A NOTE ON TRAIN ALGEBRAS

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1. Introduction

Train algebras were first introduced by Etherington in (**1**) and proved very useful in dealing with problems in mathematical genetics. The types of algebras which arose were commutative, non-associative and finite-dimensional. It proved convenient in the general theory to regard them as defined over the complex numbers. We remind the reader of some basic definitions. A *baric algebra* is one which admits a non-trivial homomorphism into its coefficient field K. A (principal) *train algebra* is baric and has a rank equation in which the coefficients of a general element x depend only on its baric value, generally called the weight of x. A *special train algebra* (STA) is a baric algebra in which the nilideal is nilpotent and all its right powers are ideals; the nilideal being the set of elements of A of weight zero. In (**2**) Etherington showed that in a baric algebra one can always take a very simple basis consisting of a distinguished element of unit weight and all other basis elements of weight zero. Finally we have the concept of a *genetic algebra* as defined by Schafer (**4**). A commutative baric algebra A is *genetic* if for any

$$T = \alpha I + f(R_{x_1}, R_{x_2}, \ldots, R_{x_n}) \quad \alpha \in K,\ x_i \in A,$$

where I is the identity, then the characteristic function of T in so far as it depends on $x_1, \ldots, x_n$ depends only on their weights. R_x represents a right linear transformation of A, $a \to ax$, $a \in A$.

Two unsolved problems in the theory of genetic algebras, which we settle affirmatively, are the following:

1. Are commutative train algebras of rank 3 necessarily special train algebras?
2. Are there commutative train algebras over fields of characteristic zero which are not genetic algebras?

For question 2 we give an example of a train algebra of rank 4 which is not a genetic algebra.

The historical origin and background to both these problems is interesting. In 1939 in (**2**) Etherington investigated very fully the properties of commutative train algebras of ranks 2 and 3. He showed that train algebras of rank 2 were STA; at the end of the paper he stated a canonical form for the multiplication table of train algebras of rank 3; this in fact was incorrect as he pointed out in his Corrigendum (2) in 1945. Had it been correct it would have disposed of question 1 and vindicated a statement in (1) that train algebras of rank 3 were special if $\lambda \neq \frac{1}{2}$, where λ is a principal train root. In his approach he assumed that $\lambda \neq \frac{1}{2}$—this is enough to ensure the existence of an idempotent element in the algebra. In fact, we do not require this assumption in our proof.

In 1949 Schafer in his paper gave an example of a train algebra over a field of characteristic 2 which was not a genetic algebra. He stated that this was not the most satisfying example. Ideally what was needed was a construction based on a commutative nilalgebra which was not nilpotent. The definition of a nilalgebra needs some clarification since the powers of an element z in A are in general not well-defined. For our purposes if $z^k = z^{k-1}z = 0$ for all $z \in A$, for some integer k independent of z, where k is minimal in this respect, then we say that A is a *nilalgebra of nilindex k*. However, the more general definition of a nilalgebra given in (3) is that every product of k factors each equal to z, in whatever association, vanishes. Clearly the former implies the latter as noted in (4) but then the nilindex is different.

For $k = 3$, however, the definitions coincide in the commutative case. A train algebra can then be obtained by adjoining an identity, and it will not be a genetic algebra since nilpotence of the nilideal is a necessary condition.

This raised a fundamental question of whether such nilalgebras existed. This question was taken up by Gerstenhaber in 1959 who made an exhaustive study of such algebras in a series of three papers. In (3) he proved that if $\dim A \leqq 3$ for a nilalgebra A then it was nilpotent, A being of characteristic zero. This led him to his conjecture that a finite-dimensional commutative nilalgebra of characteristic zero is nilpotent. This was only recently disproved by a counter-example by Suttles (6) in 1972. Suttles gave an example of a nilalgebra of nilindex 4 which was not nilpotent. We make use of this result for our counter-example. In this connection we also extend these results by showing that nilindex 3 automatically implies nilpotence, and hence that 4 is the minimum nilindex of a nilalgebra which fails to be nilpotent.

2. Train algebras of rank 3

Theorem 1. *A commutative finite-dimensional non-associative nilalgebra A of characteristic zero and nilindex* 3 *is nilpotent.*

The analogous result is clearly true for nilindex 2 but fails for nilindex 4 by Suttles' counter-example.

Proof. We linearise the identity.

$$z^3 = 0, \quad z \in A. \tag{1}$$

Put $z = \theta_1 z_1 + \theta_2 z_2 + \theta_3 z_3$ in (1), expand and collect homogeneous terms. We get

$$(z_1 z_2) z_3 + (z_2 z_3) z_1 + (z_3 z_1) z_2 = 0, \tag{2}$$

a Jacobi identity, and

$$2(z_1z_2)z_1+z_1^2z_2 = 0. \tag{3}$$

Now compare (3) to the defining relations of a commutative alternative algebra

$$(xy)x = x^2y.$$

Apart from a change of sign and a constant these define exactly the same multiplicative properties. We now can make use of a well-known theorem, Schafer (5), that an alternative nilalgebra of finite dimension is nilpotent. A similar proof holds for our algebra—in fact, it is much simplified since we have commutativity. In the proof we need to show that

$$R_z^j = R_{z^j} = 0 \quad \text{for } j = 3.$$

This result can be proved as in Schafer ((5), p. 30), but we give an alternative proof due to Gerstenhaber. If A is a commutative nilalgebra of bounded index t over a field of characteristic zero, then $R_a^{2t-3} = 0$ for all $a \in A$. Here we have $t = 3$, giving $R_a^3 = 0$.

Hence we can establish that A is nilpotent.

We note that the Jacobi identity (2) can be non-trivially satisfied. In other words, there exist nilalgebras such that $z^3 = 0$ and $A^3 \neq 0$.

Example. Take $A = (c_1, c_2, \ldots, c_7)$ with

$$c_1c_2 = \tfrac{1}{2}c_6, \quad c_1c_3 = \tfrac{1}{2}c_5, \quad c_1c_4 = \tfrac{1}{2}\mu c_7,$$

$$c_2c_3 = \tfrac{1}{2}c_4, \quad c_2c_5 = \tfrac{1}{2}\nu c_7, \quad c_3c_6 = \tfrac{1}{2}\rho c_7,$$

all other products being zero. Take $\mu+\nu+\rho = 0$, then

$$A^2 = (c_4, c_5, c_6, c_7),$$

$$A^3 = (c_7), \quad A^4 = 0 \quad \text{and} \quad z^3 = 0 \quad z \in A.$$

Theorem 2. *A commutative train algebra of rank* 3 *over a field of characteristic zero is necessarily a special train algebra.*

Proof. Suppose that the train equation is

$$x^3-(1+\lambda)\beta(x)x^2+\lambda\{\beta(x)\}^2x = 0,$$

where $\beta(x)$ is the weight of x.

Then $X = x/\beta(x)$ is normalised, i.e. has weight 1, ($\beta(x) \neq 0$) and the train equation is

$$X^3-(1+\lambda)X^2+\lambda X = 0$$

or

$$X(X-1)(X-\lambda) = 0.$$

$(1, \lambda)$ are called the *principal train roots.*

We can therefore for simplicity take the train equation to be

$$x(x-1)(x-\lambda) = 0,$$

where x is a general normalised element in A, and we do not assume that an idempotent necessarily exists.

Let us take a canonical basis for the baric algebra such that c_0 has weight 1 and all other basis elements are of weight zero. These then constitute the nil-ideal Z.

Then $A = \{c_0\} \cup Z$.

Suppose $c_0^2 = c_0 + z_0$, then

$$c_0^3 = c_0 + z_0 + c_0 z_0 = c_0 + z_0 + z_0'.$$

Lower case elements z belong to Z above and in the subsequent proof. We linearise the train equation. Consider $x = c_0 + \theta z$ then

$$x^2 = (c_0 + \theta z)^2 = c_0 + z_0 + 2c_0 \theta z + \theta^2 z^2$$

$$x^3 = c_0 + z_0 + \theta c_0 z + z_0 c_0 + \theta z_0 z + 2\theta (c_0 z) c_0 + 2\theta^2 (c_0 z) z + \theta^2 z^2 c_0 + \theta^3 z^3.$$

Now $x^3 - (1+\lambda)x^2 + \lambda x = 0$, and equating coefficients of homogeneous terms in θ we obtain

$$z^3 = 0. \tag{4}$$

$$2(c_0 z) z + z^2 c_0 - (1+\lambda) z^2 = 0. \tag{5}$$

$$z_0 z + 2(c_0 z) c_0 - (2\lambda + 1) c_0 z + \lambda z = 0. \tag{6}$$

$$z_0 c_0 - \lambda z_0 = 0. \tag{7}$$

Now we linearise (5).

Put $z = \theta_1 z_1 + \theta_2 z_2$ and equate the coefficients of homogeneous terms in $\theta_1 \theta_2$ to zero, then

$$2(c_0 z_1) z_2 + 2(c_0 z_2) z_1 + 2(z_1 z_2) c_0 - 2(1+\lambda) z_1 z_2 = 0 \tag{8}$$

and since $c_0 z_i \in Z$ we have $c_0(z_1 z_2) \in Z^2$ all z_1, z_2, thus proving that Z^2 is an ideal. It follows, by induction, that $Z^n = Z^{n-1} Z$ is an ideal in A for all n.

Take $z_1 \in Z^{n-1}$, $z_2 \in Z$ and suppose Z^{n-1} is an ideal in A. It is sufficient to show that $c_0 Z^n \subset Z^n$. Now $(c_0 z_1) z_2 \in Z^n$, $(c_0 z_2) z_1 \in Z^n$, $z_1 z_2 \in Z^n$ and so $c_0(z_1 z_2) \in Z^n$ from (8). Thus $c_0 Z^n \subset Z^n$, and therefore Z^n is an ideal in A.

By Theorem 1, since $z^3 = 0$, we have that Z is nilpotent. Hence A is a special train algebra.

We note that there exist train algebras of rank 3 without non-trivial idempotents. For example if $c_0^2 = c_0 + \alpha c_1$, $c_0 c_1 = \frac{1}{2} c_1$, $c_1^2 = 0$ and $\alpha \neq 0$, there is no non-trivial idempotent in the algebra $A = (c_0, c_1)$ and the train equation for a normalised element x is $x(x-1)(x-\frac{1}{2}) = 0$.

3. A counterexample

Suttles (6) gave the following counter-example to the conjecture of Gerstenhaber mentioned in Section 1.

Consider the commutative algebra $Z = (c_1, c_2, \ldots, c_5)$ such that

$$c_1c_2 = c_3, \quad c_1c_3 = c_4, \quad c_1c_5 = -c_3, \quad c_2c_3 = c_5, \quad c_2c_4 = c_3,$$

all other products being zero.

Then Z is a commutative (power-associative) nilalgebra of nilindex 4 which is not nilpotent but is solvable, since

$$z^4 = 0$$

and

$$Z^2 = (c_3, c_4, c_5)$$

$$Z^3 = Z^2$$

but

$$Z^2 . Z^2 = 0.$$

Adjoin an identity to Z and consider the algebra $A = \{1\} \cup Z$. Then if $x = \beta(x).1 + \Sigma x_i c_i$ the train equation of A is

$$x(x-\beta(x))^4 = 0.$$

For normalised x this gives $x(x-1)^4 = 0$. This equation is clearly minimal with respect to degree for linear dependence of the principal powers of a general, normalised element. Hence A is a train algebra with principal train root 1 (multiplicity 4), and rank 5.

However, A is not a genetic algebra since Z is not nilpotent. Now, we can decrease the rank of A by 1 if, instead of an identity, we adjoin an element c_0 to Z such that

$$c_0^2 = c_0, \quad c_0 c_i = \tfrac{1}{2} c_i \quad i = 1, \ldots, 5.$$

If $x = c_0 + x_1c_1 + x_2c_2 + \ldots + x_5c_5$,

$$\begin{aligned} x^2 - x &= (2x_1x_2 - 2x_1x_5 + 2x_2x_4)c_3 + 2x_1x_3c_4 + 2x_2x_3c_5 \\ &= \lambda_3c_3 + \lambda_4c_4 + \lambda_5c_5. \end{aligned}$$

Then

$$(x^2-x)x = (\tfrac{1}{2}\lambda_3 + \lambda_4x_2 - \lambda_5x_1)c_3 + (\lambda_3x_1 + \tfrac{1}{2}\lambda_4)c_4 + (\lambda_3x_2 + \tfrac{1}{2}\lambda_5)c_5,$$

hence

$$\begin{aligned} (x^2-x)(x-\tfrac{1}{2}) &= (\lambda_4x_2 - \lambda_5x_1)c_3 + \lambda_3x_1c_4 + \lambda_3x_2c_5 \\ &= \lambda_3x_1c_4 + \lambda_3x_2c_5, \end{aligned}$$

since $\lambda_4x_2 - \lambda_5x_1 = 0$,

$$\begin{aligned} (x^2-x)(x-\tfrac{1}{2})x &= \tfrac{1}{2}(\lambda_3x_1c_4 + \lambda_3x_2c_5) + \lambda_3x_1x_2c_3 - \lambda_3x_2x_1c_3 \\ &= \tfrac{1}{2}(\lambda_3x_1c_4 + \lambda_3x_2c_5). \end{aligned}$$

Hence $x(x-1)(x-\tfrac{1}{2})^2 = 0$. This equation is clearly minimal with respect to degree for linear dependence of the principal powers of a general, normalised element. Therefore the algebra $A = \{c_0\} \cup Z$ is a train algebra of rank 4 which is not a genetic algebra.

Thus we can conclude that for finite-dimensional commutative not necessarily associative algebras over a field of characteristic zero:

1. Nilalgebras of nilindex $\leqq 3$ are nilpotent.
2. Train algebras of rank $\leqq 3$ are special train algebras.
3. Train algebras of rank >3 are not necessarily genetic algebras.

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