suffering individuals*. At the International Remuneration Conference of 1885, representatives of the workers were angered at claims that they were better off than ever, saying that this was contradicted by their personal experience. Goldman argues that such cases proved the limitations of working only with measures of central tendency, and it was Galton's introduction of standard deviation as a measure of variation that allowed the more marginalised to be better treated. Towards the end Goldman considers two competing conceptualisations of data, either as simple descriptors free of the biases of conjecture or theory, or as embodying by their choice and deployment the mindset and intent of those who use them. (The latter is characteristic of a mindset, even an orthodoxy in some circles, that views the collection of data, like the creation of maps and even general interest in other civilisations—cf. Said's Orientalism—as an assertion of supremacy.) He then says,

> The argument of this book ... is that numbers are plastic and malleable, tools to be used for good or ill, whose essence cannot be captured by this kind of binary division.

I can't say that I have been very convinced by the author's argument—indeed, I am not sure that it goes much beyond presentation of the different uses. The book as a whole is obviously researched to high academic standards and it is very well produced (though the author's punctuation could sometimes be clarified). But at the end of it I was unconvinced that I had learnt much of vital historical or statistical significance—much, that is, that goes beyond the merely interesting. More crucially, my lasting impression was that statistics used for social or political purposes still retains plenty of problems for the non-specialist populace, and that is a matter that should concern all professional statisticians.

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The story of proof: logic and the history of mathematics by John Stillwell, pp 442, £38, ISBN 978-0-69123-6437-3, Princeton University Press (2022)

There are many books on the history of mathematics, a few of which focus on proof, but what this remarkable piece of writing sets out to do is show how *essential* the process of proving is to the development of the subject. As a result the book has an enormous *range*—from Euclid to the current day—and a *seriousness* which makes it both a delight and a challenge. It is not an easy read, and the author is never going to descend to the level of a 'history of mathematics' which I read recently which inserted flippant footnotes of the 'You're just going to have to believe me' type whenever the going got tough.

Its fundamental thesis is that any branch of mathematics sets out with problems which require a new realm of *objects* and a battery of new techniques for using them. An illustrative example of this is the use of infinitesimals for calculating areas and gradients of curves using infinite geometrical processes and

* But Raymond Williams, in a famous book, said that while Dickens "turned his mocking invective" on "the procedure of systematic enquiry, ... Carlyle makes no such trivial error. He criticises imperfect statistics, but his demand, rightly, is for the evidence, for rational enquiry." [*Culture and Society* (1958), Part 1, chapter 4]

their limits. Examples are the calculations, at either end of the sixteenth century, of the area and circumference of a disc by Leonardo and the properties of an equiangular spiral by Thomas Harriot. At the outset the ability to undertake such calculations accurately was a praiseworthy triumph. Eventually, however, it became necessary to underpin this process by rigorous statements of of the nature of the new objects of study and their properties. This was, of course, what Euclid did in his systematisation of geometry, with definitions, common notions, properties and axioms.

What distinguishes mathematics from all other fields of human enquiry is the need for *proof*. This was the case for the systematisers of analysis and the calculus in the 1500s just as for geometers in classical antiquity and for set theorists in the twentieth century. The history of mathematics can therefore be read as a process of development of new *theoretical objects* and of new ways of establishing their properties by means of rigorous argument. Eventually, in the late twentieth century, it would be *logical proof* itself which became the object for mathematical investigation, and a consequence of this was the development of metamathematics as a discipline with its own rules and procedures.

This is a lengthy book with ambitious scope and it will be difficult to do more than cherry-pick some of its outstanding insights. The narrative begins with Euclid and his account of the Pythagorean theorem. This was understood as a result about *areas*, and it led to the consideration of 'common notions', such as the fact that things equal to the same thing are equal to one another, and that if equals are added to equals the results are equal, which have the flavour of an 'algebra of geometry'. Euclid attempted to define a set of axioms which were 'undeniably true' and would lead to a rigorous proof of theorems. As is well known, the controversial—and probably the most significant—of these was that concerning parallel lines. At the same time the congruence results were obtained, and Stillwell makes a point of showing how to derive one of these from another, essentially using proof by contradiction and an appeal to parallelism. For generations Euclid was treated as the yardstick of systematic proof, but then individuals began to question whether there were unstated axioms which were not discussed in *The Elements*. These included the controversial parallel postulate, as well axioms of incidence and axioms of betweenness, which, when they were proposed, would give rise to new areas of the subjects such as spherical, elliptic, hyperbolic and projective geometry. The axiomisation of geometry was not properly addressed until Hilbert tried to fill in the gaps two thousand years later.

The *Elements* is not just a geometry text, which is hardly surprising given the 'algebraic' feel of the original set of axioms. It addresses what would be later known as number theory and also elements of analysis. A good example is the irrationality of $\sqrt{2}$, for which Euclid used *anthyphaeresis*, or repeated subtraction, which is at the heart of the algorithm which bears his name. This led to Eudoxus with his *theory of proportions* and the method of *exhaustion*, and eventually culminated in the Archimedean axiom, which says that, if a and b are non-zero lengths, there is a natural number n so that $na > b$. Alternatively, it can be stated as the fact that there is no ratio $\frac{a}{b}$ such that $0 < \frac{a}{b} < \frac{1}{n}$ for each natural number n; in other words, there are no infinitesimals. This remarkable sequence of consequences shows how the Greek concentration on proof led, often centuries later, to discoveries in arithmetic, algebra and number theory.

In subsequent chapters, the author introduces abstract objects such as groups, rings, fields and vector spaces with their own axiomatic basis and rules of inference. Number theory is developed from its beginnings through Fermat and Euler to

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Gaussian integers and the work of Galois. Topology, graph theory, combinatorics and algebraic curves involve new mathematical entities which will require their own logical underpinning.

The last third of the book is devoted to metamathematics—set theory, logic and computation. An extended discussion of the Zermelo-Fraenkel axioms, including the Axiom of Choice, leads to consequences for measure theory and analysis. A discussion of issues concerning consistency and completeness, where Kurt Gödel inevitably makes an appearance, to be joined later by Brouwer and König, when issues of consistency and completeness are interrogated. The final pages are about computability theory and the work of Turing and Post. I was pleased to see mention of a theorem which is not provable in Peano Arithmetic, as I knew both authors over forty years ago!

This is very tough reading indeed, but it is typical of the author of this book to make an attempt to explain it. This is, I think, a very special piece of work, and it will take me a long time to unpick more of its riches. I can thoroughly recommend it to readers of the *Gazette*.

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The nuts and bolts of proofs by Antonella Cupillari, pp 416, £45.95 (paper), ISBN 978-0-32399-020-2, Elsevier/Academic Press (2023)

This is a fifth edition, so this book must have found favour quite extensively. New are a chapter on ‘more challenging mathematical material’ (sets, groups, functions, limits and infinite sets), and a more extensive selection of exercises and solutions. So the targeted readership in the UK would be first-year undergraduates. Nevertheless, the book could be of use to sixth-formers of high ability thinking of university maths, and second-year undergraduates who may for whatever reason have missed some or all of these basic proof techniques.

Before Chapter 1 there is a page of symbols and lists of ‘facts and properties’ about numbers and functions. Chapter 1 itself, ‘Getting Started’, begins conventionally by discussing what proof is, its relationship with logic, and a paragraph each on terms commonly appearing in mathematical writing: statement, tautology, paradox, hypothesis, conclusion, definition, proof, theorem, lemma, corollary and example. These clarified, the chapter ends with a couple of examples of taking the first step toward proving something, which is to be quite clear about what is being claimed (the conclusion) and what assumptions (the hypotheses) justify it. All fairly trivial, you may think, but in my experience often necessary.

Many claims fit into the “If ... then ...” category, so chapter 2 covers basic techniques for proving such results, after recognising that often such claims are not expressed colloquially in exactly this way. The proof methods discussed and illustrated are direct proofs, negation, contrapositive and contradiction. The examples are mainly from integer, rational and real arithmetic and elementary real functions.

Chapter 3 follows the same pattern to cover ‘special kinds of theorems’, including if-and-only-if, existence theorems, uniqueness theorems, theorems with multiple hypotheses and/or conclusions, and proving that two numbers are equal, backed up by discussion of the logical truth tables involved.