1} for $k = L-2, \ldots, 1$ so that

$$\varphi_k \parallel (X_k \cup \cdots \cup X_L) = df_k \parallel (X_k \cup \cdots \cup X_L),$$

where $f_k: X \longrightarrow \mathbb{R}^n$ is a C^1 immersion on a neighborhood U_k of $X_k \cup \cdots \cup X_L$ in X. The final monomorphism φ_0 is the one sought; completing the proof of the theorem. ∎

7. **Concluding remarks.** If we regard the prospective definition of a quasibundle as still unsettled (ours differs from [15], which in turn differs from others in use), the

moral of our tale is that perhaps the existence of enough sections pointwise should be built into the definition of a left quasibundle. In this connection, we note that the property of having enough sections pointwise is equivalent to the following lifting property:

LIFTING PROPERTY. A left quasibundle $\xi: X \longrightarrow \mathcal{G}_\ell(N)$ has enough sections pointwise if and only if there exists a continuous map $S: X \longrightarrow \mathcal{M}(q,q)$, for some $q \geq N$, so that the following diagram commutes.

$$\begin{array}{ccccc} & & & \overset{S}{\nearrow} & \mathcal{M}(q,q) \\ & & & & \big\downarrow \text{coker} \\ X & \underset{\xi}{\longrightarrow} & \mathcal{G}_\ell(N) & \hookrightarrow & \mathcal{G}_\ell(q) \end{array}$$

This property may be the natural one for the definition of a *stable* left quasibundle. The corresponding definition for stable right quasibundles is consistent and presents an interesting problem:

PROBLEM. If α and β are equivalent quasibundles, are $\alpha^\perp \oplus \mathbb{R}^n$ and $\beta^\perp \oplus \mathbb{R}^n$ equivalent for large n?

An affirmative answer to this problem implies that if ξ is a left quasibundle, then for large n the left quasibundle $\xi \oplus \mathbb{R}^n$ has enough sections pointwise.

We note that the extension Theorem 3.5 is a first step towards formulating an appropriate generalization of the classical Covering Homotopy Theorem to quasibundles. With such a generalization, perhaps one may be able to extend to quasibundles the link between equivalence of vector bundles and homotopy of their classifying maps.

REFERENCES

1. J. Dugundji, *Topology,* Allyn and Bacon, Boston, 1966.

2. H. Federer, *Geometric Measure Theory,* Springer-Verlag, New York, 1969.

3. G. Glaeser, *Étude de quelque algèbres tayloriennes,* J. Analyse Math. **6**(1958), 1–124 erratum, insert to (2) **6**(1958).

4. M. Goresky and R. MacPherson, *Stratified Morse Theory,* Springer-Verlag, New York 1988.

5. M. Gromov, *Partial Differential Relations,* Springer-Verlag, New York, 1986.

6. S. Helgason, *Differential Geometry and Symmetric Spaces,* Academic Press, New York, 1964.

7. M. Henon, *A two dimensional mapping with a strange attractor,* Comm. Math. Phys. **50**(1976), 69–77.

8. M. Hirsch, *Immersions of manifolds,* Trans. Amer. Math. Soc. **93**(1959), 242–276.

9. W. Hurewicz and H. Wallman, *Dimension Theory,* Princeton University Press, 1941.

10. D. Husemoller, *Fibre Bundles,* McGraw-Hill, 1966.

11. R. Mansfield, H. Movahedi-Lankarani and R. Wells, *Smooth finite dimensional embeddings,* preprint.

12. F. Morgan, *Geometric Measure Theory—A Beginners Approach,* Academic Press, San Diego, 1988.

13. A. Phillips, *Submersions of open manifolds,* Topology (2) **6**(1967), 170–206.

14. Publications I.R.M.A.—U.S.T.L. Flandres Artois, 1987, Volume 6, Numéro Spécial, Journées M.-H. Schwartz.

15. J. Samsonowicz, *Images of vector bundles morphism,* Bull. Polish Acad. Sci. Math **34**(1986), 599–607.

16. M. H. Schwartz, *Classes et caractéres de Chern-Mather des espaces linéaires,* C. R. Acad. Sci. Paris Sér. I Math. (5) **295**(1982), 399–402.

17. R. G. Swan, *Vector bundles and projective modules,* Trans. Amer. Math. Soc. **105**(1962), 264–277.

Department of Mathematics
Penn State Altoona Campus
Altoona, Pennsylvania 16601
U.S.A.
e-mail: hml@math.psu.edu

Department of Mathematics
Penn State University
University Park, Pennsylvania 16802
U.S.A.
e-mail: wells@math.psu.edu