SOME CYCLIC INEQUALITIES

by V. J. BASTON

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In this note we prove some cyclic inequalities which are generalisations of known results. We shall assume throughout that $a_{i+n} = a_i \geqq 0$ for all i, that no denominator in the statement of a result vanishes and finally that p, m and q are positive integers. We shall also use $A(i, m)$ to denote $\sum_{j=1}^{m} a_{i+j}$ with the convention that $A(i, 0) = 0$. The most interesting of our results is probably Theorem 2 since, in the special case $p = 1$, $m = 2$, $r = 0$, it gives a lower bound of $\frac{1}{3}n$ for the Shapiro sum $\sum_{i=1}^{n} \frac{a_i}{A(i,2)}$. Although it is by no means best possible, see (2), our method implicitly gives a really simple way of obtaining this lower bound which, incidentally, is an improvement on Rankin's original result (5).

In (1) Boarder and Daykin established the following results:

$$\inf_n \inf \frac{1}{n} \sum_{i=1}^{n} \frac{a_{i+1}+a_{i+4}}{A(i, 4)} \leqq \tfrac{1}{3}, \tag{1}$$

$$\inf_n \inf \frac{1}{n} \sum_{i=1}^{n} \frac{a_{i+1}+a_{i+2}+a_{i+4}}{A(i, 4)} \leqq \tfrac{1}{2}, \tag{2}$$

$$\inf_n \inf \frac{1}{n} \sum_{i=1}^{n} \frac{a_{i+1}+a_{i+2}+a_{i+4}}{a_{i+1}+a_{i+3}+a_{i+4}} \leqq \tfrac{1}{2}, \tag{3}$$

where the second infs are evaluated over all choices of $a_1, \ldots, a_n$. It follows from Theorem 1 below that equality holds in all three cases; in the inequality for the upper bound, take $p = q = 1$, $m = 2$ for (1) and $p = 2$, $m = 1 = q$ for (2) and (3).

Theorem 1.

$$\frac{m}{\left[\frac{p+q+m-1}{n}\right]+1} \leqq \sum_{i=1}^{n} \frac{A(i+p, m)}{A(i, p+m+q)} \leqq \frac{m}{m+\min(p, q)}\, n$$

where $\left[\frac{p+q+m-1}{n}\right]$ *denotes the integral part of* $\frac{p+q+m-1}{n}$.

Proof. For the lower bound let $w = \left[\frac{p+q+m-1}{n}\right]$ and $s = \sum_{i=1}^{n} a_i$ then

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$A(i, p+m+q) \leqq (w+1)s$ and we have

$$\sum_{i=1}^{n} \frac{A(i+p, m)}{A(i, p+m+q)} \geqq \frac{1}{(w+1)s} \sum_{i=1}^{n} A(i+p, m) = \frac{m}{w+1}.$$

Equality clearly holds when $p+m+q$ is a multiple of n. By considering, for large x, $a_i = x^i$ when $1 \leqq i \leqq n$ the lower bound is also seen to be best possible when $p+m+q<n$.

We now prove the upper bound result. Let $k = \min(p, q)$. If for $a_1, \ldots, a_n$ there is an i such that $A(i, m+k) = 0$ then, by omitting an appropriate number of zeros, we can obtain a subsequence $b_1, \ldots, b_r$ such that

$$\sum_{i=1}^{n} \frac{A(i+p, m)}{A(i, p+m+q)} \leqq \sum_{i=1}^{r} \frac{b_{i+p+1}+\ldots+b_{i+p+m}}{b_{i+p-k+1}+\ldots+b_{i+p+m+k}}$$

where $b_{i+r} = b_i$ and $b_{i+1}+\ldots+b_{i+m+k}>0$ for all i. Hence we need only consider those $a_1, \ldots, a_n$ for which $A(i, m+k)>0$ for all i. Furthermore, since

$$\sum_{i=1}^{n} \frac{A(i+p, m)}{A(i, p+m+q)} \leqq \sum_{i=1}^{n} \frac{A(i+k, m)}{A(i, m+2k)}$$

it is now sufficient to prove the result for the case when $p = q$.

Let $\quad p+m = em+f$ where $0 \leqq f<m$,

f_r be the integral part of $\dfrac{rf}{m}$,

$u_0 = 1$ and $u_r = \dfrac{rf}{m} - f_r$ for $r \geqq 1$,

$e_0 = 0$ and $e_r = e-1+u_1-u_r+u_{r-1}$ for $r \geqq 1$,

$v_r = \sum_{s=0}^{r} e_s.$

We then have $\quad 1-u_{r-1}+e_r+u_r = \dfrac{p+m}{m}$ for $r \geqq 1$,

e_r is a non-negative integer for $r \geqq 0$,

$1 \leqq v_r \leqq p$ for $1 \leqq r<m$ and $v_m = p+1$.

Thus

$$\frac{A(i+p, m)}{A(i, m+2p)} = \frac{m}{p+m} \sum_{r=1}^{m} a_{i+p+r} \frac{1-u_{r-1}+e_r+u_r}{A(i, 2p+m)}$$

$$\leqq \frac{m}{+m} \sum_{r=1}^{m} \left\{ \frac{(1-u_{r-1})a_{i+p+r}}{A(i+p+1-v_{r-1}, p+m)} + \sum_{j=0}^{e_r-1} \frac{a_{i+p+r}}{A(i+p-j-v_{r-1}, p+m)} + \frac{u_r a_{i+p+r}}{A(i+p-v_r, p+m)} \right\}.$$

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Using

$$\sum_{i=1}^{n} \frac{a_{i+t}}{A(i-j,\, p+m)} = \sum_{i=1}^{n} \frac{a_{i+j+t}}{A(i,\, p+m)},$$

we have

$$\sum_{i=1}^{n} \frac{A(i+p,\, m)}{A(i,\, 2p+m)} \leqq \frac{m}{p+m} \sum_{r=1}^{m} \sum_{i=1}^{n} \frac{1}{A(i,\, p+m)} \Big\{(1-u_{r-1})a_{i+r+v_{r-1}-1}$$

$$+\sum_{j=0}^{e_r-1} a_{i+r+j+v_{r-1}} + u_r a_{i+r+v_r}\Big\} = \frac{m}{p+m} \sum_{i=1}^{n} \frac{A(i,\, p+m)}{A(i,\, p+m)} = \frac{m}{p+m}\, n.$$

For the case $p = q$ we can see that the bound is attained when $n = r(p+m)$ by considering $a_{1+j(p+m)} = 1$ for $j = 0, 1, \ldots, r-1$ and $a_i = 0$ otherwise.

The next theorem is both a generalisation and a sharpening of an inequality of Diananda (3, Theorem 1).

Theorem 2.

$$\frac{2m}{2m-p-r}\, n \leqq \sum_{i=1}^{n} \frac{A(i,\, p+m+r)}{A(i+p,\, m)} \quad \textit{if } p+r \leqq m.$$

Proof. By repeated use of the arithmetic-geometric mean inequality we have, for $p+r \leqq m$,

$$\sum_{i=1}^{n} \frac{A(i,\, p+m+r)}{A(i+p,\, m)} \geqq \sum_{i=1}^{n} \frac{A(i,\, p+m+r)}{A(i+p,\, m)} \frac{2\{A(i,\, m).A(i+p+r,\, m)\}^{\frac{1}{2}}}{A(i,\, m)+A(i+p+r,\, m)}$$

$$\geqq 2n \left\{ \prod_{i=1}^{n} \frac{A(i,\, p+m+r)}{A(i,\, m)+A(i+p+r,\, m)} \right\}^{1/n}$$

$$= 2n \left\{ \prod_{i=1}^{n} \left(1 + \frac{A(i+p+r,\, m-p-r)}{A(i,\, p+m+r)}\right) \right\}^{-1/n}$$

$$\geqq 2n^2 \left\{ n + \sum_{i=1}^{n} \frac{A(i+p+r,\, m-p-r)}{A(i,\, p+m+r)} \right\}^{-1}$$

$$\geqq \frac{2n^2}{n + \dfrac{m-p-r}{m}\, n} = \frac{2m}{2m-p-r}\, n \text{ (by Theorem 1).}$$

In generalising an inequality of Zulauf (6), Daykin (4) proved the following theorem for the special cases $m = 1$ and $q = 1$.

Theorem 3.

$$m \leqq \sum_{i=1}^{n} \frac{A(i,\, m)}{A(i,\, m+q)} \leqq n-q \quad \textit{if } m+q \leqq n.$$

Proof.

$$\sum_{i=1}^{n} \frac{A(i,\, m)}{A(i,\, m+q)} \geqq \sum_{i=1}^{n} \frac{A(i,\, m)}{A(i,\, n)} = m \frac{A(i,\, n)}{A(i,\, n)} = m.$$

$$\sum_{i=1}^{n} \frac{A(i,\, m)}{A(i,\, m+q)} = \sum_{i=1}^{n} \left(1 - \frac{A(i+m,\, q)}{A(i,\, m+q)}\right) \leqq n-q.$$

For the upper bound consider, for large x, $a_i = 0$ when $1 \leqq i \leqq m-1$ and $a_i = x^{n-i}$ when $m \leqq i \leqq n$. For the lower bound consider, for large x, $a_i = x^i$ when $1 \leqq i \leqq n$.

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The University
Southampton SO9 5NH