DETERMINACY AND INDETERMINACY OF GAMES PLAYED ON COMPLETE METRIC SPACES

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Abstract

Schmidt's game is a powerful tool for studying properties of certain sets which arise in Diophantine approximation theory, number theory and dynamics. Recently, many new results have been proven using this game. In this paper we address determinacy and indeterminacy questions regarding Schmidt's game and its variations, as well as more general games played on complete metric spaces (for example, fractals). We show that, except for certain exceptional cases, these games are undetermined on certain sets. Judging by the vast numbers of papers utilising these games, we believe that the results in this paper will be of interest to a large audience of number theorists as well as set theorists and logicians.

1. Introduction

In 1966, Schmidt [9] introduced a two-player game referred to thereafter as Schmidt's game. Schmidt invented the game primarily as a tool for studying certain sets which arise in number theory and Diophantine approximation theory. These sets are often exceptional with respect to both measure and category. The most significant example is the following. Let $\mathbb{Q}$ denote the set of rational numbers. A real number x is said to be *badly approximable* if there exists a positive constant $c = c(x)$ such that $|x - (p/q)| > c/q^2$ for all $p/q \in \mathbb{Q}$. We denote the set of badly approximable numbers by BA. This set plays a major role in Diophantine approximation theory, and is well known to be both meagre and Lebesgue null. Nonetheless, using his game, Schmidt was able to prove the following remarkable result.

THEOREM 1.1 (Schmidt [9]). *Let* $(f_n)_{n=1}^{\infty}$ *be a sequence of* C^1 *diffeomorphisms of* $\mathbb{R}$. *Then the Hausdorff dimension of the set* $\bigcap_{n=1}^{\infty} f_n^{-1}(\text{BA})$ *is* 1. *In particular,* $\bigcap_{n=1}^{\infty} f_n^{-1}(\text{BA})$ *is uncountable.*

Before turning our attention to the determinacy aspects of the game, we devote the first subsection of the introduction to describing the game as well as recent variations

introduced by McMullen [7], both of which will be the focus of this paper. In the second subsection of the introduction we state the main results of this paper.

1.1. Schmidt's game and its variations. Let (X, d) be a complete metric space. In what follows, we denote by $B(x, \rho)$ the closed ball in the metric space (X, d) centred at x of radius ρ, that is,

$$B(x, \rho) \overset{\text{def}}{=} \{y \in X : d(x, y) \leq \rho\}. \tag{1.1}$$

Let $\Omega \overset{\text{def}}{=} X \times \mathbb{R}_+$ be the set of formal balls in X, and define a partial ordering on Ω by letting

$$(x_2, \rho_2) \leq_s (x_1, \rho_1) \quad \text{if} \quad \rho_2 + d(x_1, x_2) \leq \rho_1.$$

We associate to each pair (x, ρ) a ball in (X, d) via the 'ball' function $B(\cdot, \cdot)$ as in (1.1). Note that the inequality $(x_2, \rho_2) \leq_s (x_1, \rho_1)$ clearly implies (but is not necessarily implied by) the inclusion $B(x_2, \rho_2) \subseteq B(x_1, \rho_1)$. Nevertheless, the two conditions are equivalent when (X, d) is a Banach space.

Fix $\alpha, \beta \in (0, 1)$ and $S \subseteq X$. The set S will be called the *target set*. Schmidt's (α, β, S)-game is played by two players, whom we shall call Alice and Bob. The game starts with Bob choosing a pair $\omega_1 = (x_1, \rho_1) \in \Omega$. Alice and Bob then take turns choosing pairs $\omega'_n = (x'_n, \rho'_n) \leq_s \omega_n$ and $\omega_{n+1} = (x_{n+1}, \rho_{n+1}) \leq_s \omega'_n$, respectively. These pairs are required to satisfy

$$\rho'_n = \alpha \rho_n \quad \text{and} \quad \rho_{n+1} = \beta \rho'_n. \tag{1.2}$$

Since the game is played on a complete metric space and since the diameters of the nested balls

$$B(\omega_1) \supseteq \cdots \supseteq B(\omega_n) \supseteq B(\omega'_n) \supseteq B(\omega_{n+1}) \supseteq \cdots \tag{1.3}$$

tend to zero as $n \to \infty$, the intersection of these balls is a singleton $\{x_\infty\}$. Call Alice the winner if $x_\infty \in S$; otherwise Bob is declared the winner. A *strategy* consists of a description of how one of the players should act based on the opponent's previous moves; see Section 2 for a precise formulation. A strategy is *winning* if it guarantees the player a win regardless of the opponent's moves. If Alice has a winning strategy for Schmidt's (α, β, S)-game, we say that S is an (α, β)-*winning* set. If S is (α, β)-winning for all (equivalently, for all sufficiently small) $\beta \in (0, 1)$, we say that S is an α-*winning* set. If S is α-winning for some (equivalently, for arbitrarily small) $\alpha \in (0, 1)$, we say that S is *winning*. (To see that 'for all' and 'for some' may be replaced by 'for all sufficiently small' and 'for arbitrarily small', respectively, see [9, Lemmas 8 and 9].)

We now describe two variations of Schmidt's game introduced by McMullen [7]. The first of these variations is the *strong winning* game. Given $\alpha, \beta \in (0, 1)$ and $S \subseteq X$, the rules of the (α, β, S)-strong winning game are the same as the rules of Schmidt's (α, β, S)-game, except that (1.2) is replaced by the inequalities

$$\rho'_n \geq \alpha \rho_n \quad \text{and} \quad \rho_{n+1} \geq \beta \rho'_n. \tag{1.4}$$

However, since (1.4) is insufficient to ensure that the diameters of the nested balls (1.3) tend to zero, it may happen that the intersection $I \overset{\text{def}}{=} \bigcap_n B(\omega_n)$ is not a singleton. If this

occurs, we call Alice the winner if $I \cap S \neq \emptyset$; otherwise Bob is declared the winner. If Alice has a winning strategy for the (α, β)-strong winning game with a given target set S, then S is called (α, β)-*strong winning*. Similarly, as above one can talk about α-*strong winning* and *strong winning* sets.

McMullen's second variation is called the *absolute winning* game. Although he defined it only on $\mathbb{R}^d$, it can be played on much more general spaces. Fix $c > 0$. A metric space (X, d) is said to be *c-uniformly perfect* if, for every pair $(x, \rho) \in \Omega$ with $\rho \leq 1$,

$$B(x, \rho) \setminus B(x, c\rho) \neq \emptyset. \tag{1.5}$$

If (X, d) is c-uniformly perfect for some $c > 0$, then we say that (X, d) is *uniformly perfect*. Examples of uniformly perfect spaces include $\mathbb{R}^d$ as well as many fractal subsets including the Cantor ternary set, the Sierpinski triangle and the von Koch snowflake curve.

Let (X, d) be a complete c-uniformly perfect metric space, and fix $0 < \beta < c/5$ and $S \subseteq X$. The (β, S)-*absolute winning* game is played as follows. As before, Bob begins by choosing a pair $\omega_1 \in \Omega$. After Bob's nth move $\omega_n = (x_n, \rho_n)$, Alice chooses a pair $\omega'_n = (x'_n, \rho'_n) \in \Omega$ satisfying $\rho'_n \leq \beta \rho_n$. She is said to 'delete' the associated ball $B(\omega'_n)$. Bob must then choose a ball $\omega_{n+1} = (x_{n+1}, \rho_{n+1}) \leq_s \omega_n$ such that

$$\rho_{n+1} \geq \beta \rho_n \quad \text{and} \quad B(\omega_{n+1}) \cap B(\omega'_n) = \emptyset. \tag{1.6}$$

Such a choice is always possible due to (1.5); see [1, Lemma 4.3]. Bob's choices result in a decreasing sequence of sets

$$B(\omega_1) \supseteq \cdots \supseteq B(\omega_n) \supseteq B(\omega_{n+1}) \supseteq \cdots .$$

As before, let $I = \bigcap_n B(\omega_n)$, and call Alice the winner if $I \cap S \neq \emptyset$. If Alice has a winning strategy for the (β, S)-absolute winning game, then S is called β-*absolute winning*. If S is β-absolute winning for all $0 < \beta < c/5$, then we say that S is *absolute winning*.

REMARK 1.2. If (X, d) is a Banach space, then the (β, S)-absolute winning game may be played for any $0 < \beta < 1/3$; the hypothesis $\beta < c/5$ is not necessary. For a proof, see for example the proof of Theorem 1.7(iv).

REMARK 1.3. In the absolute winning game, Alice has rather limited control over the situation, since she can block very few of Bob's possible moves at each step. Thus, being absolute winning is a rather strong property of a set.

The following proposition summarises some important properties of winning, strong winning and absolute winning subsets of a complete uniformly perfect metric space (X, d).

PROPOSITION 1.4.

(i)   *Winning (respectively, strong winning, absolute winning) sets are dense.*
(ii)   *Absolute winning implies strong winning, and strong winning implies winning.*

(iii)   *The countable intersection of α-winning (respectively, α-strong winning, absolute winning) sets is again α-winning (respectively, α-strong winning, absolute winning).*

PROOF. Indeed, (i) follows directly from the definition. The proof of (ii) is fairly straightforward (see [7, pages 2–3]), and the basic idea of the proof of (iii) can be found in [9, Theorem 2]. □

REMARK 1.5. In [9, Theorem 7], Schmidt proved that for a large class of games including the above games, the existence of a winning strategy implies the existence of a *positional* winning strategy, that is, a winning strategy in which a player's moves depend only on his opponent's previous move, and not on the entire history.

1.2. Determinacy of games. Inspired by the Banach–Mazur game, infinite games have been studied for almost a century. In 1953, the notion of two-player zero-sum infinite games with perfect information was introduced and systematically studied by Gale and Stewart [2]. As stated above, the main purpose of this paper is to study the determinacy of Schmidt's game and its variations. (For set-theoretic results of a somewhat different flavour concerning the game on the real line, see [5].) A two-player game is called *determined* if one of the players has a winning strategy.

As a corollary of Martin's celebrated Borel determinacy theorem [6], we deduce the following theorem (see Theorem 3.1).

THEOREM 1.6. *Let (X, d) be a complete metric space. Fix $0 < \alpha, \beta < 1$ and $S \subseteq X$. If S is Borel, then Schmidt's (α, β, S)-game, the (α, β, S)-strong winning game and the (β, S)-absolute winning game are determined. (In the last case, we assume that X is c-uniformly perfect with $\beta < c/5$.)*

We may ask whether these games remain determined when S is not a Borel set. In this paper, we consider the worst-case scenario: when S is a *Bernstein set*, meaning that every closed perfect subset of X intersects both S and $X \setminus S$. Bernstein sets are pathological in that they are not measurable with respect to any nonatomic regular measure and do not have the Baire property [8, Theorem 5.4]. Every uncountable Polish space contains a Bernstein subset [8, Theorem 5.3]. (Both of the above results are stated for $\mathbb{R}$ but easily generalise. In the latter result, the assumption that X is an uncountable Polish space guarantees that the number of closed subsets of X is equal to #(X).) We may now state our main theorem. See Section 3.2 for the proof.

THEOREM 1.7. *Let (X, d) be a complete c-uniformly perfect metric space and let $S \subseteq X$ be a Bernstein subset. Fix $0 < \alpha, \beta < 1$.*

(i)   *If $\alpha, \beta < c/(1 + 2c)$, then Schmidt's (α, β, S)-game and the (α, β, S)-strong winning game are undetermined.*

(ii)   *If (X, d) is a Banach space and if $1 + \alpha\beta > 2\max(\alpha, \beta)$, then Schmidt's (α, β, S)-game and the (α, β, S)-strong winning game are undetermined.*

(iii)   *If $\beta < (c/5)^2$, then the (β, S)-absolute winning game is undetermined.*
(iv)   *If (X, d) is a Banach space and if $\beta < 1/3$, then the (β, S)-absolute winning game is undetermined.*

REMARK 1.8. The hypotheses of Theorem 1.7 are irrelevant in applications, since one generally cares only about small values of α and β (see the definitions of α-winning and winning sets). Moreover, if $1 + \alpha\beta \le 2\max(\alpha, \beta)$, then Schmidt's (α, β, S)-game and the (α, β, S)-strong winning game are determined for all S by [9, Lemmas 5 and 6]. Thus, if (X, d) is a Banach space, then the hypotheses of Theorem 1.7 cannot be weakened.

REMARK 1.9. One may define *Schmidt's S-game* as follows. First, Alice chooses $\alpha \in (0, 1)$, then Bob chooses $\beta \in (0, 1)$ and then Alice and Bob play Schmidt's (α, β, S)-game. Note that a set S is winning if and only if Alice has a winning strategy for Schmidt's S-game. It follows immediately from Theorem 1.7 that Schmidt's S-game is undetermined whenever S is a Bernstein set. Similar statements may be made about the strong and absolute winning games.

2. A more general context

To prove Theorems 1.6 and 1.7, we will introduce a larger class of games which includes the games of Section 1.1. Before defining this class, we introduce an auxiliary class, the class of *Gale–Stewart* games.

2.1. Gale–Stewart games.   Let E be a nonempty set, and let E^* and $E^{\mathbb{N}}$ denote the sets of finite and infinite words with alphabet E, respectively. Elements of $E^* \cup E^{\mathbb{N}}$ will be called *plays*. For each play $\omega \in E^* \cup E^{\mathbb{N}}$, we will denote the length of ω by $|\omega|$, so that $|\omega| = \infty$ if $\omega \in E^{\mathbb{N}}$.

Let $\mathcal{P}^*(E)$ denote the collection of nonempty subsets of E and fix a map $\Gamma : \coprod_{n \in \mathbb{N}} E^{n-1} \to \mathcal{P}^*(E)$. The map Γ will be called the *ruleset*. A play $\omega \in E^* \cup E^{\mathbb{N}}$ will be called *legal* if $\omega_n \in \Gamma(\omega_1, \ldots, \omega_{n-1})$ for all $n \le |\omega|$. The set of all legal plays will be denoted $E^*_\Gamma \cup E^{\mathbb{N}}_\Gamma$.

Fix a set $\mathcal{S} \subseteq E^{\mathbb{N}}_\Gamma$ (the *target set*). Alice and Bob play the $(E, \Gamma, \mathcal{S})$-*Gale–Stewart game* as follows: Alice and Bob alternate choosing elements $\omega_1, \omega_2, \ldots \in E$, with the odd elements chosen by Bob and the even elements by Alice. The elements they choose must satisfy $(\omega_1, \ldots, \omega_n) \in E^*_\Gamma$ for all $n \in \mathbb{N}$. Such a choice is always possible. If the resulting play $\omega = (\omega_1, \omega_2, \ldots) \in E^{\mathbb{N}}$ is in $\mathcal{S}$, then Alice wins; otherwise Bob wins.

An *Alice-strategy* for a Gale–Stewart game $(E, \Gamma, \mathcal{S})$ is a map $\sigma_A : \coprod_{n=0}^{\infty} E^{2n+1}_\Gamma \to E$ such that $\sigma_A(\omega) \in \Gamma(\omega)$ for all $\omega \in \coprod_{n=0}^{\infty} E^{2n+1}_\Gamma$. Similarly, a *Bob-strategy* is a map $\sigma_B : \coprod_{n=0}^{\infty} E^{2n}_\Gamma \to E$ such that $\sigma_B(\omega) \in \Gamma(\omega)$ for all $\omega \in \coprod_{n=0}^{\infty} E^{2n}_\Gamma$. Given any two strategies σ_A and σ_B, there is a unique infinite play $\omega \in E^{\mathbb{N}}_\Gamma$ such that, for each $n \in \mathbb{N}$,

$$\omega_n = \begin{cases} \sigma_A(\omega_1, \ldots, \omega_{n-1}) & \text{if } n \text{ is even,} \\ \sigma_B(\omega_1, \ldots, \omega_{n-1}) & \text{if } n \text{ is odd.} \end{cases} \tag{2.1}$$

This play represents the game which results if the players Alice and Bob play according to the strategies σ_A and σ_B, respectively. It will be denoted (σ_A, σ_B). If $(\sigma_A, \sigma_B) \in \mathcal{S}$ for every Bob-strategy σ_B, then the Alice-strategy σ_A is said to be *winning* for the $(E, \Gamma, \mathcal{S})$-Gale–Stewart game. We will equivalently say that σ_A *ensures that* $\phi \in \mathcal{S}$. Here we think of ϕ as denoting the outcome of an arbitrary game. Conversely, if $(\sigma_A, \sigma_B) \notin \mathcal{S}$ for every Alice-strategy σ_A, then the Bob-strategy σ_B is said to be winning for the $(E, \Gamma, \mathcal{S})$ game, and we say that the strategy ensures that $\phi \notin \mathcal{S}$. Clearly, both players cannot possess a winning strategy. The $(E, \Gamma, \mathcal{S})$-Gale–Stewart game is said to be *determined* if one of the players possesses a winning strategy.

Alternatively, we may characterise the notion of winning strategies in terms of compatible plays.

NOTATION 2.1. Given $\omega, \tau \in E^* \cup E^{\mathbb{N}}$, we write $\omega \leq \tau$ if τ is an extension of ω, that is, if $\omega_n = \tau_n$ for all $n \leq |\omega|$.

DEFINITION 2.2. Let σ_A be an Alice-strategy. A play $\omega \in E_\Gamma^* \cap E_\Gamma^{\mathbb{N}}$ is called σ_A-*compatible* if $\omega_{2n} = \sigma_A(\omega_1, \ldots, \omega_{2n-1})$ whenever $2n \leq |\omega|$. Similarly, given a Bob-strategy σ_B, a play $\omega \in E_\Gamma^* \cup E_\Gamma^{\mathbb{N}}$ is called σ_B-*compatible* if $\omega_{2n+1} = \sigma_B(\omega_1, \ldots, \omega_{2n})$ whenever $2n + 1 \leq |\omega|$.

Given a strategy σ, denote by E_σ^* (respectively, $E_\sigma^{\mathbb{N}}$) the set of σ-compatible plays in E_Γ^* (respectively, $E_\Gamma^{\mathbb{N}}$).

REMARK 2.3. Under this definition, an Alice-strategy σ_A is winning if and only if $E_{\sigma_A}^{\mathbb{N}} \subseteq \mathcal{S}$, and a Bob-strategy σ_B is winning if and only if $E_{\sigma_B}^{\mathbb{N}} \cap \mathcal{S} = \emptyset$.

One of the main problems in game theory is to classify which Gale–Stewart games are determined. The best general result in this regard is the following theorem.

THEOREM 2.4 (Martin [6]). *Let E be a nonempty set and let $\mathcal{S} \subseteq E_\Gamma^{\mathbb{N}}$ be a Borel set, where $E_\Gamma^{\mathbb{N}}$ is viewed as a subspace of $E^{\mathbb{N}}$ equipped with the product topology (viewing E as a discrete topological space). Then the $(E, \Gamma, \mathcal{S})$-Gale–Stewart game is determined.*

On the other hand, the following result is well known (for example, [4, page 137, paragraph 8]).

PROPOSITION 2.5. *If $\mathcal{S}$ is a Bernstein set and if $\Gamma(\omega) = E$ for all $\omega \in \coprod_{n \in \mathbb{N}} E^{n-1}$, then the $(E, \Gamma, \mathcal{S})$-Gale–Stewart game is undetermined.*

2.2. The games of Section 1.1 as Gale–Stewart games. Based on Theorem 2.4 and Proposition 2.5, one might guess that the games described in Section 1.1 exhibit the same behaviour, that is, that they are determined on Borel sets but undetermined on Bernstein sets. And, indeed, Theorems 1.6 and 1.7 show that this is essentially correct. To prove these theorems, we will relate the games of Section 1.1 to Gale–Stewart games.

Let (X, d) be a complete metric space. Fix $\alpha, \beta \in (0, 1)$ and $S \subseteq X$, and consider Schmidt's (α, β, S)-game. We will consider a corresponding Gale–Stewart game which

is determined if and only if Schmidt's (α, β, S)-game is determined. Let $E = \Omega$ be the set of formal balls in X. Define the ruleset $\Gamma = \Gamma_{\alpha,\beta} : \coprod_{n \in \mathbb{N}} E^n \to E$ according to the rules of Schmidt's (α, β)-game; namely, $E \ni \omega_n \in \Gamma(\omega_1, \ldots, \omega_{n-1})$ if and only if

$$\omega_n \leq_s \omega_{n-1} \quad \text{and} \quad \rho_n = \begin{cases} \alpha \rho_{n-1} & \text{if } n \text{ is even,} \\ \beta \rho_{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Let

$$\mathcal{S}_{\alpha,\beta} = \left\{ \omega \in E_{\Gamma}^{\mathbb{N}} : \bigcap_{n \in \mathbb{N}} B(\omega_{2n+1}) \cap S \neq \emptyset \right\}.$$

Then the $(E, \Gamma_{\alpha,\beta}, \mathcal{S}_{\alpha,\beta})$-Gale–Stewart game is equivalent to Schmidt's (α, β, S)-game, in the sense that every winning Alice-strategy (respectively, Bob-strategy) for the $(E, \Gamma_{\alpha,\beta}, \mathcal{S}_{\alpha,\beta})$-Gale–Stewart game corresponds to a winning Alice-strategy (respectively, Bob-strategy) for Schmidt's (α, β, S)-game, and *vice versa*. In particular, each game is determined if and only if the other is.

Similarly, for the (α, β, S)-strong winning game and the (β, S)-absolute winning game, there exist rulesets $\Gamma_{\alpha,\beta}^{\text{strong}}$ and $\Gamma_{\beta}^{\text{absolute}}$ and sets $\mathcal{S}_{\alpha,\beta}^{\text{strong}}$ and $\mathcal{S}_{\beta}^{\text{absolute}}$ such that the $(E, \Gamma_{\alpha,\beta}^{\text{strong}}, \mathcal{S}_{\alpha,\beta}^{\text{strong}})$-Gale–Stewart game is equivalent to the (α, β, S)-strong winning game and the $(E, \Gamma_{\beta}^{\text{absolute}}, \mathcal{S}_{\beta}^{\text{absolute}})$-Gale–Stewart game is equivalent to the (β, S)-absolute winning game.

2.3. Games played on complete metric spaces. We can generalise the ideas of Section 2.2 to the setting of *games played on complete metric spaces*. Such games will depend on a nonempty set E, a ruleset $\Gamma : \coprod_{n \in \mathbb{N}} E^{n-1} \to \mathcal{P}^*(E)$, a complete metric space (X, d) and two additional parameters (I, S).

The parameter I is the *topological interpretation* of the game, and it is a map $I : E_{\Gamma}^* \to \mathcal{K}^*(X)$, where $\mathcal{K}^*(X)$ denotes the set of all nonempty closed subsets of X, which is order preserving in the sense that $\omega \leq \tau$ implies $I(\tau) \subseteq I(\omega)$. Given a topological interpretation $I : E_{\Gamma}^* \to \mathcal{K}^*(X)$, we can define $I : E_{\Gamma}^{\mathbb{N}} \to \mathcal{K}^*(X)$ by the formula $I(\omega) = \bigcap_{n=0}^{\infty} I(\omega_1, \ldots, \omega_n)$.

The parameter S is the *target set*, and it is a subset of X. The (E, Γ, I, S)-game is played as follows. Players Alice and Bob take turns playing legal moves, resulting in a play $\omega \in E_{\Gamma}^{\mathbb{N}}$ which defines the *outcome* $I(\omega) \in \mathcal{K}^*(X)$. If $I(\omega) \cap S \neq \emptyset$, then Alice wins; otherwise Bob wins. We say that the (E, Γ, I, S)-game is a *game played on the metric space* (X, d).

Based on the above description, it is easy to see that each (E, Γ, I, S)-game is equivalent to some Gale–Stewart game, namely, the Gale–Stewart game $(E, \Gamma, \mathcal{S}_{I,S})$, where

$$\mathcal{S}_{I,S} = \{ \omega \in E_{\Gamma}^{\mathbb{N}} : I(\omega) \cap S \neq \emptyset \}. \tag{2.2}$$

Moreover, the games of Section 1.1 are all examples of games played on complete metric spaces, with $I(\omega) = \bigcap_{n < |\omega|/2} B(\omega_{2n+1})$.

We mention in passing that the well-known Banach–Mazur game is also an example of a game played on a complete metric space. It satisfies the hypotheses of the two

main theorems in the next section, and is therefore determined on Borel sets and undetermined on Bernstein sets. However, both these results are already known as a consequence of the fact that Alice has a winning strategy for the Banach–Mazur game if and only if the target set is comeagre; see [4, page 152, last paragraph and page 153, first paragraph] and [8, page 29, last paragraph], respectively. In fact, the former reference shows that the Banach–Mazur game is determined on analytic sets. The corresponding question of whether the games of Section 1.1 are determined on analytic sets remains open.

3. Determinacy and indeterminacy of games played on complete metric spaces

In this section, we state and prove generalisations of the theorems of Section 1.2.

3.1. Determinacy on Borel sets. We will prove the following generalisation of Theorem 1.6.

THEOREM 3.1. *Let* (E, Γ, I, S) *be a game played on a complete metric space* (X, d) *and suppose that* $S \subseteq X$ *is Borel. Suppose in addition that*

$$\text{For all } \omega \in E_\Gamma^\mathbb{N}, \text{ either } \text{diam}(I(\omega_1, \ldots, \omega_n)) \underset{n}{\to} 0 \text{ or } I(\omega) \cap S \neq \emptyset. \qquad (3.1)$$

Then the (E, Γ, I, S)*-game is determined.*

REMARK 3.2. For the games of Section 1.1, (3.1) holds as long as S is dense.

PROOF. Fix $\omega \in E_\Gamma^\mathbb{N}$ and, for each n, let $\omega_n = (x_n, \rho_n)$. By assumption, we have $(x_{2m+1}, \rho_{2m+1}) \leq_s (x_{2n+1}, \rho_{2n+1})$ for all $m \geq n$, so

$$d(x_{2m+1}, x_{2n+1}) \leq \rho_{2n+1} - \rho_{2m+1}. \qquad (3.2)$$

Since the sequence $(\rho_{2n+1})_0^\infty$ is decreasing and bounded, it is Cauchy and therefore the right-hand side of (3.2) tends to zero as $m, n \to \infty$. Thus, the sequence $(x_{2n+1})_0^\infty$ is Cauchy. Since X is complete, we have $x_{2n+1} \to x$ for some $x \in X$. Then $I(\omega) \supseteq B(x, \rho)$, where $\rho = \lim_{n \to \infty} \rho_n$. Since S is dense, we have $I(\omega) \cap S \neq \emptyset$ if $\rho > 0$. □

On the other hand, if S is not dense, then Bob can win on the first turn. Thus, Theorem 3.1 implies Theorem 1.6.

PROOF OF THEOREM 3.1. By Theorem 2.4, it suffices to show that the set $\mathcal{S}_{I,S}$ defined in (2.2) is Borel. Let

$$Z = \{\omega \in E_\Gamma^\mathbb{N} : \text{diam}(I(\omega_1, \ldots, \omega_n)) \underset{n}{\to} 0\}. \qquad (3.3)$$

Clearly, Z is Borel. On the other hand, $E_\Gamma^\mathbb{N} \setminus Z \subseteq \mathcal{S}_{I,S}$ by (3.1). So, it suffices to show that $\mathcal{S}_{I,S} \cap Z$ is Borel. Since X is complete, we have $\#(I(\omega)) = 1$ for all $\omega \in Z$. Thus, we may define a map $\iota : Z \to X$ by letting $\{\iota(\tau)\} = I(\tau)$. We observe that $\mathcal{S}_{I,S} \cap Z = \iota^{-1}(S)$. Thus, to complete the proof, it suffices to show the following lemma.

LEMMA 3.3. *The map* ι *is continuous.*

PROOF. Suppose that U is an open subset of X, and fix $x \in U$ and $\omega \in \iota^{-1}(x)$. Since $\omega \in Z$, we have $\mathrm{diam}(I(\omega_0, \ldots, \omega_n)) < d(x, X \setminus U)$ for some $n \in \mathbb{N}$. Then the set $V = \{\tau \in E_\Gamma^\mathbb{N} : \tau_i = \omega_i, 0 \le i \le n\}$ satisfies $\omega \in V \cap Z \subseteq \iota^{-1}(U)$; moreover, V is open in the topology of $E_\Gamma^\mathbb{N}$. Since x and ω were arbitrary, ι is continuous. □

Now the proof of Theorem 3.1 is complete. □

3.2. Indeterminacy on Bernstein sets. To state the appropriate generalisation of Theorem 1.7 to the setting of games played on complete metric spaces, we need to introduce some more terminology.

DEFINITION 3.4. Let $(E, \Gamma, \mathcal{S})$ be a Gale–Stewart game. Given $\tau \in E_\Gamma^*$, an Alice-strategy σ_A and a Bob-strategy σ_B, there is a unique infinite play $\omega \ge \tau$ satisfying (2.1) for all $n > |\tau|$. This play represents the game which results if the players Alice and Bob first make the moves $\tau_1, \ldots, \tau_{|\tau|}$ and then play according to the strategies σ_A and σ_B, respectively. It will be denoted $(\tau, \sigma_A, \sigma_B)$. If $(\tau, \sigma_A, \sigma_B) \in \mathcal{S}$ for every Bob-strategy σ_B, then the Alice-strategy σ_A is said to be *winning relative to τ* for the $(E, \Gamma, \mathcal{S})$-Gale–Stewart game. We say that the strategy σ_A *ensures that $\phi \in \mathcal{S}$ assuming $\phi \ge \tau$*. Conversely, if $(\sigma_A, \sigma_B) \notin \mathcal{S}$ for every Alice-strategy σ_A, then the Bob-strategy σ_B is said to be winning relative to τ for the $(E, \Gamma, \mathcal{S})$-Gale–Stewart game, and we say that σ_B ensures that $\phi \notin \mathcal{S}$ assuming $\phi \ge \tau$.

DEFINITION 3.5. Let (E, Γ, I, S) be a game played on a complete metric space (X, d). Let Z be defined as in (3.3) and, for each $x \in X$, let

$$N_x = \{\omega \in E_\Gamma^\mathbb{N} : x \notin I(\omega)\}.$$

(i) The (E, Γ, I, S)-game is *shrinking* if, for every $\omega \in E_\Gamma^*$, both players have strategies ensuring $\phi \in Z$ assuming $\phi \ge \omega$.

(ii) The (E, Γ, I, S)-game is *nondegenerate* if, for every $\omega \in E_\Gamma^*$ and $x \in X$, both players have strategies ensuring $\phi \in N_x$ assuming $\phi \ge \omega$.

In other words, the game is shrinking if each player can ensure that the outcome is in Z, even if for finitely many moves that player does not have free will. Similarly, the game is nondegenerate if each player can ensure that the outcome is in N_x, under the same stipulations. More informally, the game is shrinking if each player can force the diameters to tend to zero, and is nondegenerate if each player can avoid an arbitrary specified point. (However, see Remark 3.8.)

THEOREM 3.6. *Let (E, Γ, I, S) be a shrinking nondegenerate game played on a complete metric space (X, d). If Alice (respectively, Bob) has a winning strategy for the (E, Γ, I, S)-game, then S (respectively, $X \setminus S$) contains a closed perfect set. In particular, if S is a Bernstein set, then the game is undetermined.*

PROOF. In fact, we will demonstrate the following slightly weaker assertion: if Alice (respectively, Bob) has a strategy ensuring that $\phi \in Z$ implies $\iota(\phi) \in S$ (respectively, that $\phi \in Z$ implies $\iota(\phi) \notin S$), then S (respectively, $X \setminus S$) contains a closed perfect set.

The advantage of proving the weaker assertion is that it is symmetric with respect to interchanging Alice and Bob, so without loss of generality we suppose that Alice has a strategy σ_A which ensures that $\phi \in Z$ implies $\iota(\phi) \in S$. Recall that $E_{\sigma_A}^*$ denotes the set of finite σ_A-compatible plays (see Definition 2.2).

LEMMA 3.7. *For every $\omega \in E_{\sigma_A}^*$ and $r > 0$, there exist $\tau_1, \tau_2 \geq \omega$ in $E_{\sigma_A}^*$ such that* $\mathrm{diam}(I(\tau_i)) < r$ *and* $I(\tau_1) \cap I(\tau_2) = \emptyset$.

PROOF. Since (E, Γ, I, S) is shrinking, Bob has a strategy σ_B ensuring that $\phi \in Z$ assuming $\phi \geq \omega$; it follows that the play $\tau_1^\infty \stackrel{\mathrm{def}}{=} (\omega, \sigma_A, \sigma_B)$ is σ_A-compatible and satisfies $\omega \leq \tau_1^\infty \in Z$. Since $\tau_1^\infty \in Z$ and X is complete, $I(\tau_1^\infty)$ is a singleton, say $I(\tau_1^\infty) = \{x\}$. Since (E, Γ, I, S) is nondegenerate, Bob has a strategy $\widetilde{\sigma}_B$ ensuring that $\phi \in N_x$ assuming $\phi \geq \omega$; it follows that the play $\tau_3^\infty \stackrel{\mathrm{def}}{=} (\omega, \sigma_A, \widetilde{\sigma}_B)$ is σ_A-compatible and satisfies $\omega \leq \tau_3^\infty \in N_x$. Since $x \notin I(\tau_3^\infty) = \bigcap_{\tau_3 \leq \tau_3^\infty} I(\tau_3)$, there exists a finite play $\omega \leq \tau_3 \leq \tau_3^\infty$ satisfying $x \notin I(\tau_3)$. Again using the fact that (E, Γ, I, S) is shrinking, there is a σ_A-compatible play $\tau_2^\infty \in Z$ satisfying $\tau_3 \leq \tau_2^\infty$. Again, $I(\tau_2^\infty)$ is a singleton, say $I(\tau_2^\infty) = \{y\}$. We have $x \neq y$. Let $\rho = \min((1/2)d(x, y), r) > 0$, and let $\tau_1 \leq \tau_1^\infty$ and $\tau_2 \leq \tau_2^\infty$ satisfy $\mathrm{diam}(I(\tau_i)) < \rho$. Since $I(\tau_i^\infty) \subseteq I(\tau_i)$, this completes the proof. □

Let $B^{\mathbb{N}}$ be the set of all infinite binary sequences and B^* be the set of all finite binary sequences. We will denote the concatenation operator on B^* by the symbol $*$. We inductively define a map $g : B^* \to E_{\sigma_A}^*$ sending the empty string of B^* to the empty string of E^* via the following rule.

> If $\omega = g(\theta)$ for some $\theta \in B^*$, choose σ_A-compatible plays $\tau_1, \tau_2 \geq \omega$ satisfying $\mathrm{diam}(I(\tau_i)) < 2^{-|\theta|}$ and $I(\tau_1) \cap I(\tau_2) = \emptyset$, as guaranteed by Lemma 3.7. Let $g(\theta * i) \stackrel{\mathrm{def}}{=} \tau_i$ for $i = 1, 2$.

Let $\overline{g} : B^{\mathbb{N}} \to E_{\sigma_A}^{\mathbb{N}}$ denote the natural extension of g and observe that $\overline{g}$ is continuous. Moreover, $\overline{g} : B^{\mathbb{N}} \to Z$. By Lemma 3.3, $\iota \circ \overline{g} : B^{\mathbb{N}} \to X$ is continuous and, by construction, $\iota \circ \overline{g}$ is injective and thus $\iota \circ \overline{g}$ is a homeomorphic embedding. On the other hand, since σ_A ensures that $\phi \in Z$ implies $\iota(\phi) \in S$, we have $\iota \circ \overline{g}(B^{\mathbb{N}}) \subseteq S$. Thus, S contains the closed perfect set $\iota \circ \overline{g}(B^{\mathbb{N}})$. □

Having completed the proof of Theorem 3.6, we proceed to use it to deduce Theorem 1.7. This consists of verifying that each of the games in Section 1.1 is shrinking and nondegenerate under the hypotheses of Theorem 1.7.

PROOF OF THEOREM 1.7 USING THEOREM 3.6. (i) Schmidt's (α, β, S)-game and the (α, β, S)-strong winning game are both shrinking for every complete metric space (X, d). Indeed, either player may choose the smallest possible radius on each turn, guaranteeing that the diameters of the balls tend to zero and thus that $\phi \in Z$.

Now suppose that (X, d) is c-uniformly perfect and that $\alpha, \beta < c/(1 + 2c)$. We claim that both Schmidt's (α, β, S)-game and the (α, β, S)-strong winning game are nondegenerate. Indeed, suppose that one of the players has just made a move $B(x, \rho)$ and suppose the other player, without loss of generality Alice, wants to avoid the

point $y \in X$. If $y \notin B(x, \alpha\rho)$, then Alice may choose the ball $B(x, \alpha\rho)$ and thus avoid the point y. So, suppose $y \in B(x, \alpha\rho)$. Since X is c-uniformly perfect, we have $B(y, (1 - 2\alpha)\rho) \setminus B(y, c(1 - 2\alpha)\rho) \neq \emptyset$; fix $z \in B(y, (1 - 2\alpha)\rho) \setminus B(y, c(1 - 2\alpha)\rho)$. Then $B(z, \alpha\rho) \subseteq B(x, \rho)$, since $d(z, x) \leq (1 - \alpha)\rho$. So, $B(z, \alpha\rho)$ is a legal move for Alice, and it avoids the point y because $d(z, y) \geq c(1 - 2\alpha)\rho > \alpha\rho$ by the condition on α.

(ii) If (X, d) is a Banach space, then Schmidt's (α, β, S)-game is nondegenerate as long as $2 \max(\alpha, \beta) < 1 + \alpha\beta$; this follows directly from [9, Lemma 7].

(iii) For any $0 < \beta < c/5$, the β-absolute winning game is shrinking on any complete c-uniformly perfect metric space. Indeed, Bob can force the diameters to go to zero by always choosing the smallest possible radius for his ball. Next we will describe Alice's strategy to force the diameters to go to zero. Suppose Bob has just made his nth move $B(x_n, \rho_n)$. Alice's strategy will be to remove the centre $\{x_n\}$ (or a small neighbourhood thereof). This forces Bob's $(n + 1)$ ball $\omega_{n+1} = (x_{n+1}, \rho_{n+1}) \leq_s \omega_n = (x_n, \rho_n)$ to satisfy $x_n \notin B(x_{n+1}, \rho_{n+1})$, that is, $\rho_{n+1} < d(x_n, x_{n+1})$. On the other hand, by the definition of formal inclusion, we have $d(x_n, x_{n+1}) \leq \rho_n - \rho_{n+1}$. Combining these inequalities gives $\rho_{n+1} < \rho_n/2$, which implies that $\rho_n \to 0$.

Now suppose that $\beta < (c/5)^2$. We claim that the (β, S)-absolute winning game is nondegenerate. Indeed, Alice can avoid a point $x \in X$ by simply deleting a neighbourhood of it. Now suppose that Alice has just deleted the ball $B(x', \rho')$, following Bob's move $B(x, \rho)$. To see that Bob can avoid the point y, note that by [1, Lemma 4.3] there is a ball $B(z, (c/5)\rho) \subseteq B(x, \rho) \setminus B(x', \rho')$; applying [1, Lemma 4.3] again gives a ball $B(w, (c/5)^2\rho) \subseteq B(z, (c/5)\rho) \setminus \{y\}$. By making this move, Bob avoids the point y.

(iv) If (X, d) is a Banach space, then, for $0 < \beta < 1/3$, the argument of (iii) shows that the β-absolute winning game is shrinking. We must show that it is also nondegenerate. Indeed, Alice can avoid a point by simply deleting a neighbourhood of it. We will now describe Bob's strategy to avoid a point $\mathbf{x}_0 \in X$. The strategy will consist of two phases. We will describe the second phase first. Fix a unit vector $\mathbf{v} \in X$. Suppose that Bob has just made the move $B(\mathbf{x}_n, \rho_n)$ and that Alice has deleted the ball $B(\mathbf{x}'_n, \rho'_n)$ with $\rho'_n \leq \beta\rho_n$. Consider the balls

$$B(\mathbf{x}_n + (1 - \beta)\rho_n\mathbf{v}, \beta\rho_n) \quad \text{and} \quad B(\mathbf{x}_n - (1 - \beta)\rho_n\mathbf{v}, \beta\rho_n). \tag{3.4}$$

It is readily computed that the distance between these balls is precisely $2(1 - 2\beta)\rho$. In particular, since $\beta < 1/3$, this distance is strictly greater than $2\beta\rho$, which is in turn greater than the diameter of Alice's ball $B(\mathbf{x}'_n, \rho'_n)$. Thus, Alice's ball $B(\mathbf{x}'_n, \rho'_n)$ intersects at most one of the balls (3.4). Bob's strategy will be to choose whichever ball Alice's ball does not intersect, that is, whichever one is legal.

The result of Bob's strategy will be that the outcome of the game takes the form

$$\mathbf{x}_n + \sum_{m=0}^{\infty} \epsilon_m (1 - \beta)\beta^m \rho_n \mathbf{v}, \tag{3.5}$$

where $\epsilon_m = \pm 1$ for all $m \in \mathbb{N}$. So, during the first phase of his strategy, Bob's goal will be to choose a ball $B(\mathbf{x}_n, \rho_n)$ such that $\mathbf{x}_0$ cannot be written in the form (3.5) or,

equivalently, that

$$\mathbf{x}_n \notin S_{\rho_n} := \left\{ \mathbf{x}_0 + \sum_{m=0}^{\infty} \epsilon_m (1-\beta)\beta^m \rho_n \mathbf{v} : \epsilon_m \in \{-1, +1\} \right\}.$$

We remark that if $\dim(X) \geq 2$, this is easy; Bob may simply choose the ball $B(\mathbf{x}_n, \rho_n)$ so that $\mathbf{x}_n$ does not lie on the line $\mathbf{x}_0 + \mathbb{R}\mathbf{v}$. This is possible since the set of legal centres for Bob's balls always contains a nonempty open set; see below for details.

On the other hand, suppose that $\dim(X) = 1$, that is, $X = \mathbb{R}$. Then, for each $\rho > 0$, the set S_ρ has Hausdorff dimension $\log_\beta(1/2) < 1$ (see [3]) and thus Lebesgue measure zero. Alternatively, we may compute directly that $\lambda(S_\rho) = 0$:

$$\lambda(S_\rho) \leq \sum_{\epsilon_0 = \pm 1} \cdots \sum_{\epsilon_{M-1} = \pm 1} \lambda \left\{ \mathbf{x}_0 + \sum_{m=0}^{\infty} \epsilon_m (1-\beta)\beta^m \rho_n \mathbf{v} : \epsilon_m \in \{-1, +1\} \ \forall m \geq M \right\}$$

$$= 2^M \lambda \left\{ \sum_{m=M}^{\infty} \epsilon_m (1-\beta)\beta^m \rho_n \mathbf{v} : \epsilon_m \in \{-1, +1\} \ \forall m \geq M \right\}$$

$$\leq 2^M \operatorname{diam} \left\{ \sum_{m=M}^{\infty} \epsilon_m (1-\beta)\beta^m \rho_n \mathbf{v} : \epsilon_m \in \{-1, +1\} \ \forall m \geq M \right\}$$

$$= 2^{M+1} \sum_{m=M}^{\infty} (1-\beta)\beta^m \rho_n = (2\beta)^M 2\rho_n \xrightarrow[M]{} 0.$$

In particular, S_ρ has empty interior. Now suppose that Bob has just made the move $B(\mathbf{x}_{n-1}, \rho_{n-1})$ (the last move where he 'does not have free will') and that Alice has just deleted the set $B(\mathbf{x}'_{n-1}, \rho'_{n-1})$. Then the set of $\mathbf{x}$ for which Bob can legally play the ball $B(\mathbf{x}, \beta\rho_{n-1})$ is the set

$$B(\mathbf{x}_{n-1}, (1-\beta)\rho_{n-1}) \setminus B(\mathbf{x}'_{n-1}, \beta\rho_{n-1} + \rho'_{n-1}). \tag{3.6}$$

Since $\beta < 1/3$ and $\rho'_{n-1} \leq \beta\rho_{n-1}$, we have $(1-\beta)\rho_{n-1} > \beta\rho_{n-1} + \rho'_{n-1}$. It follows that the set (3.6) contains a nonempty open set, so Bob can choose his centre not to lie in the set $S_{\beta\rho_{n-1}}$. In this way he avoids the point $\mathbf{x}_0$. $\qquad\square$

REMARK 3.8. It is *not* true that for a shrinking game, if a player has a strategy ensuring $\phi \in \mathcal{S}$, then he or she also has a strategy ensuring $\phi \in \mathcal{S} \cap Z$. For example, let $X = B^{\mathbb{N}}$ with the metric

$$d(\theta, \psi) = 4^{-\min\{k : \theta_k \neq \psi_k\}}$$

and consider the set

$$S = \{\theta \in X : \theta_k = 1 \text{ for infinitely many } n \in \mathbb{N}\}.$$

As we have seen above, the $(1/2, 1/2, S)$-strong winning game is shrinking. However, Alice has a strategy ensuring $\phi \in \iota^{-1}(S) \cup (E_\Gamma^{\mathbb{N}} \setminus Z)$, but she does not have a strategy

ensuring $\phi \in \iota^{-1}(S)$. Indeed, the only turns on which the choices of players affect the outcome of the game are those turns n for which

$$\rho_n < 4^{-k} \leq \rho_{n-1} \text{ for some } k. \tag{3.7}$$

Either player can ensure that he or she controls all such turns by simply choosing $\rho_n = \rho_{n-1}$ unless it is possible to choose ρ_n satisfying (3.7), in which case he or she does that instead. This ensures that either the outcome is not in Z or it is precisely what the player wanted, except for the finitely many values which were determined by Bob's initial ball. In particular, Alice has a strategy ensuring that $\phi \in \iota^{-1}(S) \cup (E_{\Gamma}^{\mathbb{N}} \setminus Z)$ and Bob has a strategy ensuring that $\phi \in \iota^{-1}(X \setminus S) \cup (E_{\Gamma}^{\mathbb{N}} \setminus Z)$. The existence of the latter strategy guarantees that Alice cannot have a strategy ensuring $\phi \in \iota^{-1}(S)$.

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