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A REMARK ON THE GROTHENDIECK-LEFSCHETZ THEOREM ABOUT THE PICARD GROUP

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Let K be an algebraically closed field of arbitrary characteristic. The term "variety" always means here an irreducible algebraic variety over K. The notations and the terminology are borrowed in general from EGA [4].

Let X be a projective non-singular variety embedded in the projective space $P^n = P$, and let Y be a closed subvariety. Throughout this note we shall assume that $\dim(Y) \geqslant 2$ and that Y is a scheme-theoretic complete intersection of X with some hypersurfaces $H_1, \cdots, H_r$ of P, where $r = \operatorname{codim}_X(Y)$. Sometimes we shall simply say that Y is complete intersection in X.

First of all recall the following result (see [5], [7]):

THEOREM A (Grothendieck-Lefschetz). *In the above hypotheses, assume moreover that K is the complex field and that Y is non-singular of dimension $\geqslant 3$. Then the natural homomorphism of restriction of Picard groups*

$$\mathrm{Pic}(X) \to \mathrm{Pic}(Y) \tag{1}$$

is an isomorphism.

Note. There is in fact a more precise statement than the above theorem, asserting that even the corresponding morphism between Picard schemes $\underline{\mathrm{Pic}}(X) \to \underline{\mathrm{Pic}}(Y)$ is an isomorphism.

The above theorem implies in particular that the following homomorphisms of restriction are also isomorphisms (the hypotheses being the same as in theorem A):

$$\mathrm{Pic}(X)/Z[O_X(1)] \to \mathrm{Pic}(Y)/Z[O_Y(1)] \tag{2}$$

$$\mathrm{Pic}^{\tau}(X) \to \mathrm{Pic}^{\tau}(Y) \tag{3}$$

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$$\text{(4)} \qquad \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y)$$

$$\text{(5)} \qquad NS(X) \to NS(Y)$$

where $Z[O_X(1)]$ is the subgroup of $\mathrm{Pic}(X)$ generated by the class $[O_X(1)]$ of the invertible sheaf $O_X(1)$ associated with the hyperplane section of X via the embedding $X \subset P$, $O_Y(1)$ is the restriction to Y of $O_X(1)$, $\mathrm{Pic}^\tau(X)$ (resp. $\mathrm{Pic}^0(X)$) is the subgroup of $\mathrm{Pic}(X)$ consisting of all classes of invertible sheaves numerically (resp. algebraically) equivalent to zero (see [9]), and $NS(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is the Néron-Severi group of X.

This result is no longer true in general if Y is a surface (take for instance P^3 as X and P, and any non-singular quadric or cubic surface in P^3 as Y). However one knows the following result (see [3]):

THEOREM B. *Let Y be a non-singular surface which is complete intersection in the projective space P^n over an algebraically closed field of arbitrary characteristic. Then* $\mathrm{Pic}^\tau(Y) = 0$ *and the class of* $O_Y(1)$ *is not divisible in* $NS(Y)$.

The proof of theorem B given in [3] is a consequence of a careful study of the ℓ-adic cohomology of Y as well as of Hodge cohomology $H^q(Y, \Omega^p_Y)$ (Ω^p_Y denoting the sheaf of germs of algebraic differential forms of degree p on Y). In other words theorem B asserts the following: $\mathrm{Pic}(Y)$ has no torsion and $\mathrm{Pic}(Y)/Z[O_Y(1)]$ has also no torsion.

The aim of this note is to find informations about the maps (1), (2), (3), (4) and (5) when Y is a surface, complete intersection in X.

THEOREM. *In the hypotheses stated at the beginning, assume moreover that Y is a normal surface and the following condition is fulfilled:*

$$(*) \qquad H^q(X, O_X(m)) = 0 \quad \text{for every } m < 0 \text{ and for every } 1 \leqslant q < \dim(X)\,.$$

Then the maps (1), (2) *and* (5) *are injective and have cokernels all isomorphic to the same group E, which is free of finite rank if* $\mathrm{char}(K) = 0$, *and e-torsion-free of finite type if* $\mathrm{char}(K) = p > 0$, *where e is any positive integer prime to p. Moreover, the map* (4) *is always an isomorphism and the map* (3) *injective and even bijective if* $\mathrm{char}(K) = 0$.

Proof. It is based on the standard Lefschetz theory in

Grothendieck's form (see [5], [7]), and on the theory of Picard schemes (see [6], [11]). We divide it into several steps.

Step 1. (Lefschetz theory). According with the notations and the context of [7], let $\hat{X}$ be the formal completion of X along Y; by [7], chap. IV, theorem (1.5) and the proof of theorem (3.1), the map Pic (X) $\to$Pic (Y) factorizes through Pic $(X) \to$ Pic $(\hat{X}) \to$ Pic (Y) and the map

$$\text{Pic}(X) \to \text{Pic}(\hat{X}) \tag{6}$$

is an isomorphism.

On the other hand, if $Y_n = (Y, O_X/I^n)$ (I being the coherent sheaf of ideals of Y in X), by a general statement (see loc. cit.) we have

$$\text{Pic}(\hat{X}) = \text{inv}\lim_n \text{Pic}(Y_n) \tag{7}$$

Step 2. Condition (*) implies that $H^1(Y, O_Y(m)) = 0$ for every $m < 0$.

Indeed, by induction on r one can easily assume $r = 1$, i.e. $Y = X \cap H$ (scheme-theoretically), where H is a hypersurface of P of degree d of equation $f = 0$. We then get the exact sequence:

$$0 \longrightarrow O_X(m-d) \xrightarrow{f} O_X(m) \longrightarrow O_Y(m) \longrightarrow 0$$

which induces the exact sequence of cohomology

$$H^q(X, O_X(m)) \to H^q(Y, O_Y(m)) \to H^{q+1}(X, O_X(m-d)) .$$

For $1 \leqslant q < \dim(Y)$ the first and the third cohomology group vanish, so that the middle one also vanishes.

Step 3. $H^1(Y, I^n/I^{n+1}) = 0$ for every $n \geqslant 1$.

Indeed, because Y is complete intersection of X with the hypersurfaces $H_1, \cdots, H_r$ of degree $d_1, \cdots, d_r$, the conormal sheaf I/I^2 is isomorphic to $\bigoplus_{i=1}^r O_Y(-d_i)$ (see [7], page 106). Moreover, for every $n \geqslant 1$ the sheaf I^n/I^{n+1} is isomorphic to the n^{th} symmetric power $S^n(I/I^2)$, and therefore is again a direct sum of line bundles of the form $O_Y(m)$ with $m < 0$. So by step 2 we get the conclusion of step 3.

Step 4. Consider the standard exact sequence (see [5], [7]):

$$0 \to I^n/I^{n+1} \to O^*_{Y_{n+1}} \to O^*_{Y_n} \to 1$$

which induces the following exact sequence of cohomology:

$$0 \overset{\text{by step 3}}{=} H^1(Y, I^n/I^{n+1}) \to \operatorname{Pic}(Y_{n+1}) \to \operatorname{Pic}(Y_n) \to H^2(Y, I^n/I^{n+1})$$

in which $H^2(Y, I^n/I^{n+1})$ is a torsion-free group if $\operatorname{char}(K) = 0$, and an e-torsion-free group if $p > 0$, being in fact a K-vector space. Thus, one can identify $\operatorname{Pic}(Y_n)$ to a subgroup of $\operatorname{Pic}(Y)$ ($= \operatorname{Pic}(Y_1)$) in such a manner that $[O_X(1)]$ and $[O_Y(1)]$ become equal and the quotient group $\operatorname{Pic}(Y_n)/\operatorname{Pic}(Y_{n+1})$ $(n \geqslant 1)$ is:

-torsion-free if $\operatorname{char}(K) = 0$,

-e-torsion-free if $p > 0$, for any e prime to p.

Then (6) and (7) from step 1 can be restated as follows:

$$\operatorname{Pic}(X) = \bigcap_{n=1}^{\infty} \operatorname{Pic}(Y_n) . \tag{8}$$

Now it is clear that (1) and (2) are injective maps and have the same cokernel E which is:

-torsion-free if $\operatorname{char}(K) = 0$ (by (8) and the torsion-freeness of the groups $\operatorname{Pic}(Y_n)/\operatorname{Pic}(Y_{n+1})$ for every $n \geqslant 1$),

-e-torsion-free if $p > 0$, where e is any integer prime to p (in the same way).

Step 5. For simplicity, let us set $p = 1$ if $\operatorname{char}(K) = 0$, and

$$e\text{-Tors}(F) = \{x \in F / ex = 0\} \qquad \text{for every abelian group } F .$$

Since $\operatorname{Pic}(X)/\operatorname{Pic}^{\tau}(X)$ and $\operatorname{Pic}(Y)/\operatorname{Pic}^{\tau}(Y)$ have no torsion, one sees that in the commutative diagram

$$\begin{array}{ccc} e\text{-Tors}(\operatorname{Pic}^{\tau}(X)) & \longrightarrow & e\text{-Tors}(\operatorname{Pic}(X)) \\ \downarrow & & \downarrow \\ e\text{-Tors}(\operatorname{Pic}^{\tau}(Y)) & \longrightarrow & e\text{-Tors}(\operatorname{Pic}(Y)) \end{array}$$

the horizontal arrows are isomorphisms, and, by step 4, the right vertical arrow is also an isomorphism, hence

$$e\text{-Tors}(\operatorname{Pic}^{\tau}(X)) \cong e\text{-Tors}(\operatorname{Pic}^{\tau}(Y)) \tag{9}$$

By a theorem of Matsusaka (see [9]) $\operatorname{Pic}^{\tau}(X)/\operatorname{Pic}^0(X)$ and $\operatorname{Pic}^{\tau}(Y)/\operatorname{Pic}^0(Y)$ are finite groups. Considering the injective homomorphism

$$e\text{-Tors}(\operatorname{Pic}^0(X)) \to e\text{-Tors}(\operatorname{Pic}^0(Y)) . \tag{10}$$

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I claim it has finite cokernel of cardinality at most $\lambda = \text{order}(\text{Pic}^\tau(X)/\text{Pic}^0(X))$.

Indeed, consider the commutative diagram with exact lines

$$\begin{array}{ccccccccc} 0 & \longrightarrow & e\text{-Tors}(\text{Pic}^0(X)) & \longrightarrow & e\text{-Tors}(\text{Pic}^\tau(X)) & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow u & & \\ 0 & \longrightarrow & e\text{-Tors}(\text{Pic}^0(Y)) & \longrightarrow & e\text{-Tors}(\text{Pic}^\tau(Y)) & \longrightarrow & D & \longrightarrow & 0 \end{array}$$

Taking into account of (9), one deduces that the cokernel of (10) is isomorphic to Ker (u). But the exact sequence (e-Tors is a left exact functor!)

$$0 \to e\text{-Tors}(\text{Pic}^0(X)) \to e\text{-Tors}(\text{Pic}^\tau(X)) \to e\text{-Tors}(\text{Pic}^\tau(X)/\text{Pic}^0(X))$$

shows that $\text{Ker}(u) \subset C \subset \text{Pic}^\tau(X)/\text{Pic}^0(X)$, hence the claim.

Step 6. Since X is non-singular and Y normal, the group-schemes $\underline{\text{Pic}}^0(X)$ and $\underline{\text{Pic}}^0(Y)$ are proper over K (see [6], No. 236, theorem 2.1), therefore it makes sense to speak about the abelian varieties $\underline{\text{Pic}}^0(X)_{red}$ and $\underline{\text{Pic}}^0(Y)_{red}$. By [12] chap. II, § 6, one gets:

$$\text{order}(e\text{-Tors}(\text{Pic}^0(X))) = e^{2g}, \qquad g = \dim \underline{\text{Pic}}^0(X),\ e \text{ prime to } p,$$
$$\text{order}(e\text{-Tors}(\text{Pic}^0(Y))) = e^{2g'}, \qquad g' = \dim \underline{\text{Pic}}^0(Y).$$

By the injectivity of the map $\underline{\text{Pic}}^0(X) \to \underline{\text{Pic}}^0(Y)$ and by step 5 we get the inequalities

$$g \leqslant g' \quad \text{and} \quad e^{2g'} \leqslant \lambda e^{2g}.$$

Remarking that one can choose e arbitrarily large (and prime to p), these inequalities imply $g = g'$, i.e. the morphism $\underline{\text{Pic}}^0(X) \to \underline{\text{Pic}}^0(Y)$ is bijective, since it is injective and $\underline{\text{Pic}}^0(X)$ and $\underline{\text{Pic}}^0(Y)$ are abelian schemes of the same dimension. Consequently, we deduce that (4) is an isomorphism.

Step 7. The isomorphism (4), the injectivity of (1) and the e-torsion-freeness of E show that (5) is also injective and that $NS(Y)/NS(X)$ (which can be identified with E) is e-torsion-free.

Step 8 (For char $(K) = 0$). By step 7, $\text{Tors}(NS(X)) \cong \text{Tors}(NS(Y))$, where if F is an abelian group, $\text{Tors}(F)$ denotes the torsion subgroup of F. But taking into account of (4) and the equalities

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$$\text{Tors}\,(NS(X)) = \text{Pic}^{\tau}(X)/\text{Pic}^{0}(X) \quad \text{and} \quad \text{Tors}\,(NS(Y)) = \text{Pic}^{\tau}(Y)/\text{Pic}^{0}(Y)$$

one concludes that (3) is an isomorphism.

Step 9. By Néron-Severi theorem for surfaces, $NS(Y)$ is finitely generated, hence E is also finitely generated, since, by step 7, E is a quotient of $NS(Y)$. If $\text{char}\,(K) = 0$, E results therefore free of finite rank. q.e.d.

COROLLARY 1. *In the hypotheses stated at the beginning, assume that Y is a normal surface and that the ground field is the complex field* $\boldsymbol{C}$.

a) *The maps* (1), (2) *and* (5) *are injective and yield the isomorphisms* $\text{Pic}\,(Y) \cong \text{Pic}\,(X) \oplus Z^s$, $\text{Pic}\,(Y)/Z[O_Y(1)] \cong \text{Pic}\,(X)/Z[O_X(1)] \oplus Z^s$ *and* $NS(Y) \cong NS(X) \oplus Z^s$, *where s is a non-negative integer.*

b) *The class of $O_Y(1)$ is not divisible in* $\text{Pic}\,(Y)$ *(resp. in* $NS(Y)$*) if and only if the class of $O_X(1)$ is not divisible in* $\text{Pic}\,(X)$ *(resp. in* $NS(X)$*).*

c) *The morphism* $\text{Pic}\,(X) \to \text{Pic}\,(Y)$ *between Picard schemes yields the isomorphisms* $\text{Pic}^{\tau}(X) \to \text{Pic}^{\tau}(Y)$ *and* $\text{Pic}^{0}(X) \to \text{Pic}^{0}(Y)$, *and in particular* $\dim H^1(X, O_X) = \dim H^1(Y, O_Y)$.

Proof. First observe that condition (*) of the theorem is always fulfilled over the complex field, by Kodaira's vanishing theorem. a) is then a direct consequence of the theorem; b) results easily using the isomorphism

$$\text{(11)} \qquad \text{Tors}\,(\text{Pic}\,(X)/Z[O_X(1)]) \cong \text{Tors}\,(\text{Pic}\,(Y)/Z[O_Y(1)])$$

(resp. $\text{Tors}\,(NS(X)/Z\{O_X(1)\}) \cong \text{Tors}\,(NS(Y)/Z\{O_Y(1)\})$) combined with the injectivity of (1) (resp. of (5)). The proof of c) is contained in step 6 of the proof of the theorem, because an abelian scheme in characteristic zero is always reduced (this is a theorem of Cartier, see [11], lecture 25).

Remarks. 1. First of all recall the following theorem of Lefschetz about singular cohomology (see [1]):

THEOREM C (Lefschetz). *Assume that Y is a non-singular complete intersection in the projective space $P^n = P$ over the complex field. Then the natural homomorphism*

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$$H^q(P, Z) \to H^q(Y, Z)$$

induced by inclusion $Y \subset P$, *is*

(i) *bijective for* $q < d = \dim(Y)$, *and*

(ii) *injective for* $q = d$, *and the quotient group* $H^d(Y, Z)/H^d(P, Z)$ *has no torsion.*

As it is known (see [8]), theorem A can be deduced from part (i) of theorem C, if we restrict ourselves to the case where Y is a non-singular complete intersection in P^n (i.e. $X = P^n$) and $\dim(Y) \geqslant 3$. Proceeding analogously as in loc. cit., we shall see bellow that one can deduce from part (ii) of theorem C the following special case of corollary 1: if Y is a non-singular surface which is complete intersection in $P^n = P$, then the map $\mathrm{Pic}(P) \to \mathrm{Pic}(Y)$ is injective and $\mathrm{Pic}(Y)/\mathrm{Pic}(P)$ has no torsion, i.e. theorem B for the complex field.

Indeed, using the exponential sequence for P and Y, one gets the commutative diagram with exact lines:

$$\begin{array}{ccccccc}
H^1(P, O_P^h) & \longrightarrow & H^1(P, (O_P^h)^*) & \longrightarrow & H^2(P, Z) & \longrightarrow & H^2(P, O_P^h) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(Y, O_Y^h) & \longrightarrow & H^1(Y, (O_Y^h)^*) & \longrightarrow & H^2(Y, Z) & \longrightarrow & H^2(Y, O_Y^h)
\end{array}$$

where O^h denotes the sheaf of germs of holomorphic functions, and (O^h)—the sheaf of germs of nowhere—vanishing holomorphic functions. By GAGA [16] and FAC [15], $H^1(P, O_P^h) = H^2(P, O_P^h) = H^1(Y, O_Y^h) = 0$, $\mathrm{Pic}(P) = H^1(P, (O_P^h)^*)$ and $\mathrm{Pic}(Y) = H^1(Y, (O_Y^h)^*)$. Hence the map $\mathrm{Pic}(P) \to \mathrm{Pic}(Y)$ is injective since by (ii) of theorem C the map $H^2(P, Z) \to H^2(Y, Z)$ is so, and $\mathrm{Pic}(Y)/\mathrm{Pic}(P)$ may be identified to a subgroup of $H^2(Y, Z)/H^2(P, Z)$, and this last group has no torsion by (ii) again.

2. If we restrict ourselves to the complex case, we see that corollary 1 may be regarded as a more general statement than theorem B.

3. By Lefschetz's principle corollary 1 is also valid over any algebraically closed field of characteristic zero.

4. By corollary 1 and the above remark, the map (3) turns out to be an isomorphism, provided that $\mathrm{char}(K) = 0$. We do not know if (3) is always an isomorphism in the assumptions of the theorem and $\mathrm{char}(K) > 0$. However, the answer is affirmative if Y is a non-singular surface, complete intersection in P^n, as shows theorem B.

Corollary 2 (Kleiman [9]). *It is sufficient to prove Néron-Severi's theorem for projective non-singular surfaces in order to deduce it for projective non-singular varieties of arbitrary dimension (with arbitrary* char (K)).

Proof. Let X be a projective non-singular veriety; we have to prove that $NS(X)$ is finitely generated if we assume this for projective non-singular surfaces. Since Néron-Severi's theorem is trivial for curves, we may assume $\dim(X) \geqslant 3$. Choose a projective embedding $i: X \hookrightarrow P^n$; by FAC [15], § 75, theorem 3, $H^q(X, O_X(m)) = 0$ for $1 \leqslant q < \dim(X)$, provided that m sufficiently small. Therefore composing eventually i with a Veronese embedding $v_d: P^n \hookrightarrow P^N\left(N = \binom{n+d}{n} - 1\right)$ of sufficiently high degree d, we get another embedding which satisfies this time condition $(*)$ of the theorem. Hence, without loss of generality, we may assume that i satisfies $(*)$. Let then Y be the intersection of X with a general linear subspace of P^n of codimension $= \dim(X) - 2$. By Bertini's theorem, Y is a non-singular surface. Hence, by the theorem, $NS(X)$ can be identified to a subgroup of $NS(Y)$, which was assumed to be finitely generated, so that $NS(X)$ is also finitely generated.

Remark. Instead of the injectivity of (5), Kleiman used Hodge index theorem (see [11], lecture 18) to get the statement of corollary 2. Actually, Kleiman showed more, namely that Néron-Severi's theorem is true for every complete variety (which may have singularities), assuming it for projective non-singular surfaces, by using moreover Chow's lemma and the resolution of singularities for surfaces.

Corollary 3. *Let X be a projective non-singular and arithmetically Cohen-Macaulay subvariety of the projective space P^n over an algebraically closed field of arbitrary characteristic, and let Y be a normal surface which is a complete intersection of X with some hypersurfaces of P^n. Then the same conclusions as in the theorem hold, and moreover* $\mathrm{Pic}^0(X)$ *and* $\mathrm{Pic}^0(Y)$ *are both trivial.*

Proof. We have to observe that the arithmetic Cohen-Macaulay-ness of X in P^n implies that condition $(*)$ of the theorem is fulfilled on one hand, and that Y is also arithmetically Cohen-Macaulay in P^n, on the other hand. In particular we get that $H^1(X, O_X) = H^1(Y, O_Y) = 0$, hence $\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(Y)$ are trivial. Hence we see that a direct proof of corollary 3 does not need the discussion from steps 5 and 6, i.e. does not need the theory of Picard schemes.

COROLLARY 4. *In the hypotheses of corollary* 3, *assume moreover that the characteristic of the ground field is zero and that the group* $\mathrm{Pic}(X)/Z[O_X(1)]$ *has no torsion. Then every integral curve on* Y *whose sheaf of ideals in* Y *is invertible and which is set-theoretic complete intersection of* Y *with a hypersurface of* P^n, *is actually a scheme-theoretic complete intersection of* Y *with a hypersurface of* P^n.

Proof. Using corollary 3 and isomorphism (11), one gets that the group $\mathrm{Pic}(Y)/Z[O_Y(1)]$ has no torsion. Moreover, since Y is arithmetically Cohen-Macaulay in P^n, the Serre homomorphism $\alpha(Y): S(Y) \to \oplus_{s=0}^{\infty} \Gamma(Y, O_Y(s))$ (see EGA II or FAC) is an isomorphism, where by $S(Y)$ we mean the homogeneous coordinates ring of Y in P^n. Corollary 4 follows now from the following lemma of Robbiano (see [13]):

LEMMA. *Let* $V \subset P^n$ *be a closed subvariety of* P^n *such that the Serre map* $\alpha(V)$, *is an isomorphism and* $\mathrm{Pic}(V)/Z[O_V(1)]$ *has no torsion. Then every closed integral subscheme* D *of* Y *whose sheaf of ideals in* Y *is invertible and which is set-theoretic complete intersection of* V *with a hypersurface of* P^n. *is actually a scheme-theoretic complete intersection of* V *with a hypersurface of* P^n.

Proof. D can be regarded as a Cartier divisor on Y; since D is integral and set-theoretic complete intersection of V with a hypersurface of P^n, there are two positive integers α and β such that $O_V(\alpha D) \cong O_V(\beta)$ ($O_V(\alpha D)$ denoting the invertible O_V-module associated with the divisor αD). The torsion-freeness of $\mathrm{Pic}(V)/Z[O_V(1)]$ shows that there is a positive integer γ such that $O_V(D) \cong O_V(\gamma)$. Now the divisor D corresponds to a section $f \in \Gamma(V, O_V(\gamma))$, and since $\Gamma(V, O_V(\gamma)) \cong S(V)_r$, we see that D is a complete intersection of V with the hypersurface H of equation $F = 0$, where F is an homogeneous form of degree γ in $n + 1$ variables representing f in $S(V)$.

Remark. Corollary 4 extends a result of Robbiano (see [14]), which essentially is the special case of corollary 4 in which X is moreover

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arithmetically factorial in P^n (modulo the fact that in [14] the ground field is not assumed to be algebraically closed; but also in our case this restriction is not really necessary, if we look carefully at the direct proof of corollary 3). The arithmetic factoriality of X in P^n implies that $\mathrm{Pic}(X)/Z[O_X(1)] = 0$ and that every hypersurface of X is cut out by a hypersurface of P^n.

Before giving an example, let us mention an elementary interpretation of the group $\mathrm{Pic}(V)/Z[O_V(1)]$ associated with a projective embedding $V \subset P^n$ of a non-singular variety V, namely: it is canonically isomorphic to the divisor class group of the normal graded ring $\bigoplus_{s=0}^{\infty} \Gamma(V, O_V(s))$ (see [2], [10]).

EXAMPLE. Let $X = P^m \times P^{m'}$ and embed X in the projective space P^n by:

$$(a_0, \cdots, a_m; b_0, \cdots, b_{m'}) \to (\cdots, a_0^{\alpha_0} \cdots a_m^{\alpha_m} \cdot b_0^{\beta_0} \cdots b_{m'}^{\beta_{m'}}, \cdots)$$

with $\alpha_0 + \cdots + \alpha_m = s, \beta_0 + \cdots + \beta_{m'} = t, \alpha_i \geqslant 0, \beta_j \geqslant 0, s > 0, t > 0$ and $n = \binom{m+s}{m} \cdot \binom{m'+t}{m'} - 1$. This embedding is always Cohen-Macaulay, and by above interpretation, $\mathrm{Pic}(X)/Z[O_X(1)] \cong Z \oplus Z/dZ$, where d is the greatest common divisor of s and t (since $O_X(1) \cong p_1^*O(s) \otimes p_2^*O(t)$, where p_1 and p_2 are the canonical projections of our product). Therefore, if $m + m' \geqslant 3$ and s and t are relatively prime each other, this embedding satisfies the hypotheses of corollary 4.

We do not know in general if the restriction about the characteristic of K to be zero is really necessary in corollary 4. However, theorem B and Robbiano's lemma above allow us to deduce the following result, which extends to arbitrary characteristic the result of Robbiano from [13]:

COROLLARY 5. *Let Y be a projective non-singular surface, complete intersection in the projective space P^n over an algebraically closed field K of arbitrary characteristic. Then every integral curve D on Y which is set-theoretic complete intersection of Y with a hypersurface of P^n, is actually a complete intersection of Y with a hypersurface of P^n, and hence D is a complete intersection in P^n.*

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