ON CONJUGATE FUNCTIONS

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1. Introduction. In a previous paper **(3)** generalizations of M. Riesz's theorem by the method of asymptotic approximation have been given. The present paper is concerned with further generalizations for even and odd functions. In Section 3, we consider a generalization of a theorem due to Zygmund: If $|f| \log^+ |f| \in L(-\pi, \pi)$, then $\bar{f} \in L(-\pi, \pi)$ **(10**, p. 254**)**. The results in this paper include generalizations of results by K. K. Chen **(2)** and T. M. Flett **(4**, Theorems 1, 2**)** as important special cases.

K. I. Babenko **(1)** proved that if $|f(x)|^p |x|^\alpha \in L(-\pi, \pi)$, for $p > 1$, $-1 < \alpha < p - 1$, then

$$\int_{-\pi}^{\pi} |\bar{f}(x)|^p |x|^\alpha dx \leqslant A \int_{-\pi}^{\pi} |f(x)|^p |x|^\alpha dx, \tag{1.1}$$

where $\bar{f}$ is the conjugate function of $f(x)$, and A depends on p and α only. This result has been generalized in **(3)**. Concerning even and odd functions, stronger results have been obtained by K. K. Chen **(2)** and T. M. Flett **(4)**. We shall consider some more general results (Theorems 1 and 2). By complex methods, V. F. Gapoškin **(5)** has generalized (1.1) in the following form:

$$\int_{-\pi}^{\pi} |\bar{f}(x)|^p \phi(x)\, dx \leqslant A_{p,c} \int_{-\pi}^{\pi} |f(x)|^p \phi(x)\, dx, \tag{1.2}$$

where $\phi(x) \in L(-\pi, \pi)$, $\phi(x) > 0$, and

$$\phi(r, x) \geqslant c\, |\psi(r, x)|, \tag{1.3}$$

$$\phi(z) = \phi(r, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t) \frac{1 - r^2}{1 + r^2 - 2r\cos(t - x)} dt, \tag{1.4}$$

$0 < r \leqslant 1$, $z = re^{it}$, and $\psi(z) = \psi(r, x)$ is the conjugate function of $\phi(r,x)$,

$$c > 0 \quad (p \geqslant 2), \qquad c > |\tan \tfrac{1}{2} p\pi| \quad (1 < p \leqslant 2).$$

The following classes of asymptotic approximation functions have been defined in **(3)**, but for the sake of completeness, we now reproduce the simplified definitions and notation here.

(i) By $\phi(x) \sim [a, b]$, $0 \leqslant a \leqslant b$, we denote any non-negative even function $\phi(x)$, such that $\phi(x)/x^a$ is non-decreasing and $\phi(x)/x^b$ is non-increasing as x

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is increasing in $(0, \infty)$. The class of functions $\phi(x) \sim [a, b]$, where $a \leqslant b \leqslant 0$, may be defined in a similar way.

(ii) By $\phi(x) \sim \langle a, b\rangle$, where $0 \leqslant a < b$ or $a < b \leqslant 0$, we mean there exists a small positive constant ϵ, such that $\phi(x) \sim [a + \epsilon, b - \epsilon]$. Similarly $\phi(x) \sim \langle a, b]$ and $\phi(x) \sim [a, b\rangle$ mean $\phi(x) \sim [a + \epsilon, b]$ and $\phi(x) \sim [a, b-\epsilon]$, respectively, for some $\epsilon > 0$.

(iii) By $\phi(x) \in M(a, b)$, $1 \leqslant a < b < \infty$, we denote any non-negative continuous non-decreasing function $\phi(x)$, $0 < x < \infty$, satisfying $\phi(0) = 0$ and the following conditions:

$$\phi(2u) = O\{\phi(u)\}, \tag{1.5}$$

$$\int_u^\infty \frac{\phi(t)}{t^{b+1}}\, dt = O\left\{\frac{\phi(u)}{u^b}\right\}, \tag{1.6}$$

$$\int_1^u \frac{\phi(t)}{t^{a+1}}\, dt = O\left\{\frac{\phi(u)}{u^a}\right\}, \tag{1.7}$$

for $u \to \infty$.

(iv) By $\phi(x) \in Z(a, b) \subset M(a, b)$, we denote any $\phi(x)$ satisfying the above conditions (1.5), (1.6), (1.7) and the following conditions as $u \to +0$:

$$\phi(2u) = O\{\phi(u)\}, \tag{1.8}$$

$$\int_u^1 \frac{\phi(t)}{t^{b+1}}\, dt = O\left\{\frac{\phi(u)}{u^b}\right\}, \tag{1.9}$$

$$\int_0^u \frac{\phi(t)}{t^{a+1}}\, dt = O\left\{\frac{\phi(u)}{u^a}\right\}. \tag{1.10}$$

Gapoškin's result (1.2) is similar to a lemma in **(3)**:

$$\int_{-\pi}^{\pi} \alpha(x)\, |\bar{f}(x)|^p\, dx \leqslant K \int_{-\pi}^{\pi} \alpha(x)\, |f(x)|^p\, dx, \tag{1.11}$$

$$\int_{-\pi}^{\pi} \beta(x)\, |\bar{f}(x)|^p\, dx \leqslant K \int_{-\pi}^{\pi} \beta(x)\, |f(x)|\, dx, \tag{1.12}$$

where $p > 1$, $\alpha(x) \sim [0, p - 1\rangle$, $\beta(x) \sim \langle -1, 0]$.

Throughout this paper, K denotes a positive constant, and two different positive constants may be denoted by the same K. We shall use the functions $\phi(x) \in M(p_1, p_2)$, $\phi(x) \in Z(p_1, p_2)$, $\alpha(x) \sim [0, a\rangle$, $\beta(x) \sim \langle -b, 0]$ as defined above. In this paper, each of the inequalities involving integrals has the meaning: "If the integrals on the right are finite, then the integrals on the left exist, and the inequality holds."

2. Even and odd functions. The conjugate function $\bar{f}(x)$ of an even function $f(x) = f(-x)$ is given by

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$$\bar{f}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) \cot \tfrac{1}{2}(x-y)\, dy = \frac{1}{\pi}\int_0^{\pi}\left\{\frac{\sin x}{\cos y - \cos x}\right\} f(y)\, dy. \tag{2.1}$$

If $f(x)$ is an odd function, then the conjugate function $\bar{f}(x)$ is given by

$$\bar{f}(x) = \frac{1}{\pi}\int_0^{\pi}\left\{\frac{\sin y}{\cos y - \cos x}\right\} f(y)\, dy. \tag{2.2}$$

Moreover, when f is even, then $\bar{f}$ is odd, and when f is odd, then $\bar{f}$ is even.

THEOREM 1. *If $f(x)$ is even and belongs to $L(-\pi, \pi)$, and if $\bar{f}(x)$ is the conjugate function of $f(x)$, $\beta(x) \sim \langle -(p_1+1), 0]$, $1 < p_1 < p_2 < \infty$, $\phi(x) \in M(p_1, p_2)$, then*

$$\int_{-\pi}^{\pi} \beta(x)\, \phi\{|\bar{f}(x)|\}\, dx \leqslant K \int_{-\pi}^{\pi} \beta(x)\, \phi\{|f(x)|\}\, dx + K, \tag{2.3}$$

where the constants are independent of $f(x)$. In particular, if $\phi(x) \in Z(p_1, p_2)$, then the second term K in the right member of (2.3) *may be replaced by zero.*

Remark. If we replace the "explicit form" (2.3) by the "implicit form"

$$\int \phi\{\beta(x)\, |\bar{f}(x)|\}\, dx \leqslant K \int \phi\{\beta(x)\, |f(x)|\}\, dx + K, \qquad \beta(x) \sim \langle -1/p_2, 0],$$

the result remains true.

THEOREM 2. *If $f(x)$ is odd and belongs to $L(-\pi, \pi)$, and if $\bar{f}(x)$ is the conjugate function of $f(x)$, $\alpha(x) \sim [0, 2p_1 - 1\rangle$, $1 < p_1 < p_2 < \infty$, $\phi(x) \in M(p_1, p_2)$, then*

$$\int_{-\pi}^{\pi} \alpha(x)\, \phi\{|\bar{f}(x)|\}\, dx \leqslant K \int_{-\pi}^{\pi} \alpha(x)\, \phi\{|f(x)|\}\, dx + K, \tag{2.4}$$

where the constants are independent of $f(x)$. In particular, if $\phi(x) \in Z(p_1, p_2)$, then the second constant K on the right may be replaced by zero.

For the proofs of Theorems 1 and 2, we need the following lemmas. Lemma 1 has been proved in **(4)**.

LEMMA 1. *If $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant \pi$, then*

$$\frac{|\cos x - \cos y|}{\cos \frac{1}{2}x} \geqslant \frac{|x^2 - y^2|}{\pi^2}. \tag{2.5}$$

LEMMA 2. *If $f(x)$ is even, $\beta(x) \sim \langle -(p+1), 0]$, $p > 1$, and if $\bar{f}(x)$ is the conjugate function of $f(x) \in L(-\pi, \pi)$, then*

$$\int_{-\pi}^{\pi} \beta(x)\, |\bar{f}(x)|^p\, dx \leqslant K \int_{-\pi}^{\pi} \beta(x)\, |f(x)|^p\, dx, \tag{2.6}$$

where K is independent of $f(x)$.

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LEMMA 3. *If $f(x)$ is odd and belongs to $L(-\pi, \pi)$, and if $\bar{f}(x)$ is the conjugate function of $f(x)$, $\alpha(x) \sim [0, 2p-1\rangle$, $p > 1$, then*

$$\int_{-\pi}^{\pi} \alpha(x)\, |\bar{f}(x)|^p\, dx \leqslant K \int_{-\pi}^{\pi} \alpha(x)\, |f(x)|^p\, dx, \tag{2.7}$$

where K depends on p and $\alpha(x)$ only.

We consider a proof of Lemma 2 only, while the proof of Lemma 3 follows in a similar way, since the kernels are respectively

$$\frac{\sin x}{\cos y - \cos x} \quad \text{and} \quad \frac{\sin y}{\cos y - \cos x}.$$

We first introduce the following functions:

$$W(x) = \{\beta(x)\}^{1/p}\bar{f}(x) - V(x), \qquad U(x) = \{\beta(x)\}^{1/p} f(x), \tag{2.8}$$

where $V(x)$ is the conjugate function of $U(x)$ defined by

$$V(x) = \frac{1}{\pi}\int_0^{\pi} \left(\frac{\sin x}{\cos y - \cos x}\right) U(y)\, dy. \tag{2.9}$$

It should be remarked that if f is even, then U is even and $\bar{f}$, V, W are odd, and that if f is odd, then U is odd and the corresponding $\bar{f}$, V, W are therefore even. So we may restrict ourselves to the range $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant \pi$. Let $\mathfrak{M}_p[f]$ denote the p-norm in $L^p(-\pi, \pi)$:

$$\mathfrak{M}_p[f] = ||f(x)||_p = \left\{\int_{-\pi}^{\pi} |f(x)|^p\, dx\right\}^{1/p}. \tag{2.10}$$

(2.6) is equivalent to

$$\mathfrak{M}_p[\bar{f}\beta^{1/p}] \leqslant K\mathfrak{M}_p[U]. \tag{2.11}$$

In virtue of Minkowski's inequality and M. Riesz's theorem, we have

$$\mathfrak{M}_p[\bar{f}\beta^{1/p}] \leqslant \mathfrak{M}_p[V] + \mathfrak{M}_p[W] \leqslant K\mathfrak{M}_p[U] + \mathfrak{M}_p[W]. \tag{2.12}$$

It remains to prove that

$$\mathfrak{M}_p[W] \leqslant K\mathfrak{M}_p[U]. \tag{2.13}$$

In fact, we have

$$\begin{cases} W(x) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi}\left(\frac{\sin x}{\cos y - \cos x}\right)\{f(y)\beta^{1/p}(x) - U(y)\}\, dy, \\ \quad = \dfrac{1}{\pi}\displaystyle\int_0^{\pi}\left(\frac{\sin x}{\cos y - \cos x}\right)\{f(y)\beta^{1/p}(x) - f(y)\beta^{1/p}(y)\}\, dy. \end{cases} \tag{2.14}$$

In virtue of Lemma 1 and by Minkowski's inequality, it follows that

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$$
(2.15)\quad \begin{cases}
\mathfrak{M}_p[W] = ||W||_p = \left\{2 \int_0^\pi |W(x)|^p \, dx\right\}^{1/p} \\
\leqslant K\left\{ \int_0^\pi \left[\int_0^\pi \frac{x}{|x^2 - y^2|} |\beta^{1/p}(x) - \beta^{1/p}(y)| \, |f(y)| \, dy \right]^p dx \right\}^{1/p} \\
= K\left\{ \int_0^\pi \left[\int_0^\pi \frac{x}{|x^2 - y^2|} |1 - (\beta(x)/\beta(y))^{1/p}| \, \beta^{1/p}(y) \, |f(y)| \, dy \right]^p dx \right\}^{1/p} \\
= K\left\{ \int_0^\pi \left[\int_0^{\pi/x} |1 - t^2|^{-1} \, |1 - (\beta(x)/\beta(tx))^{1/p}| \right.\right. \\
\qquad \left.\left. \times \, \beta^{1/p}(tx) \, |f(tx)| \, dt \right]^p dx \right\}^{1/p} \\
\leqslant K \int_0^\infty \left\{ \int_0^{\min(\pi, \pi/t)} |f(tx)|^p \, \beta(tx) \, |1 - (\beta(x)/\beta(tx))^{1/p}|^p \, dx \right\}^{1/p} \\
\qquad \times \, |1 - t^2|^{-1} \, dt \\
\leqslant K \int_0^\infty \left\{ \int_0^\pi |f(u)\beta^{1/p}(u)|^p \, t^{-1} \, |1 - t^{(1+1/p-\epsilon)}|^p \, du \right\}^{1/p} |1 - t^2|^{-1} \, dt \\
\leqslant K \int_0^\infty \left\{ \int_0^\pi |U(y)|^p \, dy \right\}^{1/p} t^{-1/p} \, |1 - t^{(1+1/p-\epsilon)}| \, |1 - t^2|^{-1} \, dt \\
\leqslant K\left\{ \int_0^\infty t^{-1/p} \, |1 - t^{(1+1/p-\epsilon)}| \, |1 - t^2|^{-1} \, dt \right\} ||U||_p \\
\leqslant K\mathfrak{M}_p[U].
\end{cases}
$$

(Cf. formulas (2.28), (2.29), (2.30), (3.41), (3.42), (3.43) in **(3)**, where $\beta(x)$ is equivalent to $1/\alpha(x)$. Since $\beta(x) \sim \langle -1 - p, 0]$, there exists $\epsilon > 0$ such that $\beta(x) \sim [-1 - p + \epsilon, 0]$. If $|t| < 1$, then $\beta(tx) \geqslant \beta(x)$, $x^{1+p-\epsilon}\beta(x) \geqslant (tx)^{1+p-\epsilon}\beta(tx)$, $\beta(x)/\beta(tx) \geqslant t^{1+p-\epsilon}$, and $|1 - \beta(x)/\beta(tx)| \leqslant 1 - t^{1+p-\epsilon}$. If $t > 1$, then $1 \leqslant \beta(x)/\beta(tx) \leqslant t^{1+p-\epsilon}$, and $|1 - \beta(x)/\beta(tx)| \leqslant t^{1+p-\epsilon} - 1 = |1 - t^{1+p-\epsilon}|$.) This proves Lemma 2.

From Lemmas 2 and 3, and by a similar argument as given in **(7**; **9**; **3**, p. 405**)**, Theorems 1 and 2 follow at once.

3. Generalizations of Zygmund's theorem.

THEOREM 3. *Let $\bar{f}(x)$ be the conjugate function of $f(x) \in L(-\pi, \pi)$. If $\beta(x) \sim \langle -1, 0]$, then*

$$
(3.1)\qquad \int_{-\pi}^{\pi} \beta(x) \, |\bar{f}(x)| \, dx \leqslant K \int_{-\pi}^{\pi} \beta(x) \, |f(x)| \log^+ \beta(x) \, |f(x)| \, dx + K.
$$

If, in addition, f is even, then (3.1) *remains true for $\beta(x) \sim \langle -2, 0]$. Moreover, if, in addition, f is odd, then the inequality remains true when $\beta(x)$ is replaced*

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by $\alpha(x) \sim [0, 1]$. *The constants in the right member of* (3.1) *are independent of* $f(x)$.

We need the following lemma. Although we shall not use the full strength of the inequalities, for the sake of completeness, we consider the general case here.

LEMMA 4. *Let* $K(x, y)$ *be non-negative and* $f(x) \geqslant 0$, *and suppose that* $G(x, y) = y \,.\, K(x, y)$ *satisfies*

$$G(x, y) \leqslant K, \qquad 0 < x \leqslant y \leqslant a < \infty, \tag{3.2}$$

$$\int_0^x K(x, y)\, dy = \int_0^1 xK(x, \lambda x)\, d\lambda \leqslant K + K \log^+(1/x) \tag{3.3}$$

for $0 < x \leqslant a < \infty$. *Then*

$$\int_0^\infty dy \left\{ \int_0^a K(x, y) f(x)\, dx \right\} \leqslant K \int_0^a f(x) \log^+ f(x)\, dx + K, \tag{3.4}$$

where the constants on the right are independent of $f(x)$.

Proof. We shall use the inequality

$$uv \leqslant u \log^+ u + e^{v-1}, \qquad u \geqslant 0. \tag{3.5}$$

(See Hardy, Littlewood, and Pólya: *Inequalities* (2nd ed., Cambridge University Press, 1952), p. 61, Theorem 63, p. 107, and p. 113.) By inverting the order of integration, and from (3.5), we find that

$$\left\{\begin{aligned} I &= \int_0^a dy \left\{ \int_0^a K(x, y) f(x)\, dx \right\} \\ &= \int_0^a dy \left\{ \left(\int_0^y + \int_y^a \right) K(x, y) f(x)\, dx \right\} \\ &= \int_0^a \frac{dy}{y} \int_0^y G(x, y) f(x)\, dx + \int_0^a f(x)\, dx \int_0^x K(x, y)\, dy \\ &\leqslant K \int_0^a \frac{dy}{y} \int_0^y f(x)\, dx + K \int_0^a f(x) \log^+(1/x)\, dx + K \int_0^a f(x)\, dx \\ &\leqslant K \int_0^a f(x) \log(a/x)\, dx + K \int_0^a f(x) \log^+(1/x)\, dx + K \int_0^a f(x)\, dx \\ &\leqslant K \int_0^a \{2f\}\{\tfrac{1}{2}\log^+(1/x)\}\, dx + K \log^+ a \int_0^a f(x)\, dx + K \int_0^a f(x)\, dx \\ &\leqslant K \int_0^a 2f \log^+ 2f\, dx + K \int_0^a e^{\frac{1}{2}\log^+(1/x)}\, dx + K \int_0^a f(x)\, dx \\ &\leqslant K \int_0^a f(x) \log^+ f(x)\, dx + K. \end{aligned}\right. \tag{3.6}$$

This proves Lemma 4.

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Proof of Theorem 3. By Zygmund's theorem, it is sufficient to prove the following inequality:

$$(3.7)\qquad \int_{-\pi}^{\pi} |W(y)|\,dy \leqslant \int_{-\pi}^{\pi} \beta(x)\,|f(x)|\log^{+}\beta(x)\,|f(x)|\,dx + K,$$

where now $W(y)$ is defined by

$$(3.8)\qquad W(y) = \bar{f}(y)\beta(y) - V(y),$$

and $V(y)$ is the conjugate function of $U(y) = f(y)\beta(y)$:

$$(3.9)\qquad V(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(x)\cot\tfrac{1}{2}(y-x)\,dx.$$

We have, for even and odd functions,

$$(3.10)\qquad \begin{cases} |W(y)| = \dfrac{1}{2\pi}\left|\displaystyle\int_{-\pi}^{\pi}\{\beta(y)-\beta(x)\}f(x)\cot\tfrac{1}{2}(y-x)\,dx\right| \\ \qquad\leqslant K\displaystyle\int_{0}^{\pi} K(x,y)\{\beta(x)\,|f(x)|\}\,dx, \end{cases}$$

where the kernel $K(x, y)$ is defined by

$$(3.11)\qquad K_1(x,y) = \frac{y|1-\beta(y)/\beta(x)|}{|x^2-y^2|},\qquad \beta(x)\sim\langle -2, 0],$$

when $f(x)$ is even, and

$$(3.12)\qquad K_2(x,y) = \frac{x|1-\alpha(y)/\alpha(x)|}{|x^2-y^2|},\qquad \alpha(x)\sim[0, 1],$$

or

$$(3.13)\qquad K_3(x,y) = \frac{x|1-\beta(y)/\beta(x)|}{|x^2-y^2|},\qquad \beta(x)\sim\langle -1, 0],$$

when $f(x)$ is odd. We next show that in each case, $K_i(x, y)$ satisfies the conditions of Lemma 4.

Let us first consider $K_1(x, y)$. Then $G_1(x, y) = yK_1(x, y)$, $0 < x \leqslant y < a < \infty$. If $0 < \mu \leqslant 1$, then $y^2\beta(y) \geqslant \mu^2 y^2\beta(\mu y)$. It follows that

$$G_1(x,y) = \frac{|1-\beta(y)/\beta(\mu y)|}{|\mu^2-1|} \leqslant \frac{|1-\beta(y)/\beta(y)\mu^{-2}|}{1-\mu^2} = 1.$$

Thus $G_1(x, y)$ satisfies (3.2). Next, if $|\lambda| \leqslant 1$, let $y = \lambda x$. Then there exists $\epsilon > 0$, such that $y^{2-\epsilon}\beta(y) \leqslant x^{2-\epsilon}\beta(x)$, and therefore

$$\int_0^1 xK_1(x,\lambda x)\,d\lambda \leqslant \int_0^1 \frac{\lambda|1-\beta(\lambda x)/\beta(x)|}{1-\lambda^2}\,d\lambda$$

$$\leqslant \int_0^1 \frac{\lambda(\lambda^{-2+\epsilon}-1)}{1-\lambda^2}\,d\lambda = \int_0^1 \frac{\lambda^{-1+\epsilon}-\lambda}{1-\lambda^2}\,d\lambda = \int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^1 \leqslant \frac{2}{\epsilon} + K \leqslant K.$$

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Hence (3.3) is satisfied for $K_1(x, y)$. In a similar way, it can be readily shown that $K_2(x, y)$ also satisfies (3.2) and (3.3).

It remains to consider $K_3(x, y)$. Let $x = \lambda y$, $0 < \lambda < 1$. Then

$$G_3(x, y) = \frac{\lambda|1 - \alpha(y)/\alpha(\lambda y)|}{1 - \lambda^2} \leqslant \frac{\lambda|\lambda^{-1} - 1|}{1 - \lambda^2} = \frac{1}{1 + \lambda} < 1.$$

Thus $K_3(x, y)$ satisfies (3.2). On the other hand,

$$\int_0^x K_3(x, y)\, dy = \int_0^1 xK_3(x, \mu x)\, d\mu$$
$$= \int_0^1 \frac{|1 - \alpha(\mu x)/\alpha(x)|}{1 - \mu^2}\, d\mu \leqslant \int_0^1 \frac{1 - \mu}{1 - \mu^2}\, d\mu = \log 2.$$

Hence $K_3(x, y)$ also satisfies the condition (3.3).

We may now apply Lemma 4. By this lemma the results for even and odd functions in Theorem 3 follow at once. It remains to consider the general case when f is neither even nor odd. Let us write

$$f(x) = \tfrac{1}{2}\{f(x) + f(-x)\} + \tfrac{1}{2}\{f(x) - f(-x)\} = r(x) + s(x), \tag{3.14}$$

where $r(x)$ is even and $s(x)$ is odd. Then both $|r(x)|$ and $|s(x)|$ are bounded by $|f(x)| + |f(-x)|$. It follows that

$$\begin{cases} \displaystyle\int_{-\pi}^{\pi} \beta(x)\, |\bar{f}(x)|\, dx \leqslant \int_{-\pi}^{\pi} \beta(x)\, |\bar{r}(x)|\, dx + \int_{-\pi}^{\pi} \beta(x)\, |\bar{s}(x)|\, dx \\ \displaystyle\leqslant K \int_{-\pi}^{\pi} \beta(x)\, |r(x)| \log^+\beta(x)\, |r(x)|\, dx \\ \displaystyle\qquad + K \int_{-\pi}^{\pi} \beta(x)\, |s(x)| \log^+\beta(x)\, |s(x)|\, dx + K \\ \displaystyle\leqslant K \int_{-\pi}^{\pi} \beta(x)\{|f(x)| + |f(-x)|\} \log^+\tfrac{1}{2}\beta(x)\, \{|f(x)| + |f(-x)|\}\, dx + K \\ \displaystyle\leqslant K \int_{-\pi}^{\pi} \beta(x)\, |f(x)| \log^+\beta(x)\, |f(x)|\, dx + K. \end{cases} \tag{3.15}$$

This proves (3.1).

Finally it should be remarked that if we set the kernel in (3.10) equal to $|1 - \beta(y)/\beta(x)|/|x - y|$, then our method fails, for, in this case, (3.2) is not satisfied.

4. Analogous theorems for Hilbert transforms.

THEOREM 4. *If $f(x)$ is even, $\beta(x) \sim \langle -(p_1 + 1), 0]$, $1 < p_1 < p_2 < \infty$, $\phi(x) \in Z(p_1, p_2)$, and if*

$$g(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{x - t}\, dt = \frac{2}{\pi} \int_0^{\infty} \left(\frac{x}{x^2 - t^2}\right) f(t)\, dt \tag{4.1}$$

is the Hilbert transform of the even function $f(x)$, then

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$$\int_{-\infty}^{\infty} \beta(x)\ \phi\{|g(x)|\}\ dx \leqslant K \int_{-\infty}^{\infty} \beta(x)\ \phi\{|f(x)|\}\ dx, \tag{4.2}$$

where the positive constant K is independent of $f(x)$. (For the corresponding M. Riesz's theorem for Hilbert transforms, see Titchmarsh (**8**, pp. 132–138). In Formula (4.2) the $\mu(R)$ measure and $\nu(S)$ measure are both infinite, so we cannot take $\phi(x) \in M(p_1, p_2)$. The result is a consequence of an interpolation theorem due to S. Koizumi, cf. (**3**, p. 365, Theorem H).)

For the proof of Theorem 4, it is sufficient to consider the kernel

$$K(x, y) = \frac{|x|}{|x^2 - y^2|}, \tag{4.3}$$

which is analogous to the kernel in (2.1). In fact this is exactly similar to the case of conjugate functions of $f(x)$ and the proof is essentially the same as in Section 2. By a similar argument for odd functions, we deduce the following theorem.

THEOREM 5. *If $f(x)$ is odd, $\alpha(x) \sim [0, 2p_1 - 1\rangle$, $1 < p_1 < p_2 < \infty$, $\phi(x) \in Z(p_1, p_2)$, and*

$$g(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{x - t}\, dt = \frac{2}{\pi} \int_0^{\infty} \left(\frac{t}{x^2 - t^2}\right) f(x)\, dt \tag{4.4}$$

is the Hilbert transform of the odd function $f(x)$, then

$$\int_{-\infty}^{\infty} \alpha(x)\ \phi\{|g(x)|\}\ dx \leqslant K \int_{-\infty}^{\infty} \alpha(x)\ \phi\{|f(x)|\}\ dx, \tag{4.5}$$

where the positive constant K is independent of $f(x)$.

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