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When projective covers and injective hulls are isomorphic

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It is shown that rings in which the projective cover and injective hull of cyclic modules are isomorphic are equivalent to uniserial rings. Further, it is shown that rings for which the top and bottom of finitely generated modules are isomorphic also are equivalent to uniserial rings.

In this paper we examine the condition that the injective hull and projective cover of a module are isomorphic. Since over a quasi-Frobenius ring R, this holds for injective and projective modules, as well as $R/\text{Rad}R$, one might suppose that this holds for any R-module. However, this is not the case. Requiring this condition to be true of just finitely generated R-modules is a necessary and sufficient condition that R be uniserial.

We also consider here when the top and bottom of each R-module are isomorphic. Again this condition holds for certain modules over a quasi-Frobenius ring but is true for all R-modules only when R is uniserial. In fact this condition is equivalent to uniserial.

1. Preliminaries

Throughout this paper each ring R will be a ring with an identity element 1, and each right R-module M will be unitary in the sense that

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$x1 = x$ for all $x \in M$. Mod-R will denote the category of all right R-modules.

A ring R is *completely primary* in case it is a local ring which is right (or left) artinian. A *primary* ring is a ring which is isomorphic to a full ring of $n \times n$ matrices over a completely primary ring.

A *uniserial ring* is a ring R in which each principal indecomposable right, and left ideal has a unique composition series as an R-module and where R is a direct product of finitely many primary rings. Faith [6] has characterized these rings as right artinian rings in which the injective hull of cyclic right modules are cyclic.

A *right Köthe ring* is a ring in which every right R-module is a direct sum of cyclics. Köthe [9] has shown that every uniserial ring is a Köthe ring.

A *quasi-Frobenius ring* (QF) is a right, and left, artinian ring in which every left (respectively right) ideal is an annihilator ideal. Ikeda [8] characterized these as artinian rings which are right (or left) self injective. If R is a ring in which every cyclic right, and cyclic left, R-module is contained in a projective R-module, then Faith [5] has shown that this ring is QF.

Let J be the Jacobson radical of the ring R . Then if $M \in$ Mod-R , the *top of* M is the right R-module M/MJ . The *bottom of* M is the sum of all its simple right submodules, denoted Soc$M$.

Given a right module M , the injective hull of M will be denoted $E(M)$ and is defined as by Eckmann und Schopf [4].

2. Uniserial rings

THEOREM 2.1. *The following conditions on a ring* R *are equivalent:*

(1) R *is a uniserial ring;*

(2) *the injective hull and projective cover of each right and left finitely generated* R*-module are isomorphic;*

(3) *the top and bottom of every right and left finitely generated* R*-module are isomorphic;*

(4) *the top and bottom of every right and left* R*-module are*

isomorphic.

Proof. (1) implies (2). Since R is uniserial, it is a product of a finite number of primary rings, and hence it suffices to consider the case when R is primary. However, by the equivalence of categories Mod-R and Mod-R_n , it suffices to consider a local ring, a primary ring being a full ring of $n \times n$ matrices over a local ring. Further, since R is a Köthe ring [9], we only need to consider cyclic R-modules. Let C be a cyclic R-module, say $C \approx R/I$ for some right ideal I . R local implies that $I \subset \text{Rad}R$ and hence that R with the canonical map $R \to R/I$ is the projective cover of C .

Since R is a right, and left, principal ideal ring [1, 2] $\text{ann}_R(I) = Rx$ for some $x \in R$ and thus there exists an injection

$$0 \to R/I \to R ,$$

$$r + I \to xr , \quad r \in R .$$

Since R is self injective, R contains $E(R/I)$. So $E(C)$ is isomorphic to a summand of R and by indecomposability of R , $E(C) = R$.

(2) implies (1). By [6] it suffices to show that R is right artinian and that the injective hull of each cyclic right R-module is cyclic. Let C be any cyclic R-module. Let P be the projective cover of C . Since C is cyclic, P is cyclic. Then $E(C) \approx P$ implies that $E(C)$ is cyclic. Further, since every cyclic right and every cyclic left R-module is contained in a projective R-module (namely its projective cover), R is quasi-Frobenius by [5] and hence is right artinian.

(2) implies (3). Let $\{e_1R, \ldots, e_nR\}$ be the set of principal indecomposable right ideals. We first show that the top and bottom of each of these are isomorphic and then proceed to the case of any finitely generated module. Let $J = \text{Rad}R$. Since $e_iJ \subset J$, e_iR is the projective cover of e_iR/e_iJ . By hypothesis the projective cover and injective hull of e_iR/e_iJ are isomorphic. So there exists an injection $0 \to e_iR/e_iJ \to e_iR$. Since e_iR contains a unique simple $e_iR/e_iJ \approx \text{Soc}\,e_iR$.

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Now let A be any finitely generated R-module. Then since R/J is semisimple, A/AJ is semisimple and thus

$$A/AJ \approx \sum \oplus e_i R/e_i J$$

and the projective cover of A is

$$P = \sum \oplus e_i R \ .$$

Thus

$$\mathrm{Soc}P \approx \sum \oplus e_i R/e_i J \approx P/PJ \ .$$

Since $P \approx E(A)$,

$$\mathrm{Soc}A \approx \mathrm{Soc}P \approx \sum \oplus e_i R/e_i J \ .$$

Hence, $A/AJ \approx \mathrm{Soc}A$.

(3) implies (2). We first want to show that R is QF and hence right and left self-injective. For any finitely generated right R-module A , $A/AJ \neq 0$ and thus, $\mathrm{Soc}A \neq 0$. So every right R-module has a nonzero socle. Let $\{e_i \mid i \in I\}$ be a set of orthogonal idempotents in R . Since $\sum_{i \in I} e_i R/e_i J \subset \mathrm{Soc}R$ and $\mathrm{Soc}R$ is cyclic, this set must be finite. Thus by [3], R is right perfect. Similarly it is left perfect. The top of R/J^2 , and hence $\mathrm{Soc}R/J^2$, is cyclic. Thus $J/J^2 \approx \mathrm{Soc}R/J^2$ is finitely generated. By [10] perfect plus J/J^2 finitely generated implies that R is artinian. Further, since for any primitive idempotent $e \in R$, $eR/eJ \approx \mathrm{Soc}eR$ and $Re/Je \approx \mathrm{Soc}Re$ by [7], R is self-injective.

Let A be any finitely generated R-module. Then the projective cover of A is $P = \sum \oplus e_i R$ for some finite index set I where $A/AJ \approx \sum \oplus e_i R/e_i J \subset \mathrm{Soc}A$. Since R is self-injective, P is injective. Further, $A/AJ \approx \mathrm{Soc}A$ and $e_i R/e_i J \approx \mathrm{Soc}e_i R \subset e_i R$ implies that $P \approx E(\mathrm{Soc}A)$. Hence $P \approx E(A)$.

(3) implies (4). Since R is uniserial, every right, and left,

R-module is a direct sum of cyclics. Let $M \in \text{Mod-}R$. Then $M = \sum m_i R$, $m_i \in M$.

$$\operatorname{Soc} M = \sum \operatorname{Soc} m_i R \cong \sum m_i R / m_i J$$

$$\cong \sum m_i R \,/\, \sum m_i J = M/MJ .$$

(4) implies (3). Clear.

Note that for the proof of (1), if and only if (2), it suffices to have the injective hull and projective cover of cyclic right, and left, R-modules isomorphic.

We can derive similar characterizations using indecomposable R-modules.

COROLLARY 2.2. *R is uniserial if and only if R is noetherian and the injective hull and projective cover of each right, and left, indecomposable R-module are isomorphic.*

Proof. Let D be any indecomposable right R-module. Since R is uniserial, every module is a direct sum of cyclics. Hence D is cyclic, and by Theorem 2.1 it has isomorphic projective cover and injective hull.

Conversely, it suffices to show that the injective hull and projective cover of every cyclic R-module are isomorphic. Since R is noetherian, every cyclic can be written as a direct sum of finitely many indecomposables. The conclusion then follows from the theorem.

COROLLARY 2.3. *R is uniserial if and only if R is noetherian and the top and bottom of every indecomposable right, and left, R-module are isomorphic.*

Proof. Let R be uniserial. The result follows from the theorem.

Conversely, since R is noetherian, every finitely generated right, and left, R-module is a direct sum of finitely many indecomposables.

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