

AN EXPANSION OF BIVARIATE SPLINE FUNCTIONS

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Abstract

Let Δ denote a triangulation of a planar polygon Ω . For any positive integer $0 \leq r < k$, let $S_k^r(\Delta)$ denote the vector space of functions in C^r whose restrictions to each triangle of Δ are polynomials of total degree at most k . Such spaces, called bivariate spline spaces, have many applications in surface fitting, scattered data interpolation, function approximation and numerical solutions of partial differential equations. An important problem is to give the function expression. In this paper, we prove that, if (Δ, Ω) is type- X , then any bivariate spline function in $S_k^r(\Delta)$ can be expressed by a series of univariate polynomials and a special bivariate finite element function in $S_{k-r}^0(\Delta)$ satisfying a so-called integral conformality condition system. We also give a direct sum decomposition of the space $S_k^r(\Delta)$. In addition, the dimension of $S_{r+1}^r(\Delta)$ for a kind of triangulation has been determined.

1. Introduction

For a connected polygonal region Ω in \mathbf{R}^2 , let Δ be a triangulation of Ω , and by this, we mean that the complement of Δ relative to Ω consists of a finite number of triangles such that none of the vertices of any triangle lies on the interior of any edge of another triangle. For $0 \leq r \leq k - 1$, where r and k are integers, we define $S_k^r(\Delta)$ to be the vector space of C^r functions which are piecewise polynomials with total degree at most k over each triangle of Δ . The space $S_k^r(\Delta)$ is called a bivariate spline space with degree k and smoothness order r .

It is well-known that, in the one dimensional case, univariate spline functions are very useful in many practical applications. For example, the cubic spline, which has good smoothness and solves the optimal interpolation problem, is widely applied in solving bending problems of plates and shells and in the surface design of cars, ships and aeroplanes. In the early and mid seventies, because of the increasing

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predominance of the finite element method, people began to study spline functions in two dimensions, see [22, 25, 26]. Since then, bivariate splines have been widely applied in multivariate function approximation, surface fitting and design, the finite element methods and numerical solutions of partial differential equations. In recent years, they have been used in scattered data interpolation [1, 8] and wavelets [6, 7].

With such a widespread range of practical applications, the space of bivariate spline functions is of great interest to many researchers. It is important to determine both the dimension of such a space and the expression of the function. Since Morgan and Scott gave the first result on dimension [22] in 1975, much work has been done in determining the spaces dimension, see [2, 3, 8, 13, 21, 24]. As for research into the expression of bivariate spline functions, there are currently three main methods: Bernstein-Bézier representation (B-form, B-net) [4, 11, 12, 14, 16], the basic spline (B-spline or box spline) method [5, 10, 17] and the smoothing cofactor method [8, 9, 25].

In [19, 20], a so-called integral representation of bivariate splines is proposed by the present author. This together with the B-net method, has been used to determine the dimension of the double periodic spline space $\tilde{S}_k^1(\Delta_{mn}^{(1)})$ with $k \geq 4$ on Type-1 triangulation $\Delta_{mn}^{(1)}$ [19], the bivariate quadratic spline space $S_2^1(\Delta)$ [20] and the cubic spline space $S_3^1(\Delta)$ over stratified triangulation [21]. In this paper, an expansion of the bivariate spline function in $S_k^r(\Delta)$ is presented, a direct sum decomposition of $S_k^r(\Delta)$ is given and the dimension of $S_{r+1}^r(\Delta)$ for a kind of particular triangulation is determined.

2. An integral representation of bivariate splines

For any triangulation Δ , denote

V_I := the set of all the interior vertices of Δ ,

V_B := the set of all the boundary vertices of Δ ,

E_I := the set of all the interior edges of Δ ,

E := the set of all the edges of Δ ,

$V := V_I + V_B$.

We say (Ω, Δ) is of type- X provided that there exists a rectangular coordinate system XOY such that the slope of any edge in E equals neither 0 nor ∞ and the number of intersection points of the boundary $\partial\Omega$ and any straight line which is parallel to the X -axis is no more than two. For example, (Ω, Δ) is always type- X for any triangulation Δ if Ω is a connected convex polygonal region. We will merely consider the case that (Ω, Δ) is type- X throughout this paper.

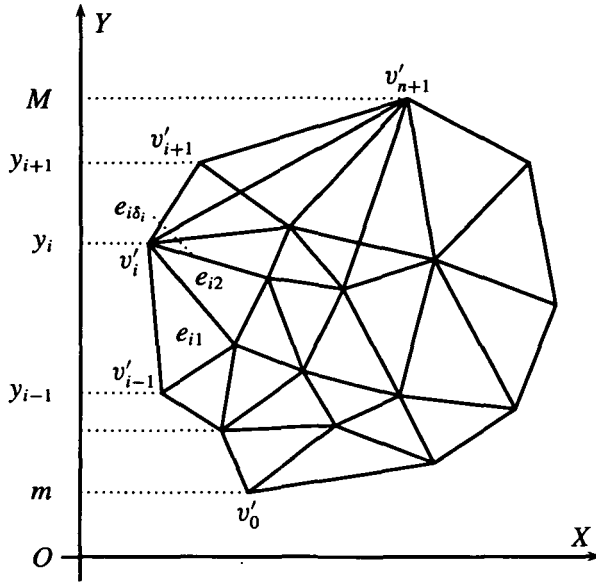


FIGURE 1. Planar polygon Ω and its triangulation Δ .

We set up a rectangular coordinate system $X O Y$ as required above. Clearly the two boundary vertices of Δ with maximum and minimum ordinates divide $\partial\Omega$ into two parts. The left part is denoted by $\partial^-\Omega$ and the set of all the vertices on $\partial^-\Omega$ except for the end points is denoted by V_B^- .

For each $v \in V_i$, the union of all the triangles with the common vertex v is called a *standard cell with interior vertex* v and is denoted by Δ_v . We denote the boundary vertices in Δ_v , in the counter-clockwise direction, by $v_j, j = 1, 2, \dots, d_v$, where d_v is the number of edges emanating from v , called the degree of v . Set $v_0 := v, e_j := v_0 v_j$ and $k_j := \text{slope}(e_j)$. Clearly, e_j can be described by the equation $x = l_j(y)$, where $l_j(y) := x_0 + (y - y_0)/k_j$, and (x_0, y_0) are the coordinates of v_0 . In addition, the triangle between e_j and e_{j+1} is denoted by $T_j, j = 1, 2, \dots, d_v$.

As shown in Figure 1, from bottom to top, we label consecutively all the vertices on $\partial^-\Omega$ $v'_i, i = 0, 1, \dots, n, n + 1$. And for $i = 1, \dots, n$, we number all the edges attached to v'_i in the counter-clockwise direction and label them $e'_{ij}, j = 1, \dots, d'_i$, where we have put $e'_{i1} := v'_{i-1} v'_i$. Note that, with the above-mentioned assumption of slope $(e) \neq 0, \infty$ for any $e \in E$, the equation of e'_{ij} can be denoted by $x = \phi_{ij}(y), i = 1, \dots, n, j = 1, \dots, d'_i$. In addition, we let T'_{ij} stand for the triangle between e'_{ij} and e'_{ij+1} .

From the assumption that (Ω, Δ) is type- X , we can see easily that the projections of all vertices $v'_i (i = 0, 1, \dots, n, n + 1)$ in the Y -axis form a univariate partition Δ_y ,

of interval $[m, M]$ as follows:

$$\Delta_y : \quad m = y'_0 < y'_1 < \cdots < y'_n < y'_{n+1} = M, \tag{1}$$

where y'_i is the ordinate of v'_i . We denote the equation of the broken line on which $\partial^-\Omega$ lies by $x = \Phi(y)$ with

$$\Phi(y) := \begin{cases} \phi_{i1}(y), & y \in [y'_{i-1}, y'_i], \quad i = 1, \dots, n \\ \phi_{na'_n}(y), & y \in [y_n, y'_{n+1}]. \end{cases}$$

For $0 \leq r \leq k - 1$, where r and k are integers, we define $S^r_k(\Delta)$ to be the vector space of C^r functions which are piecewise polynomials with total degree at most k over each triangle of Δ .

Let \mathbf{P}_k and Π_k denote the spaces of univariate and bivariate polynomials with degree at most k , respectively. As is well known [8, 9, 25], a necessary and sufficient condition for $s \in S^r_k(\Delta)$ is that the following *conformality condition*

$$\sum_{j=1}^{d_v} q_j(x, y) [x - l_j(y)]^{r+1} \equiv 0 \tag{2}$$

holds for any $v \in V_T$, where $q_j \in \Pi_{k-r-1}$ is the smoothing cofactor of s from T_{j-1} to T_j across edge e_j around v satisfying

$$s(x, y)|_{T_j} - s(x, y)|_{T_{j-1}} = q_j(x, y)[x - l_j(y)]^{r+1}. \tag{3}$$

Recently [20], another kind of identity, namely

$$\sum_{j=1}^{d_v} \int_{l_j(y)}^{l_{j+1}(y)} q_j(x, y) [x - l_j(y)]^{r+1} dx \equiv 0, \tag{4}$$

or

$$\sum_{j=1}^{d_v} \int_{l_j(y)}^{l_{j+1}(y)} s(x, y)|_{T_j} dx \equiv 0, \tag{5}$$

called *the integral conformality condition of s at the interior vertex v* , was introduced and a kind of relevant spline space $\hat{S}^r_k(\Delta)$ was defined to be composed of s such that

- (i) $s \in S^r_k(\Delta)$;
- (ii) the integral conformality condition (4) holds at each $v \in V_T$.

On the basis of this ([20] also see [21]), we obtain the following lemma.

LEMMA 1. Let $1 \leq r \leq k-1$ and (Ω, Δ) be type- X ; then a necessary and sufficient condition for $s \in S_k^r(\Delta)$ is that s can be expressed as

$$s(x, y) = p(y) + \int_{\Phi(y)}^x \eta(\tau, y) d\tau, \tag{6}$$

where $\eta(x, y) \in \hat{S}_{k-1}^{r-1}(\Delta)$ and $p(y)$ is a univariate piecewise polynomial with degree at most k on Δ_y such that

$$p_i(y) = p_0(y) + \sum_{t=1}^i \sum_{j=1}^{d'_t-1} \int_{\phi_{ij}}^{\phi_{ij+1}} \eta(x, y)|_{\tau_{ij}^r} dx, \quad i = 1, \dots, n, \tag{7}$$

with $p_i(y) := p(y)|_{[y'_i, y'_{i+1}]}$, $i = 0, 1, \dots, n$.

For convenience we here recast Lemma 1 as follows.

LEMMA 2. Let $1 \leq r \leq k-1$ and (Ω, Δ) be type- X ; then a necessary and sufficient condition for $s \in S_k^r(\Delta)$ is that s can be expressed as

$$s(x, y) = p(y) + \int_{\Phi(y)}^x \eta(\tau, y) d\tau, \tag{8}$$

where $\eta(x, y) \in \hat{S}_{k-1}^{r-1}(\Delta)$ and $p(y)$ is a univariate piecewise polynomial with degree at most k on Δ_y such that

$$p(y) = p_0(y) + \sum_{t=1}^n (y - y'_t)_+^0 \sum_{j=1}^{d'_t-1} \int_{\phi_{ij}}^{\phi_{ij+1}} \eta(x, y)|_{\tau_{ij}^r} dx, \tag{9}$$

where

$$(y - y'_t)_+^0 := \begin{cases} 1 & y > y'_t, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

3. An expansion of bivariate spline functions

Before giving our result, we define an integral projection operator $I_i : C^0(\Omega) \rightarrow C^0[m, M]$, $i = 0, 1, \dots, r-1$. For any piecewise polynomial function $f(x, y)$ defined on Δ with global smoothness at least 0 on the region Ω , denote

$$I_0[f] := \sum_{t=1}^n (y - y'_t)_+^0 \sum_{j=1}^{d'_t-1} \int_{\phi_{ij}}^{\phi_{ij+1}} f(x, y)|_{\tau_{ij}^r} dx,$$

$$I_1[f] := I_0 \left[\int_{\Phi}^x f(\tau_r, y) d\tau_r \right],$$

$$I_i[f] := I_0 \left[\int_{\Phi}^x \int_{\Phi}^{\tau_{r-i+1}} \cdots \int_{\Phi}^{\tau_{r-1}} f(\tau_r, y) d\tau_r \cdots d\tau_{r-i+2} d\tau_{r-i+1} \right], \quad i = 2, \dots, r-1.$$

THEOREM 1. *Let $1 \leq r \leq k - 1$ and (Ω, Δ) be type- X ; then a necessary and sufficient condition for $s \in S_k^r(\Delta)$ is that s has the following expansion:*

$$s(x, y) = \sum_{j=0}^{r-1} \frac{1}{j!} (x - \phi_{11}(y))^j \left(p_0^{(j)}(y) + \mathbf{I}_{r-1-j} [s^{(r)}] \right) + \int_{\Phi(y)}^{\tau_0} \int_{\Phi(y)}^{\tau_1} \dots \int_{\Phi(y)}^{\tau_{r-1}} s^{(r)}(\tau_r, y) d\tau_r \dots d\tau_2 d\tau_1, \tag{11}$$

where $\tau_0 = x, p_0^{(j)}(y)$ are arbitrary univariate polynomials in $\mathbf{P}_{k-j}, j = 0, 1, \dots, r - 1$, and $s^{(r)}(x, y) \in S_{k-r}^0(\Delta)$ satisfies the following integral conformality condition system at each interior vertex $v \in V_1$:

$$\left\{ \begin{array}{l} \sum_{j=1}^{d_v} \int_{l_1}^{l_j} q_j^{(r)}(x, y)[x - l_j] dx \equiv 0 \\ \sum_{j=1}^{d_v} \int_{l_1}^{l_j} \int_{l_j}^x q_j^{(r)}(x_1, y)[x_1 - l_j] dx_1 dx \equiv 0 \\ \sum_{j=1}^{d_v} \int_{l_1}^{l_j} \int_{l_j}^x \int_{l_j}^{x_1} q_j^{(r)}(x_2, y)[x_2 - l_j] dx_2 dx_1 dx \equiv 0 \\ \dots\dots\dots \\ \sum_{j=1}^{d_v} \int_{l_1}^{l_j} \int_{l_j}^x \int_{l_j}^{x_1} \dots \int_{l_j}^{x_{r-2}} q_j^{(r)}(x_{r-1}, y)[x_{r-1} - l_j] dx_{r-1} \dots dx_2 dx_1 dx \equiv 0, \end{array} \right. \tag{12}$$

with $q_j^{(r)} \in \Pi_{k-r}$ being the smoothing cofactor of $s^{(r)}(x, y)$ from T_j to T_{j+1} across edge e_j around v .

PROOF. From Lemma 2, we can easily see that a necessary and sufficient condition for $s^{(0)}(x, y) = s(x, y) \in S_k^r(\Delta)$ is that the following recurrence expressions

$$s^{(0)}(x, y) = p^{(0)}(y) + \int_{\Phi(y)}^x s^{(1)}(\tau_1, y) d\tau_1, \quad s^{(1)} \in \hat{S}_{k-1}^{r-1}(\Delta), \tag{13}$$

$$s^{(1)}(x, y) = p^{(1)}(y) + \int_{\Phi(y)}^x s^{(2)}(\tau_2, y) d\tau_2, \quad s^{(2)} \in \hat{S}_{k-2}^{r-2}(\Delta), \tag{14}$$

.....

$$s^{(r-1)}(x, y) = p^{(r-1)}(y) + \int_{\Phi(y)}^x s^{(r)}(\tau_r, y) d\tau_r, \quad s^{(r)} \in \hat{S}_{k-r}^0(\Delta), \tag{15}$$

hold, where

$$p^{(i)}(y) = p_0^{(i)}(y) + \mathbf{I}_0[s^{(i+1)}], \quad i = 0, 1, \dots, r - 1, \tag{16}$$

with $p_0^{(j)}$ being arbitrary univariate polynomials in \mathbf{P}_{k-j} , $j = 0, 1, \dots, r - 1$.

From (13)–(15) we have

$$s(x, y) = \sum_{i=0}^{r-1} \frac{1}{i!} [x - \Phi(y)]^i p^{(i)}(y) + \int_{\Phi(y)}^{\tau_0} \int_{\Phi(y)}^{\tau_1} \dots \int_{\Phi(y)}^{\tau_{r-1}} s^{(r)}(\tau_r, y) d\tau_r \dots d\tau_2 d\tau_1, \tag{17}$$

with $\tau_0 = x$. From the recursions (13)–(15) and equality (16), we have, for $0 \leq i < r - 1$,

$$\begin{aligned} p^{(i)}(y) &= p_0^{(i)}(y) + \mathbf{I}_0 [s^{(i+1)}] \\ &= p_0^{(i)}(y) + \mathbf{I}_0 \left[p^{(i+1)}(y) + \int_{\Phi(y)}^{\tau_{i+1}} s^{(i+2)}(\tau_{i+2}, y) d\tau_{i+2} \right] \\ &= p_0^{(i)}(y) + \sum_{t=1}^n (y - y_t')_+^0 \sum_{j=1}^{d_t-1} p^{(i+1)}(y) (\phi_{tj+1}(y) - \phi_{11}(y)) + \mathbf{I}_1 [s^{(i+2)}] \\ &= p_0^{(i)}(y) + (\Phi(y) - \phi_{11}(y)) p^{(i+1)}(y) + \mathbf{I}_1 [s^{(i+2)}] \\ &= \dots \\ &= p_0^{(i)}(y) - \sum_{j=1}^{r-i-1} \frac{(-1)^j}{j!} (\Phi(y) - \phi_{11}(y))^j p^{(i+j)}(y) + \mathbf{I}_{r-i-1} [s^{(r)}]. \end{aligned} \tag{18}$$

This gives

$$p^{(i)}(y) = \sum_{j=i}^{r-1} \frac{1}{(j-i)!} (\Phi(y) - \phi_{11}(y))^{j-i} \left(p_0^{(j)}(y) + \mathbf{I}_{r-1-j} [s^{(r)}] \right). \tag{19}$$

Substituting this expression into (17) yields

$$\begin{aligned} s(x, y) &= \sum_{i=0}^{r-1} \frac{1}{i!} (x - \Phi)^i \sum_{j=i}^{r-1} \frac{1}{(j-i)!} (\Phi - \phi_{11})^{j-i} \left(p_0^{(j)} + \mathbf{I}_{r-1-j} [s^{(r)}] \right) \\ &\quad + \int_{\Phi(y)}^{\tau_0} \int_{\Phi(y)}^{\tau_1} \dots \int_{\Phi(y)}^{\tau_{r-1}} s^{(r)}(\tau_r, y) d\tau_r \dots d\tau_2 d\tau_1 \\ &= \sum_{j=0}^{r-1} \frac{1}{j!} (x - \phi_{11}(y))^j \left(p_0^{(j)}(y) + \mathbf{I}_{r-1-j} [s^{(r)}] \right) \\ &\quad + \int_{\Phi(y)}^{\tau_0} \int_{\Phi(y)}^{\tau_1} \dots \int_{\Phi(y)}^{\tau_{r-1}} s^{(r)}(\tau_r, y) d\tau_r \dots d\tau_2 d\tau_1. \end{aligned} \tag{20}$$

Next, we need to discuss the integral conformality condition of $s^{(i)}(x, y)$, $i = 1, \dots, r$, at each interior vertex $v \in V_l$. Let $q_j^{(i)}(x, y) \in \Pi_{k-r-1}$ be the smoothing

cofactor of $s^{(i)}(x, y)$ from T_{j-1} to T_j across edge e_j around v ; then the integral conformality condition of $s^{(i)}(x, y)$, $i = 1, \dots, r$, at vertex v is equivalent to the system

$$\begin{cases} \sum_{j=1}^{d_v} \int_{l_1(y)}^{l_j(y)} q_j^{(1)}(x, y)[x - l_j(y)]^r dx \equiv 0 \\ \sum_{j=1}^{d_v} \int_{l_1(y)}^{l_j(y)} q_j^{(2)}(x, y)[x - l_j(y)]^{r-1} dx \equiv 0 \\ \dots\dots\dots \\ \sum_{j=1}^{d_v} \int_{l_1(y)}^{l_j(y)} q_j^{(r)}(x, y)[x - l_j(y)]^1 dx \equiv 0. \end{cases} \tag{21}$$

From $s^{(i)}(x, y) \in S_{k-i}^{r-i}(\Delta)$, we have, for any $i = 1, \dots, r - 1$,

$$s^{(i)}(x, y)|_{T_j} - s^{(i)}(x, y)|_{T_{j-1}} = q_j^{(i)}(x, y)[x - l_j(y)]^{r-i+1}. \tag{22}$$

For convenience, we say T_{j-1} is located at the left side of T_j if there exists y^* such that $(x_j, y^*) \in T_j$ and $(x_{j-1}, y^*) \in T_{j-1}$ with $x_{j-1} < x_j$. Without loss of generality, we can assume that T_{j-1} is located at the left side of T_j . Then by setting $s_i^{(i)} := s^{(i)}(x, y)|_{T_i}$, $i = j - 1, j$, it follows from (13)–(15) that

$$\begin{aligned} s_j^{(i)}(x, y) &= p^{(i)}(y) + \int_{\Phi(y)}^l s^{(i+1)}(\tau, y) d\tau + \int_l^{l_j} s_{j-1}^{(i+1)}(\tau, y) d\tau \\ &\quad + \int_{l_j}^x s_j^{(i+1)}(\tau, y) d\tau, \\ s_{j-1}^{(i)}(x, y) &= p^{(i)}(y) + \int_{\Phi(y)}^l s^{(i+1)}(\tau, y) d\tau + \int_l^x s_{j-1}^{(i+1)}(\tau, y) d\tau, \end{aligned}$$

where $x = l(y)$ stands for $x = l_{j-1}(y)$ or the equation of the third edge of T_{j-1} . These two equalities together with (22) yield

$$q_j^{(i)}(x, y)[x - l_j(y)]^{r-i+1} = \int_{l_j}^x [s_j^{(i+1)}(\tau, y) - s_{j-1}^{(i+1)}(\tau, y)] d\tau, \tag{23}$$

that is,

$$q_j^{(i)}(x, y)[x - l_j(y)]^{r-i+1} = \int_{l_j}^x q_j^{(i+1)}(\tau, y) [\tau - l_j(y)]^{r-i} d\tau. \tag{24}$$

Substituting this into the integral-conformality-condition system (21) yields (12). The proof of the theorem is complete.

Theorem 1 shows that any bivariate spline function $s \in S_k^r(\Delta)$ can be expressed by a series of univariate polynomials and a special bivariate finite-element function $s^{(r)} \in \tilde{S}_{k-r}^0(\Delta)$ satisfying the integral-conformality-condition system (12). Hence all discussion of the space $S_k^r(\Delta)$ can be transformed into discussion of the integral-conformality-condition system (12).

Noticing that, for all $j = 0, \dots, r-1$, $\mathbf{I}_{r-1-j} [s^{(r)}]$ is always a univariate polynomial in \mathbf{P}_{k-j} , and denoting the subspace of $\tilde{S}_{k-r}^0(\Delta)$ with its elements satisfying the integral-conformality-condition system (12) by $\tilde{S}_{k-r}^0(\Delta)$, we can re-write Theorem 1.

THEOREM 2. *Let $1 \leq r \leq k - 1$ and (Ω, Δ) be type-X; then a necessary and sufficient condition for $s \in S_k^r(\Delta)$ is that s has the following expansion:*

$$s(x, y) = \sum_{j=0}^{r-1} \frac{1}{j!} (x - \phi_{11}(y))^j p_0^{(j)}(y) + \int_{\Phi(y)}^{\tau_0} \int_{\Phi(y)}^{\tau_1} \dots \int_{\Phi(y)}^{\tau_{r-1}} s^{(r)}(\tau_r, y) d\tau_r \dots d\tau_2 d\tau_1, \tag{25}$$

where $\tau_0 = x$, $p_0^{(j)}(y)$ are arbitrary univariate polynomials in \mathbf{P}_{k-j} , $j = 0, \dots, r - 1$, and $s^{(r)}(x, y) \in \tilde{S}_{k-r}^0(\Delta)$.

In addition, we can easily prove that the expansion in Theorem 2 is unique. Hence we have the following direct sum decomposition of $S_k^r(\Delta)$.

THEOREM 3. *Let $1 \leq r \leq k - 1$ and (Ω, Δ) be type-X. If we set*

$$\mathbf{W}_j := \left\{ (x - \phi_{11}(y))^j \cdot p_0^{(j)}(y) : p_0^{(j)} \in \mathbf{P}_{k-j} \right\}, \quad j = 0, \dots, r - 1,$$

$$\mathbf{Q} := \left\{ \int_{\Phi(y)}^{\tau_0} \int_{\Phi(y)}^{\tau_1} \dots \int_{\Phi(y)}^{\tau_{r-1}} s^{(r)}(\tau_r, y) d\tau_r \dots d\tau_2 d\tau_1 : s^{(r)} \in \tilde{S}_{k-r}^0(\Delta) \right\},$$

then

$$S_k^r(\Delta) = \bigoplus_{j=0}^{r-1} \mathbf{W}_j \oplus \mathbf{Q}. \tag{26}$$

Alternatively, if we note that $\mathbf{Q} \cong \tilde{S}_{k-r}^0(\Delta)$, we have the following result.

COROLLARY 1. *Let $1 \leq r \leq k - 1$ and (Ω, Δ) be type-X; then*

$$\dim S_k^r(\Delta) = \frac{1}{2}(2k - r + 3)r + \dim \tilde{S}_{k-r}^0(\Delta). \tag{27}$$

However, we need to point out that, because the integral-conformality-condition system (12) involves the triangulation Δ , it is still very difficult to discuss, especially for the cases where the smoothness r is close to the degree k . This reflects the complexity of the bivariate spline space.

4. The bivariate spline space $S_{r+1}^r(\Delta)$

Consider the special case of $k = r + 1$. At this time, $s^{(r)}(x, y) \in S_1^0(\Delta)$ and $q_j^{(r)}(x, y) \in \Pi_0$, so we can set $c_j := q_j^{(r)}(x, y)$, $j = 0, \dots, d_v$. Then the integral-conformality-condition system (12) can be simplified to

$$\begin{cases} \sum_{j=1}^{d_v} c_j [l_1(y) - l_j(y)]^2 \equiv 0 \\ \sum_{j=1}^{d_v} c_j [l_1(y) - l_j(y)]^3 \equiv 0 \\ \dots\dots\dots \\ \sum_{j=1}^{d_v} c_j [l_1(y) - l_j(y)]^{r+1} \equiv 0, \end{cases} \tag{28}$$

that is,

$$\begin{cases} \sum_{j=1}^{d_v} c_j [k_1^{-1} - k_j^{-1}]^2 \equiv 0 \\ \sum_{j=1}^{d_v} c_j [k_1^{-1} - k_j^{-1}]^3 \equiv 0 \\ \dots\dots\dots \\ \sum_{j=1}^{d_v} c_j [k_1^{-1} - k_j^{-1}]^{r+1} \equiv 0. \end{cases} \tag{29}$$

In addition, note that, for $s^{(r)} \in S_1^0(\Delta)$, $c_j = q_j^{(r)}$ should satisfy the conformality condition (2), that is,

$$\sum_{j=1}^{d_v} c_j [x - l_j(y)] \equiv 0, \tag{30}$$

which is equivalent to

$$\begin{cases} \sum_{j=1}^{d_v} c_j = 0 \\ \sum_{j=1}^{d_v} c_j [k_1^{-1} - k_j^{-1}] = 0. \end{cases} \tag{31}$$

So, a necessary and sufficient condition for $s \in S_{r+1}^r(\Delta)$ is that s can be expanded as (11) with $s^{(r)} \in S_1^0(\Delta)$ satisfying, at each interior vertex $v \in V_I$, the following system:

$$\begin{cases} \sum_{j=1}^{d_v} c_j = 0 \\ \sum_{j=1}^{d_v} [k_1^{-1} - k_j^{-1}] c_j = 0 \\ \sum_{j=1}^{d_v} [k_1^{-1} - k_j^{-1}]^2 c_j = 0 \\ \dots\dots\dots \\ \sum_{j=1}^{d_v} [k_1^{-1} - k_j^{-1}]^{r+1} c_j = 0. \end{cases} \tag{32}$$

The coefficient matrix of the above homogeneous system is

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & \lambda_2 & \lambda_3 & \cdots & \lambda_{d_v} \\ 0 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_{d_v}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \lambda_2^{r+1} & \lambda_3^{r+1} & \cdots & \lambda_{d_v}^{r+1} \end{bmatrix} \tag{33}$$

with $\lambda_j := k_1^{-1} - k_j^{-1}, j = 2, \dots, d_v$.

It is easy to see that, if $d_v = r + 2$, then

$$\det A = \prod_{j=2}^{r+2} \lambda_j \cdot \det V[\lambda_2, \lambda_3, \dots, \lambda_{r+2}], \tag{34}$$

where $\det V[\lambda_2, \lambda_3, \dots, \lambda_{r+2}]$ is the well-known Vandermonde determinant. So $\det A \neq 0$ if and only if $k_i \neq k_j, i, j = 1, \dots, d_v, i \neq j$.

We now introduce the following result in which the so-called cross-cut grid line is introduced [25] and defined such that it divides Ω into two parts.

THEOREM 4. *Let (Ω, Δ) be type-X and, for each interior vertex v , let the number of edges with different slopes attached to v be not greater than $r + 2$. Then*

$$\dim S_{r+1}^v(\Delta) = \dim \Pi_{r+1} + N_c, \tag{35}$$

where N_c is the number of all cross-cut grid lines in Δ .

PROOF. First, we just consider the case that $\Delta = \Delta_v$. Without loss of generality, we can assume that there are only two interior edges having the same slope, for example, e_{j_0} and e_1 . Then the system (32) becomes

$$\begin{cases} \sum_{j=1}^{j_0-1} c'_j + \sum_{j=j_0+1}^{d_v} c'_j = 0 \\ \sum_{j=1}^{j_0-1} [k_1^{-1} - k_j^{-1}] c'_j + \sum_{j=j_0+1}^{d_v} [k_1^{-1} - k_j^{-1}] c'_j = 0 \\ \sum_{j=1}^{j_0-1} [k_1^{-1} - k_j^{-1}]^2 c'_j + \sum_{j=j_0+1}^{d_v} [k_1^{-1} - k_j^{-1}]^2 c'_j = 0 \\ \dots\dots\dots \\ \sum_{j=1}^{j_0-1} [k_1^{-1} - k_j^{-1}]^{r+1} c'_j + \sum_{j=j_0+1}^{d_v} [k_1^{-1} - k_j^{-1}]^{r+1} c'_j = 0, \end{cases} \tag{36}$$

with $c'_1 = c_1 + c_{j_0}$ and $c'_j = c_j, j = 2, \dots, d_v, j \neq j_0$. Since the number of edges with different slopes attached to v is not greater than $r + 2$, that is, $d_v - 1 \leq r + 2$, the system (36) has a unique zero solution:

$$c'_1 = c'_2 = \dots = c'_{j_0-1} = c'_{j_0+1} = \dots = c'_{d_v} = 0, \tag{37}$$

that is,

$$c_1 = -c_{j_0}, \quad c_2 = \cdots = c_{j_0-1} = c_{j_0+1} = \cdots = c_{d_v} = 0. \tag{38}$$

Hence the dimension of $\tilde{S}_1^0(\Delta)$ equals the sum of the dimension, 3, contributed from the so-called source-cell [8, 9, 25] and the dimension, 1, contributed from all the smoothing cofactors. Furthermore, according to Corollary 1, we can easily have

$$\dim S_{r+1}^r(\Delta) = \frac{1}{2}(r + 5)r + 4 = \dim \Pi_{r+1} + 1 = \dim \Pi_{r+1} + N_c. \tag{39}$$

Secondly, we consider the case that the number of interior vertices in Δ is more than 1. Without loss of generality, we can still assume that there is only one cross-cut grid line, denoted by L in Δ . From one end to the other, we denote all vertices which lie on L by $v^{(i)}$, $i = 0, 1, \dots, N + 1$, as shown in Figure 2, where $v^{(0)}$ and $v^{(N+1)}$ are two boundary vertices. For each $v^{(i)}$, $i = 1, \dots, N$, we denote all the edges attached to $v^{(i)}$ in the counter-clockwise direction by $e_j^{(i)}$, $j = 1, \dots, d_i$, where d_i is the degree of $v^{(i)}$. Without loss of generality, we can assume that $e_1^{(i)}$ and $e_{j_i}^{(i)}$ lie on the cross-cut grid line L and the corresponding smoothing cofactors of $s^{(r)} \in S_1^0(\Delta)$ across $e_1^{(i)}$ and $e_{j_i}^{(i)}$ around $v^{(i)}$ are denoted by $c_1^{(i)}$ and $c_{j_i}^{(i)}$, respectively. Then from the proof of the first case, we have

$$c_1^{(i)} = -c_{j_i}^{(i)}, \quad i = 1, \dots, N. \tag{40}$$

Note that $c_{j_i}^{(i+1)} = -c_1^{(i)}$ [8, 9, 25] and we have

$$c_1^{(i)} = c_1^{(1)}, \quad i = 1, \dots, N. \tag{41}$$

Now, let v be any interior vertex in V_I . Since the number of edges with different slopes attached to v is not greater than $r + 2$, it follows from the proof of the first case that, for any edge e_j attached to v , if no other edge has the same slope, then the corresponding smoothing cofactor $c_j = 0$; otherwise $c_j = -c_{j_0}$, where e_{j_0} has the same slope as edge e_j . For the latter case, e_j and e_{j_0} either lie on the unique cross-cut grid line L or a line segment L^* . We have proved that if they lie on L then both c_j and c_{j_0} are determined by $c_1^{(1)}$. Hence we need merely discuss the case that e_j and e_{j_0} lie on the line segment L^* .

Because L^* is not a cross-cut grid line, there exists at least one end point of L^* that is an interior vertex of Δ , denoted by v^* , as shown in Figure 2. From this end point v^* to v , we denote all vertices that lie on L^* by v_j^* , $j = 0, 1, \dots, N^*$, with $v^* = v_0^*$ and $v = v_{N^*}^*$. Similarly to (41), we conclude that each smoothing cofactor of $s^{(r)}(x, y)$ across edge $v_j^*v_{j+1}^*$ around v_{j+1}^* , $j = 0, 1, \dots, N^* - 1$, equals the smoothing cofactor of $s^{(r)}(x, y)$ across edge $v_0^*v_1^*$ around v_1^* . Since there is no other interior edge in $\Delta_{v_0^*}$

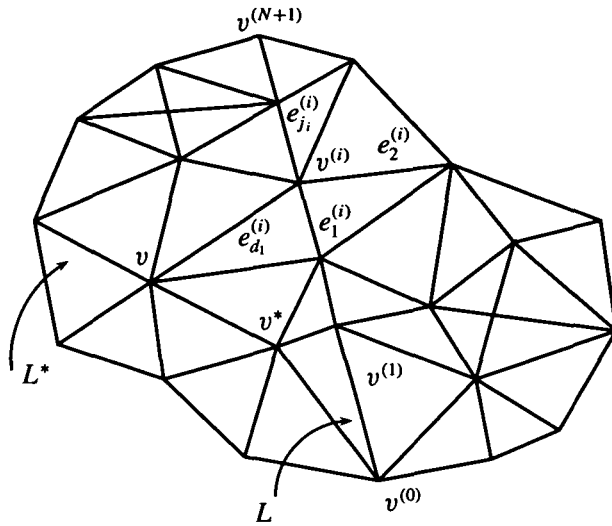


FIGURE 2. A particular triangulation Δ of Ω in Theorem 4.

having the same slope as edge $v_0^*v_1^*$, it follows from the proof of the first case that the smoothing cofactor of $s^{(r)}(x, y)$ across edge $v_0^*v_1^*$ around v_1^* equals 0. So all the smoothing cofactors of $s^{(r)}(x, y)$ across all the edges that lie on L^* are equal to 0, meaning $c_j = 0$.

So we conclude that the dimension contributed by the edges lying on the cross-cut grid line L is 1 and by other edges is 0. This results in the same result as (39). The proof of the theorem is completed.

It is worth indicating that, in the special case discussed in Theorem 4, our dimension result is not the lower bound of the dimension formula given by Schumaker [24]. In fact, if we still use e_i , as in [24], to stand for the number of edges with different slopes attached to the i th interior vertex, $i = 1, 2, \dots, |V_I|$, then [24] gives

$$\begin{aligned}
 \dim S'_{r+1}(\Delta) &\geq \frac{1}{2}(r+2)(r+3) + |E_I| - (r+2)|V_I| + \sum_{i=1}^{|V_I|} (r+2 - e_i)_+ \\
 &= \frac{1}{2}(r+2)(r+3) + |E_I| - (r+2)|V_I| + \sum_{i=1}^{|V_I|} (r+2 - e_i) \\
 &= \frac{1}{2}(r+2)(r+3) + |E_I| - (r+2)|V_I| - \sum_{i=1}^{|V_I|} e_i. \tag{42}
 \end{aligned}$$

Generally speaking, the lower bound (42) is often smaller than the dimension given

in (35).

COROLLARY 2. *Let (Ω, Δ) be type- X and, for each interior vertex v , let the number of all edges around v with different slopes be not greater than $r + 2$. If $N_c = 0$ then the spline space $S_{r+1}^r(\Delta)$ degenerates to a trivial polynomial space Π_{r+1} .*

From Corollary 1 we can easily see that, for the well-known Morgan-Scott triangulation Δ_s [23], whether it is symmetrical or not, the bivariate spline space $S_{r+1}^r(\Delta)$ is always a trivial polynomial space Π_{r+1} when $r \geq 2$.

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