SOME REMARKS ON NOETHERIAN RINGS

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In his lecture at the University of Kyoto on September 23, 1955, Professor Artin gave an important theorem on Noetherian rings, which seems to have not a few interesting consequences. It is the purpose of our present note to point out one of them. We begin by quoting a special case of the theorem.

THEOREM. *Let R be a Noetherian ring with unit element, and $\mathfrak{a}$, $\mathfrak{b}$ ideals of R. Then there exists a positive integer d such that*

$$\mathfrak{a}^n \cap \mathfrak{b} = \mathfrak{a}^{n-r}(\mathfrak{a}^r \cap \mathfrak{b}) \qquad n \geqslant r \geqslant d.$$

Proof. Let $\{a_1, \ldots, a_m\}$ be a system of generators of $\mathfrak{a}$, and consider the polynomial ring $R[x] = R[x_1, \ldots, x_m]$. Denote by A_r the set of forms of degree r in $R[x]$, and by B_r the set of all the forms $f(x)$ of degree r such that $f(a_1, \ldots, a_m) \in \mathfrak{b}$. A_r is a R-module, B_r is a submodule of A_r, and obviously $A_{n-r} \cdot B_r \subseteq B_n$ for $n \geqslant r$. We select a finite system of forms $f_i(x)$, $1 \leqslant i \leqslant l$, from $\{B_r;\ r = 0, 1, 2, \ldots\}$ such that any form $f(x)$ of $\{B_r;\ r = 0, 1, 2, \ldots\}$ may be represented as

$$f = \sum_{i=1} \phi_i \cdot f_i,$$

where ϕ_i's are forms of $R[x]$. Denote by d the maximum of the degrees of $f_i(x)$, $1 \leqslant i \leqslant l$, then for $n \geqslant r \geqslant d$, $A_{n-r} \cdot B_r = B_n$, namely $\mathfrak{a}^n \cap \mathfrak{b} = \mathfrak{a}^{n-r}(\mathfrak{a}^r \cap \mathfrak{b})$.

By taking a principal ideal for $\mathfrak{b}$, we obtain the following:

COROLLARY. *Let $\mathfrak{a}$ be an ideal of R, and a a nonzero-divisor of R, then there exists a positive integer d such that*

$$\mathfrak{a}^n : Ra = \mathfrak{a}^{n-r}(\mathfrak{a}^r : Ra) \qquad n \geqslant r \geqslant d,$$

consequently

$$\mathfrak{a}^n : Ra \subseteq \mathfrak{a}^{n-r}.$$

Though Professor Artin did not mention this corollary, the last formula $\mathfrak{a}^n : Ra \subseteq \mathfrak{a}^{n-r}$ is of some interest. This is really a satisfactory generalization of a well-known theorem (**1**, p. 699, Lemma 9; **5**, p. 38, Lemma 1). We would refer readers to a remark by Samuel on this kind of formula (**2**, p. 34). This formula enables us to sharpen one of his results (**2**, p. 23) as follows.

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THEOREM 1. *Let* $\mathfrak{a}$ *be an ideal of Noetherian ring* R. *If* $\mathfrak{a}$ *contains at least one nonzero-divisor, then there exists an element* a *of* $\mathfrak{a}$ *such that*

$$\mathfrak{a}^{n+r}: Ra = \mathfrak{a}^n$$

for sufficiently large n, *where* r *is determined by* $a \in \mathfrak{a}^r$ *and* $a \notin \mathfrak{a}^{r+1}$.

Proof. Put

$$\mathfrak{n} = \bigcap_{n=1}^{\infty} \mathfrak{a}^n, \quad {}^*R = R/\mathfrak{n}, \quad {}^*\mathfrak{a} = \mathfrak{a}/\mathfrak{n}.$$

It is easily seen e.g. by the intersection theorem (**4**, p. 180, Theorem 3) that ${}^*\mathfrak{a}$ contains at least one nonzero-divisor and that any prime ideal of the zero ideal of ${}^*R$ is closed and not open in ${}^*\mathfrak{a}$-adic topology. So Samuel's observations on the ring of forms $F({}^*\mathfrak{a}) = \sum {}^*\mathfrak{a}^i/{}^*\mathfrak{a}^{i+1}$ (**2**, p. 22–23) ensure the existence of a superficial element ${}^*a$ of some degree r with respect to ${}^*\mathfrak{a}$, which is not a zero-divisor. Hence ${}^*\mathfrak{a}^{n+r}: {}^*R{}^*a = {}^*\mathfrak{a}^n$ for sufficiently large n. Any element in the residue class ${}^*a$ will have the property required in the theorem.

COROLLARY. *Under the same assumption on* $\mathfrak{a}$, *there exist positive integers* r, n_0 *such that*

$$\mathfrak{a}^{nr+mr}: \mathfrak{a}^{mr} = \mathfrak{a}^{nr}, \qquad n \geqslant n_0.$$

We do not know whether we can always take 1 for r in this corollary, but Samuel (**3**, p. 177, Theorem 10) tells us the following:

THEOREM. *Let* A *be a local ring with the maximal ideal* $\mathfrak{m}$, *and let* $\mathfrak{q}$ *be an* $\mathfrak{m}$-*primary ideal. Suppose* $\mathfrak{m}$ *contains at least one nonzero-divisor, then*

$$\mathfrak{q}^n: \mathfrak{q} = \mathfrak{q}^{n-1} \qquad \textit{for sufficiently large } n.$$

Proof. In the case that the residue field $k = A/\mathfrak{m}$ is infinite, his assertion is substantiated by the existence of a superficial element of degree 1 with respect to $\mathfrak{q}$, which is not a zero-divisor (**2**, p. 23) .The other case that k is finite shall be reduced to the former case by the following device. Form the polynomial ring $A[x]$ in an indeterminate X, then form the ring of quotients A^* of $\mathfrak{m}A[x]$ with respect to $A[x]$. The residue field of A^* is $k(x)$, hence

$$\mathfrak{q}^nA^*: \mathfrak{q}A^* = \mathfrak{q}^{n-1}A^*.$$

Notice that

$$(\mathfrak{q}^nA^*: \mathfrak{q}A^*) \cap A = \mathfrak{q}^n: \mathfrak{q}, \quad \mathfrak{q}^{n-1}A^* \cap A = \mathfrak{q}^{n-1}.$$

Before we transform the above theorems by "globalization," we shall recall some definitions and well-known facts. Let $\mathfrak{z}$ be a prime ideal of R, and $\mathfrak{q}$ a $\mathfrak{z}$-primary ideal. The $\mathfrak{z}$-primary component of $\mathfrak{q}^n$ is called nth symbolic power of $\mathfrak{q}$, and usually denoted by $\mathfrak{q}^{(n)}$. Let $\mathfrak{a}$ be an ideal of R, and $\mathfrak{z}_1, \ldots, \mathfrak{z}_l$ be the minimal prime ideals of $\mathfrak{a}$. The intersection of the $\mathfrak{z}_i$-primary compo-

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nents ($1 \leqslant i \leqslant l$) of $\mathfrak{a}^n$ is called nth symbolic power of $\mathfrak{a}$, and denoted by $\mathfrak{a}^{(n)}$. If $\mathfrak{q}_i$ denotes the $\mathfrak{z}_i$-primary component of $\mathfrak{a}$, then as is well known

$$\mathfrak{a}^{(n)} = \mathfrak{q}_1^{(n)} \cap \ldots \cap \mathfrak{q}_l^{(n)}.$$

We denote by S the complement of

$$\bigcup_{i=1}^{l} \mathfrak{z}_i$$

in R, and form the ring of quotients R_s of S with respect to R in the Chevalley-Uzkov sense. We have then $\mathfrak{a}^{(n)} = \mathfrak{a}^n R_s \cap R$. Let

$$(0) = \mathfrak{q}_1^* \cap \ldots \cap \mathfrak{q}_t^*$$

be a primary decomposition of the zero ideal of R, and let $\mathfrak{z}_i^*$ be the prime ideal of $\mathfrak{q}_i^*$. Assume $\mathfrak{z}_i^* \cap S = \phi$ for $i = 1, \ldots, s$ and $\mathfrak{z}_i^* \cap S \neq \phi$ for $i = s + 1, \ldots, t$. Then $\mathfrak{n} = \mathfrak{q}_1^* \cap \ldots \cap \mathfrak{q}_s^*$ is the kernel of the canonical homomorphism of R into R_S. Contracting of ideals of R_S on R and extending of ideals of R to R_S both give one-to-one mappings between the set of all ideals of R_S and the set of ideals of R whose prime ideals are disjoint with S. These mappings are the inverse of each other and they are isomorphisms with respect to the ideal operations ($\cap$) and (:). We are now in a position to verify the following:

THEOREM 2. *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R. Suppose that any minimal prime ideal of $\mathfrak{a}$ is not a prime ideal of* (0). *Then there exist an element a of $\mathfrak{a}$ and a positive integer n_0 such that*

$$\mathfrak{a}^{(n+r)} : Ra = \mathfrak{a}^{(n)}, \qquad n \geqslant n_0$$

where r satisfies $a \in \mathfrak{a}^{(r)}$ and $a \notin \mathfrak{a}^{(r+1)}$. Moreover

$$\mathfrak{a}^{(n+m)} : \mathfrak{a}^{(m)} = \mathfrak{a}^{(n)}$$

for sufficiently large n and arbitrary $m \geqslant 0$.

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