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A comparison theorem for ℓ-adic cohomology

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Abstract. We show that, for certain types of rigid analytic varieties X and constructible ℓ-adic sheaves $(F_n)_n$ on X, one has $H^p_c(X,(F_n)_n) \xrightarrow{\sim} \varprojlim_n H^p_c(X,F_n)$. As an application we obtain that, for an algebraic variety X and associated rigid analytic variety X^{rig}, the ℓ-adic cohomology of X and X^{rig} agree.

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Imitating the definition of compactly supported cohomology of ℓ-adic sheaves on algebraic varieties [J], [E], one can define compactly supported cohomology of ℓ-adic sheaves on rigid analytic varieties over an algebraically closed non-archimedean field k.

In this paper we are interested in the following question

Let X be a separated rigid analytic variety over k and let $(F_n)_{n\in\mathbb{N}}$ be a constructible ℓ-adic sheaf on X with $\ell \neq \mathrm{char}\,(k^\circ/k^{\circ\circ})$. Is the natural mapping

$$\varphi\colon H^p_c(X,(F_n)_{n\in\mathbb{N}}) \longrightarrow \varprojlim_n H^p_c(X,F_n)$$

bijective?

If X is quasi-compact then φ is bijective. (This can be shown by the same arguments as in the algebraic case). But if X is not quasi-compact then φ is not bijective in general. In this paper we give some examples of non quasi-compact rigid analytic varieties X for which φ is bijective. Namely we will show

Suppose that X is an open subvariety of some quasi-compact separated rigid analytic variety Y such that X is Zariski-open in Y or $X = \{y \in Y \mid |f_1(y)| < 1,\ldots,|f_n(y)| < 1\}$ with $f_1,\ldots,f_n \in \mathcal{O}_Y(Y)$. Furthermore suppose that the constructible ℓ-adic sheaf $(F_n)_n$ on X extends to a constructible ℓ-adic sheaf on Y. Assume $\mathrm{char}(k) = 0$ and $\mathrm{char}(k^\circ/k^{\circ\circ}) \neq \ell$. Then φ is bijective.

As a consequence of this result we will obtain the following comparison theorem: Let X be a separated scheme of finite type over k and let X^{rig} be the associated

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rigid analytic variety over k. Let $(F_n)_{n\in\mathbb{N}}$ be a constructible ℓ-adic sheaf on X and let $(F_n^{\mathrm{rig}})_{n\in\mathbb{N}}$ be the associated ℓ-adic sheaf on X^{rig}. Assume $\mathrm{char}(k) = 0$ and $\mathrm{char}(k^\circ/k^{\circ\circ}) \neq \ell$. Then $H_c^p(X,(F_n)_n) \cong H_c^p(X^{\mathrm{rig}},(F_n^{\mathrm{rig}})_n)$.

In [H₁] we defined a certain type of analytic spaces which we call analytic adic spaces. The category of rigid analytic varieties is naturally isomorphic to a full subcategory of the category of analytic adic spaces. For many definitions, constructions and arguments of this paper it is more natural and sometimes even indispensable to use analytic adic spaces. Therefore we will apply the étale cohomology of adic spaces ([H]).

In Section 1 we will define compactly supported cohomology of ℓ-adic sheaves on rigid analytic varieties and analytic adic spaces. In Section 2 we will note some properties of this cohomology. In Sections 3 and 4 we will prove the results mentioned above.

For the whole paper we fix an algebraically closed non-archimedean field k and a complete discrete valuation ring R with maximal ideal $\mathfrak{m}$ such that $\mathrm{char}(R/\mathfrak{m}) > 0$.

1. Definition of cohomology with compact support for $R.$-modules

Let X be a rigid analytic variety or an adic space. By an $R.$-module on the étale site $X_{\text{ét}}$ of X we mean a projective system

$$\to F_{n+1} \to F_n \to \cdots \to F_1$$

of R-modules on $X_{\text{ét}}$ with $\mathfrak{m}^n \cdot F_n = 0$ for every $n \in \mathbb{N}$. Let $\mathrm{mod}(X_{\text{ét}} - R.)$ denote the category of $R.$-modules on $X_{\text{ét}}$.

In [H] the compactly supported cohomology for $R/\mathfrak{m}^n$-modules on analytic adic spaces is defined. In this paragraph we will define the compactly supported cohomology for $R.$-modules on analytic adic spaces. More precisely, we will define, for every taut separated adic space locally of $^+$weakly finite type over $\mathrm{Spa}(k,k^\circ)$ ([H, 1.2.1, 1.3.1, 5.1.2]) and every $R.$-module $F = (F_n)_{n\in\mathbb{N}}$ on $X_{\text{ét}}$ and every $p \in \mathbb{N}_0$, the compactly supported cohomology $H_c^p(X,F)$ of X with values in F which is an R-module. (Instead of adic spaces over $\mathrm{Spa}(k,k^\circ)$ one could consider, more generally, pseudo-adic spaces over analytic geometric points [H, 1.10.3, 2.5.1]).

Once one has defined the compactly supported cohomology for $R.$-modules on analytic adic spaces one can define the compactly supported cohomology for $R.$-modules $(F_n)_{n\in\mathbb{N}}$ on taut separated rigid analytic varieties X over k as follows: With X one can associate a taut separated adic space X^{ad} locally of finite type over $\mathrm{Spa}(k,k^\circ)$ ([H, 1.1.11]). The étale toposes of X and X^{ad} are naturally isomorphic ([H, 2.1.4]). Put $H_c^p(X,(F_n)_n) := H_c^p(X^{\mathrm{ad}},(F_n^{\mathrm{ad}})_n)$.

The definition of compactly supported cohomology for $R.$-modules on analytic adic spaces follows the algebraic pattern [J], [E].

Recall that for a quasi-compact scheme X the global section functor for $R.$-modules is defined by

$$\begin{aligned} \Gamma\colon \mathrm{mod}(X_{\text{ét}} - R.) &\longrightarrow \mathrm{mod}(R) \\ (F_n)_n &\longmapsto \Gamma(X, \varprojlim_n F_n) \end{aligned} \tag{1.1}$$

and the R-adic cohomology

$$H^p(X, (F_n)_n) = R^p\Gamma(X, (F_n)_n) \tag{1.2}$$

is the derived functor of Γ. For a separated scheme X of finite type over k one puts

$$H_c^p(X, (F_n)_n) := H^p(\bar{X}, (j_!F_n)_n), \tag{1.3}$$

where $j\colon X \hookrightarrow \bar{X}$ is a compactification of X.

Now we come to the analytic adic situation. A separated adic space X locally of $^+$weakly finite type over $\mathrm{Spa}(k, k^\circ)$ is called complete if X is quasi-compact and the structure morphism $X \to \mathrm{Spa}(k, k^\circ)$ is universally closed ([H, 1.3.2]) and it is called partially complete if, for every quasi-compact subset T of X, the closure $\bar{T}$ of T in X is complete ([H, 1.3.3, 1.3.4, 1.3.13]). For every taut separated adic space X locally of $^+$weakly finite type over $\mathrm{Spa}(k, k^\circ)$ there exists an open embedding $j\colon X \to \bar{X}$ where $\bar{X}$ is an adic space which is partially complete over $\mathrm{Spa}(k, k^\circ)$ ([H, 5.1.5]). j can be chosen to be quasi-compact. Moreover, if X is quasi-compact then $\bar{X}$ can be chosen to be complete. Therefore for quasi-compact X one can define $H_c^p(X, (F_n)_{n\in\mathbb{N}})$ analogously to (1.1)–(1.3). But one can also define $H_c^p(X, (F_n)_{n\in\mathbb{N}})$, more generally, for taut X. For this one only has to replace the global section functor for $R.$-modules on complete adic spaces by the global section functor with compact support for $R.$-modules on partially complete adic spaces. To be precise, the definition is as follows. First let X be a partially complete adic space over $\mathrm{Spa}(k, k^\circ)$. The global section functor with compact support for $R.$-modules on $X_{\text{ét}}$ is defined according to (1.1) by

$$\begin{aligned} \Gamma_c\colon \mathrm{mod}(X_{\text{ét}} - R.) &\longrightarrow \mathrm{mod}(R) \\ (F_n)_n &\longmapsto \Gamma_c\left(X, \varprojlim_n F_n\right), \end{aligned}$$

where $\Gamma_c(X, \varprojlim_n F_n)$ denotes the R-module of all global sections $s \in \Gamma(X, \varprojlim_n F_n)$ whose support $\mathrm{supp}(s) \subseteq X$ is complete over $\mathrm{Spa}(k, k^\circ)$ ([H, 5.2.1]). According to (1.2), the compactly support cohomology

$$H_c^p(X, (F_n)_n) := R^p\Gamma_c(X, (F_n)_n)$$

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is defined as the derived functor of Γ_c. Now let X be a taut separated adic space locally of $^+$weakly finite type over $\mathrm{Spa}(k, k^\circ)$. Then we choose a quasi-compact open embedding $j: X \to \bar{X}$ where $\bar{X}$ is partially complete over $\mathrm{Spa}(k, k^\circ)$, and we put according to (1.3)

$$H_c^p(X, (F_n)_n) := H_c^p(\bar{X}, (j_! F_n)_n).$$

Up to natural isomorphism, this definition is independent of the choice of the quasi-compact open embedding $j: X \to \bar{X}$, see Lemma (1.4) below. (But in order to get this independence we have to restrict ourselves to quasi-compact j, see Example (2.7.v) below). At the end of this paragraph we will compare the above definition of compactly supported cohomology for R.-modules with Berkovich's definition in [B_1].

LEMMA 1.4. *Let X, P, Q be adic spaces over $\mathrm{Spa}(k, k^\circ)$ with P, Q partially complete, let $a: X \hookrightarrow P$ and $b: X \hookrightarrow Q$ be quasi-compact open embeddings, and let $(F_n)_n$ be an R.-module on $X_{\text{ét}}$. Then for every $m \in \mathbb{N}_0$ there is a natural isomorphism*

$$H_c^m(P, (a_! F_n)_n) \cong H_c^m(Q, (b_! F_n)_n).$$

Proof. We may assume that $a: X \hookrightarrow P$ is a universal partial compactification of X ([H, 5.1.5]). So there is a unique morphism $g: P \to Q$ of adic spaces over $\mathrm{Spa}(k, k^\circ)$ with $b = g \circ a$. We have

(I) g induces a homeomorphism from P onto the closure $\overline{b(X)}$ of $b(X)$ in Q. For every $p \in P$ the mapping between the residue fields $k(g(p)) \to k(p)$ induces an isomorphism between the completions $k(g(p))^\wedge \to k(p)^\wedge$.

Proof of (I): First we show that g is quasi-compact and injective. Let U be a quasi-compact open subset of Q. Since every point of P is a specialization of a point of $a(X)$, we obtain $g^{-1}(U) \subseteq \overline{a(b^{-1}(U))}$. Since b is quasi-compact, $a(b^{-1}(U))$ is quasi-compact and then by [H, 1.3.13] $\overline{a(b^{-1}(U))}$ is quasi-compact. Let V be a quasi-compact open subset of X which contains $\overline{a(b^{-1}(U))}$. By [H, 1.3.14.ii] the restriction $g|V: V \to g(V)$ is a homeomorphism. $g(V)$ is a pro-constructible subset of Q and hence $U \cap g(V)$ is quasi-compact. Thus we obtain that $g^{-1}(U)$ is quasi-compact and that g is injective.

Since P is partially complete and Q is separated, g is partially proper ([H, 1.10.17.vi]). Every quasi-compact partially proper morphism is proper ([H, 1.3.4]). Hence g is proper, in particular g is closed. Thus we see that $g(P) = \overline{b(X)}$ and that $g: P \to g(P)$ is a homeomorphism.

Let p be a point of P. There is a point $p' \in a(X)$ which specializes to p. Then $g(p')$ specializes to $g(p)$ and so we have $k(p)^\wedge \xrightarrow{\sim} k(p')^\wedge$ and $k(g(p))^\wedge \xrightarrow{\sim} k(g(p'))^\wedge$ ([H, 1.1.10.iii]). Hence $k(g(p))^\wedge \xrightarrow{\sim} k(p)^\wedge$. This completes the proof

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of (I).

Put $Z := g(P)$. Let $\mathrm{mod}(Z_{\text{ét}} - R.)$ denote the category of $R.$-modules on the étale site of the pseudo-adic space $Z = (Q, Z)$, and let $H_c^m(Z, -)$ be the m-th derived functor of the functor $\mathrm{mod}(Z_{\text{ét}} - R.) \to \mathrm{mod}(R)$, $(L_n)_n \mapsto \Gamma_c(Z, \varprojlim_n L_n)$.

Let $r: P \to Z$ be the morphism of pseudo-adic spaces given by g. From (I) and [H, 2.3.7] we obtain that the morphism of toposes $(r^*, r_*): P_{\text{ét}}^{\sim} \to Z_{\text{ét}}^{\sim}$ is an equivalence. A subset S of P (resp. Z) is complete if and only if S is quasi-compact and closed in P (resp. Z), because P, Q are partially complete and Z is closed in Q. Since r is a homeomorphism, we obtain that a subset $T \subseteq P$ is complete if and only if $r(T) \subseteq Z$ is complete. Hence, for every $R.$-module $(E_n)_n$ on $P_{\text{ét}}$ and every $m \in \mathbb{N}_0$, we have

(II) $\quad H_c^m(P, (E_n)_n) = H_c^m(Z, (r_* E_n)_n).$

Let $i: Z \to Q$ be the inclusion. Since Z is closed in Q, we have for every $R.$-module $(L_n)_{n \in \mathbb{N}}$ on $Z_{\text{ét}}$ and every $m \in \mathbb{N}_0$,

(III) $\quad H_c^m(Z, (L_n)_n) = H_c^m(Q, (i_* L_n)_n).$

From (II) and (III) we obtain

$$H_c^m(P, (E_n)_n) = H_c^m(Q, (g_* E_n)_n).$$

In particular we have

$$H_c^m(P, (a_! F_n)_n) = H_c^m(Q, (b_! F_n)_n).$$

□

In [B] Berkovich defines k-analytic spaces and a functor $X \mapsto X_0$ from the category of hausdorff strictly k-analytic spaces to the category of rigid analytic varieties over k. With every étale sheaf F on X one can associate an étale sheaf F_0 on X_0. In [B_1] Berkovich defines cohomology with compact support for $R.$-modules on k-analytic spaces. Let X be a hausdorff strictly k-analytic space and let $(F_n)_n$ be an $R.$-module on $X_{\text{ét}}$. Then we get the rigid analytic variety X_0 and the $R.$-module $(F_{n,0})_n$ on $(X_0)_{\text{ét}}$. In general, $H_c^p(X, (F_n)_n)$ and $H_c^p(X_0, (F_{n,0})_n)$ are not isomorphic ([H, 0.7.16]). But if X is closed ([B, 1.5.3.iii]) then there is a natural isomorphism $H_c^p(X, (F_n)_n) \xrightarrow{\sim} H_c^p(X_0, (F_{n,0})_n)$ as is shown in the following proposition.

PROPOSITION 1.5. *Let X be a hausdorff strictly k-analytic space which is closed and let $(F_n)_n$ be an $R.$-module on $X_{\text{ét}}$. Then for every $p \in \mathbb{N}_0$,*

$$H_c^p(X, (F_n)_n) = H_c^p(X_0, (F_{n,0})_n).$$

Proof. The rigid analytic variety X_0 associated with X is partially complete. Let $\mathbb{U}$ be the set of all open subsets U of X such that the closure $\bar{U}$ is compact and the morphism of rigid analytic varieties $U_0 \to X_0$ is an open embedding. By [B$_1$], $H_c^p(X, -)$ is the p-th derived functor of

$$\text{(I)}\qquad \begin{aligned} \Gamma_c\colon \mathrm{mod}(X_{\text{ét}} - R.) &\longrightarrow \mathrm{mod}(R) \\ (F_n)_n &\longmapsto \varinjlim_{U\in\mathbb{U}} \varprojlim_{n\in\mathbb{N}} \Gamma_c(U, F_n). \end{aligned}$$

For every $U \in \mathbb{U}$ we have $\Gamma_c(U, F_n) \xrightarrow{\sim} \Gamma_c(U_0, F_{n,0})$ ([H, 8.3.6]). Therefore $\Gamma_c(X, (F_n)_n) \xrightarrow{\sim} \Gamma_c(X_0, (F_{n,0})_n)$ (see (2.2.2) below). Hence if ε denotes the functor $\mathrm{mod}(X_{\text{ét}} - R.) \to \mathrm{mod}(X_0)_{\text{ét}} - R.), (F_n)_n \mapsto (F_{n,0})_n$ then the functor Γ_c from (I) is naturally isomorphic to the functor $\Gamma_c \circ \varepsilon$. Thus Proposition (1.5) is proved once we have seen that $R^+(\Gamma_c \circ \varepsilon) \xrightarrow{\sim} R^+\Gamma_c \circ R^+\varepsilon = R^+\Gamma_c \circ \varepsilon$. For every $R.$-module $(F_n)_n$ on $X_{\text{ét}}$, there is a monomorphism from $(F_n)_n$ to an $R.$-module $(I_n)_n$ such that there exists a family $(J_n, n \in \mathbb{N})$ such that each J_n is an injective $R/\mathfrak{m}^n$-module, $I_n = \prod_{s=1}^n J_s$ and $I_n \to I_{n-1}$ is the projection. For every $p, n \in \mathbb{N}$ and $U \in \mathbb{U}$, $H_c^p(U, I_n) = 0$. Then by [H, 8.3.6] $H_c^p(U_0, I_{n,0}) = 0$. Hence by Corollary (2.4.iii) below, $\varepsilon((I_n)_n) = (I_{n,0})_n$ is Γ_c-acyclic. □

2. Some general properties of the compactly supported cohomology of $R.$-modules

In the following proposition we describe some functorial properties.

PROPOSITION 2.1. *Let X and Y be taut separated adic spaces locally of $^+$weakly finite type over Spa(k, k°).*

(i) *If $f\colon X \to Y$ is a proper morphism then, for every $R.$-module $(F_n)_n$ on $Y_{\text{ét}}$ and every $p \in \mathbb{N}_0$, there is a natural morphism $H_c^p(Y, (F_n)_n) \to H_c^p(X, (f^*F_n)_n)$.*
(ii) *If $f\colon X \to Y$ is an open embedding then, for every $R.$-module $(F_n)_n$ on $Y_{\text{ét}}$ and every $p \in \mathbb{N}_0$, there is a natural morphism $H_c^p(X, (f^*F_n)_n) \to H_c^p(Y, (F_n)_n)$.*
(iii) *If $f\colon X \to Y$ is an open embedding then, for every $R.$-module $(F_n)_n$ on $X_{\text{ét}}$ and every $p \in \mathbb{N}_0$, there is a natural morphism $H_c^p(X, (F_n)_n) \to H_c^p(Y, (f_!F_n)_n)$.*
(iv) *Let $\mathbb{U}$ be an open covering of X such that every $U \in \mathbb{U}$ is taut and for every $U, V \in \mathbb{U}$ there exists a $W \in \mathbb{U}$ with $U \cup V \subseteq W$. Then, for every $R.$-module $(F_n)_n$ on $X_{\text{ét}}$ and every $p \in \mathbb{N}_0$, the mapping induced by (ii)*

$$\varinjlim_{U\in\mathbb{U}} H_c^p(U, (F_n|U)_n) \longrightarrow H_c^p(X, (F_n)_n)$$

is bijective.

Proof. (i) is obvious if Y is partially complete. For general Y, use universal partial compactifications of X and Y.

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The mapping in (ii) can easily be constructed if either f is quasi-compact or X is partially complete. Obviously assertion (iv) holds if X and all $U \in \mathbb{U}$ are partially complete.

Now we prove (iv) under the assumption that every $U \in \mathbb{U}$ is quasi-compact. Let $j: X \hookrightarrow \bar{X}$ be a partial compactification of X such that j is quasi-compact, and put $(G_n)_n := (j_! F_n)_n$. Let $\mathbb{V}$ be the set of all quasi-compact open subsets of $\overline{X}$. For every $V \in \mathbb{V}$, $V \cap X$ is quasi-compact and we have $H_c^p(V \cap X, (F_n|V \cap X)_n) = H_c^p(V, (G_n|V)_n)$. Therefore, we have to show that

$$\varinjlim_{V \in \mathbb{V}} H_c^p(V, (G_n|V)_n) \longrightarrow H_c^p(\overline{X}, (G_n)_n)$$

is bijective. Let $\mathbb{W}$ be the set of all partially complete open subsets of $\overline{X}$ which are contained in a quasi-compact open subset of $\overline{X}$. Then for every $V \in \mathbb{V}$ there is a $W \in \mathbb{W}$ with $V \subseteq W$ ([H, 5.3.3.ii]). Since we already know (iv) in the partially complete case, the mapping

$$\varinjlim_{W \in \mathbb{W}} H_c^p(W, (G_n|W)_n) \longrightarrow H_c^p(\overline{X}, (G_n)_n)$$

is bijective. Hence $\varinjlim_{V \in \mathbb{V}} H_c^p(V, (G_n|V)_n) \to H_c^p(\overline{X}, (G_n)_n)$ is bijective. Thus we have proved that (iv) holds if every $U \in \mathbb{U}$ is quasi-compact.

Now we can construct, for an arbitrary open embedding $f: X \to Y$, the mapping of (ii). Let $(X_i)_{i \in I}$ be the family of all quasi-compact open subsets of X. Since the morphisms $f|X_i: X_i \to Y$ are quasi-compact, we have the mappings $H_c^p(X_i, (f^*F_n|X_i)_n) \to H_c^p(Y, (F_n)_n)$ which induce a mapping $H_c^p(X, (f^*F_n)_n) = \varinjlim_{i \in I} H_c^p(X_i, (f^*F_n|X_i)_n) \to H_c^p(Y, (F_n)_n)$. With this construction, obviously (iv) holds.

The mapping in (iii) can easily be constructed if X (and hence f) is quasi-compact. For general X, cover X by quasi-compact open subsets and apply (iv). $\square$

2.2. Let X be a partially complete adic space over $\mathrm{Spa}(k, k^\circ)$ and let $\Gamma_c: \mathrm{mod}(X_{\text{ét}} - R.) \to \mathrm{mod}(R)$ be the global section functor with compact support for $R.$-modules on $X_{\text{ét}}$ as defined in Paragraph 1. This functor can be factorized as follows: First we introduce some categories and functors. Let $\mathrm{mod}(R.) = \mathrm{mod}(\mathrm{Spa}(k, k^\circ)_{\text{ét}} - R.)$ be the category of projective systems of R-modules $F_1 \leftarrow F_2 \leftarrow \ldots$ with $\mathfrak{m}^n F_n = 0$ for every $n \in \mathbb{N}$, let *Ind*(mod$(R.)$) be the Ind-category of mod$(R.)$, and let $\mathbb{U}$ be an open covering of X such that every $U \in \mathbb{U}$ is taut and is contained in a quasi-compact open subset of X and such that for every $U, V \in \mathbb{U}$ there is a $W \in \mathbb{U}$ with $U \cup V \subseteq W$ (thus $\mathbb{U}$ gives a filtered category). We have the functors

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$$\pi_* = (\pi_X)_*\colon \mathrm{mod}(X_{\text{ét}} - R.) \longrightarrow \mathrm{mod}(X_{\text{ét}} - R), (F_n)_n \longmapsto \varprojlim_n F_n$$

$$\varrho := (\pi_{\mathrm{Spa}(k,k^\circ)})_*\colon \mathrm{mod}(R.) \longrightarrow \mathrm{mod}(R), (F_n)_n \longmapsto \varprojlim_n F_n$$

$$\mathit{Ind}\,(\varrho)\colon \mathit{Ind}\,(\mathrm{mod}(R.)) \longrightarrow \mathit{Ind}\,(\mathrm{mod}(R))$$

$$\varinjlim\colon \mathit{Ind}\,(\mathrm{mod}(R)) \longrightarrow \mathrm{mod}(R), (F_i)_{i\in I} \longmapsto \varinjlim_i F_i$$

$$\sigma\colon \mathrm{mod}(X_{\text{ét}} - R.) \longrightarrow \mathit{Ind}\,(\mathrm{mod}(R.)), (F_n)_n \longmapsto (\Gamma_c(U, F_n)_n)_{U\in\mathbb{U}}$$

$$\Gamma_! = \Gamma_c\colon \mathrm{mod}(X_{\text{ét}} - R) \longrightarrow \mathrm{mod}(R), F \longmapsto \Gamma_c(X, F).$$

The functor Γ_c: $\mathrm{mod}(X_{\text{ét}} - R.) \to \mathrm{mod}\ (R)$ has the following two factorizations,

$$\Gamma_c = \Gamma_! \circ \pi_* \tag{2.2.1}$$

$$\Gamma_c = \varinjlim \circ \mathit{Ind}\,(\varrho) \circ \sigma. \tag{2.2.2}$$

LEMMA 2.3. *In the situation of (2.2) we have*

(i) $R^+\Gamma_c = R^+\Gamma_! \circ R^+\pi_*$ *and* $R^+\Gamma_c = R^+ \varinjlim \circ R^+ Ind\,(\varrho) \circ R^+\sigma$.

(ii) *For every* $R.$*-module* $F = (F_n)_n$ *on* $X_{\text{ét}}$ *and every* $p \in \mathbb{N}_0$*, we have* $R^p\sigma(F) = (H^p_c(U, F_n)_n)_{U\in\mathbb{U}}$.

(iii) *For every* $p \in \mathbb{N}_0$, $R^p(\mathit{Ind}\,(\varrho)) = \mathit{Ind}\ (R^p\varrho)$.

Proof. (iii) is a general fact on Ind-functors.

(ii) For every $p \in \mathbb{N}_0$, let T^p denote the functor $\mathrm{mod}(X_{\text{ét}} - R.) \to$ *Ind* (mod $(R.)), (F_n)_n \mapsto (H^p_c(U, F_n)_n)_{U\in\mathbb{U}}$. The family of functors $(T^p)_{p\in\mathbb{N}_0}$ is an exact δ-functor. For every object $(F_n)_n$ of mod $(X_{\text{ét}} - R.)$, there is a monomorphism $(F_n)_n \to (I_n)_n$ from $(F_n)_n$ to an object $(I_n)_n$ of $\mathrm{mod}(X_{\text{ét}} - R.)$ of the following type: There is a family $(J_n, n \in \mathbb{N})$ where each J_n is an injective $R/\mathfrak{m}^n$-module on $X_{\text{ét}}$ such that $I_n = \prod_{s=1}^n J_s$ and $I_n \to I_{n-1}$ is the projection.

For every $U \in \mathbb{U}$ there is a $U' \in \mathbb{U}$ and a partially complete open subspace V of X with $U \subseteq V \subseteq U'$ ([H, 5.3.3.ii]). For every $p \in \mathbb{N}$ and $n \in \mathbb{N}$, we have $H^p_c(V, I_n) = 0$. Therefore $T^p((I_n)_n) = 0$ for every $p \in \mathbb{N}$. This shows that T^p is the p-th derived functor of $T^0 = \sigma$.

(i) Let $(I_n)_n$ with $I_n = \prod_{s=1}^n J_s$ be as in the proof of (ii). Then $(I_n)_n$ is an injective object of $\mathrm{mod}(X_{\text{ét}} - R.)$. Furthermore, $\pi_*((I_n)_n) = \prod_{s\in\mathbb{N}} J_s$ is $\Gamma_!$-acyclic ([H, 5.3.6]) and $\sigma((I_n)_n)$ is $Ind(\varrho)$-acyclic (by (iii)). This shows that $R^+(\Gamma_! \circ \pi_*) = R^+\Gamma_! \circ R^+\pi_*$ and $R^+(\mathit{Ind}\,(\varrho) \circ \sigma) = R^+\ \mathit{Ind}\,(\varrho) \circ R^+\sigma$. □

COROLLARY 2.4. *Let* X *be a taut separated adic space locally of* $^+$*weakly finite type over* Spa(k, k°) *and let* $(F_n)_n$ *be an* $R.$*-module on* $X_{\text{ét}}$.

(i) *If* X *is quasi-compact then, for every* $p \in \mathbb{N}_0$*, there is an exact sequence*

$$0 \longrightarrow \varprojlim_n{}^{(1)}\ H^{p-1}_c(X, F_n) \longrightarrow H^p_c(X, (F_n)_n) \longrightarrow \varprojlim_n H^p_c(X, F_n) \longrightarrow 0.$$

In particular, if $H_c^{p-1}(X, F_n)$ is a finitely generated R-module for every $n \in \mathbb{N}$ then

$$H_c^p(X, (F_n)_n) \xrightarrow{\sim} \varprojlim_n H_c^p(X, F_n)$$

(cf. (3.1)).

(ii) *For every $p > 2 \cdot \dim.\mathrm{tr}(X/k) + 1$, $H_c^p(X, (F_n)_n) = 0$. If, for every quasi-compact open subset U of X and every $n \in \mathbb{N}$, the R-module $H_c^r(U, F_n)$ with $r = 2 \cdot \dim.\mathrm{tr}(X/k)$ is finitely generated, then $H_c^p(X, (F_n)_n) = 0$ for every $p > 2 \cdot \dim.\mathrm{tr}(X/k)$.*

(iii) *Suppose there exists an open covering $\mathbb{U}$ of X such that, for every $U, V \in \mathbb{U}$, there is a $W \in \mathbb{U}$ with $U \cup V \subseteq W$ and, for every $U \in \mathbb{U}$, U is taut and is contained in a quasi-compact open subset of X and $H_c^p(U, F_n) = 0$ for every $p, n \in \mathbb{N}$ and $H_c^0(U, F_{n+1}) \to H_c^0(U, F_n)$ is surjective for every $n \in \mathbb{N}$. Then $H_c^p(X, (F_n)_n) = 0$ for every $p \in \mathbb{N}$.*

Proof. (i) We choose a compactification $j: X \hookrightarrow \bar{X}$ of X such that j is quasi-compact. Applying Lemma (2.3) to the R.-module $(j_! F_n)_n$ on $\bar{X}_{\text{ét}}$ and the covering $\mathbb{U} = \{\bar{X}\}$, we obtain the assertion.

(ii) For every taut open subset U of X and every $n \in \mathbb{N}$ and every $p > 2 \cdot \dim.\mathrm{tr}(X/k)$, we have $H_c^p(U, F_n) = 0$ ([H, 5.5.8]). Hence for quasi-compact X the assertion follows from (i). For arbitrary X, apply Proposition (2.1.iv).

(iii) We choose a dense quasi-compact open embedding $j: X \hookrightarrow \bar{X}$ where $\bar{X}$ is partially complete. There is an open covering $\mathbb{V}$ of $\bar{X}$ such that, for every $U, V \in \mathbb{V}$, there is a $W \in \mathbb{V}$ with $U \cup V \subseteq W$ and, for every $V \in \mathbb{V}$, V is taut and is contained in a quasi-compact open subset of $\bar{X}$ and $V \cap X \in \mathbb{U}$. Put $G_n := j_! F_n$. Then $H_c^p(V, G_n) = H_c^p(V \cap X, F_n)$ for every $V \in \mathbb{V}$, $n \in \mathbb{N}$, $p \in \mathbb{N}_0$. Hence, for every $V \in \mathbb{V}$, $H_c^p(V, G_n) = 0$ for every $p, n \in \mathbb{N}$ and $H_c^0(V, G_{n+1}) \to H_c^0(V, G_n)$ is surjective for every $n \in \mathbb{N}$. Then Lemma (2.3) implies $0 = H_c^p(\bar{X}, (G_n)_n) = H_c^p(X, (F_n)_n)$ for every $p \in \mathbb{N}$. $\square$

PROPOSITION 2.5. *Let X be a taut separated adic space locally of $^+$weakly finite type over Spa(k, k°) with $\dim.\mathrm{tr}(X/k) < \infty$, let $(U_i)_{i \in I}$ be a covering of X by taut open subsets, and let $F = (F_n)_n$ be an R.-module on $X_{\text{ét}}$. For every finite subset J of I, put $U_J := \bigcap_{i \in J} U_i$ and let $e_J: U_J \to X$ be the inclusion.*

(i) *There is a spectral sequence* $(p \leqslant 0, q \geqslant 0)$

$$E_1^{pq} = \bigoplus_{\substack{J \subseteq I \\ |J| = -p+1}} H_c^q(U_J, F|U_J) \Rightarrow H_c^{p+q}(X, F).$$

(ii) *If $(U_i)_{i \in I}$ is locally finite then there is a spectral sequence* $(p \leqslant 0, q \geqslant 0)$

$$E_1^{pq} = \bigoplus_{\substack{J \subseteq I \\ |J| = -p+1}} H_c^q(X, (e_{J!}(F_n|U_J))_{n \in \mathbb{N}}) \Rightarrow H_c^{p+q}(X, F).$$

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Proof. (i) We choose a total ordering on the index set I. Let $\mathfrak{V} = (V_i)_{i \in I}$ be a family of subsets of X such that each V_i is a quasi-compact open subset of U_i and $V_i \neq \emptyset$ for only finitely many $i \in I$. By the arguments of [SGA 4, XVII.6.2.10] there is a spectral sequence

$$E_1^{pq} = \bigoplus_{\substack{J \subseteq I \\ |J| = -p+1}} H_c^q(V, d_{J!}(F|V_J)) \Rightarrow H_c^{p+q}(V, F|V),$$

where $V := \bigcup_{i \in I} V_i$, $V_J := \bigcap_{i \in J} V_i$, d_J denotes the open embedding $V_J \to V$ and $d_{J!}(F|V_J)$ denotes the $R.$-module $(d_{J!}(F_n|V_J))_{n \in \mathbb{N}}$ on $V_{\text{ét}}$. Since the open embedding $d_J\colon V_J \to V$ is quasi-compact, we have $H_c^q(V_J, F|V_J) \xrightarrow{\sim} H_c^q(V, d_{J!}(F|V_J))$. So we get a spectral sequence

$$E_1^{pq} = \bigoplus_{\substack{J \subseteq I \\ |J| = -p+1}} H_c^q(V_J, F|V_J) \Rightarrow H_c^{p+q}(V, F|V). \qquad (S_{\mathfrak{V}})$$

By Proposition (2.1.iv) the inductive limit $\varinjlim_{\mathfrak{V}} S_{\mathfrak{V}}$ is the desired spectral sequence.

(ii) For every finite subset J of I put $F_J := (e_{J!}(F_n|U_J))_{n \in \mathbb{N}} \in \mathrm{mod}(X_{\text{ét}} - R.)$. Then for every quasi-compact open subset V of X we have a spectral sequence

$$E_1^{pq} = \bigoplus_{\substack{J \subseteq I \\ |J| = -p+1}} H_c^q(V, F_J|V) \Rightarrow H_c^{p+q}(V, F|V). \qquad (S_V)$$

Again take the inductive limit $\varinjlim_{V} S_V$. $\square$

PROPOSITION 2.6. *Let X be a taut separated adic space locally of $^+$weakly finite type over* Spa(k, k°), *let $(F_n)_n$ be an $R.$-module on $X_{\text{ét}}$, let U be a taut open subspace of X and put $Z := X - U$.*

(i) *Let $j\colon U \to X$ and $i\colon Z \to X$ be the inclusions. Then there is an exact sequence*

$$\begin{aligned} \cdots &\to H_c^p(X, (j_! j^* F_n)_n) \to H_c^p(X, (F_n)_n) \to H_c^p(X, (i_* i^* F_n)_n) \\ &\to H_c^{p+1}(X, (j_! j^* F_n)_n) \to \cdots \end{aligned}$$

(In Section 1 we defined the compactly supported cohomology for $R.$-modules on adic spaces. Analogously one can define the compactly supported cohomology for $R.$-modules on pseudo-adic spaces. Then $H_c^p(X, (i_* i^* F_n)_n) \xrightarrow{\sim} H_c^p(Z, (i^* F_n)_n)$.)

(ii) *Assume that X and U are partially complete. Let $R^+\pi_*$ be the derived functor of π_*: mod$(X_{\text{ét}} - R.) \to$ mod $(X_{\text{ét}} - R), (F_n)_n \mapsto \varprojlim F_n$, let $H_c^p(Z, -)$ be the compactly supported cohomology for R-modules on $Z_{\text{ét}}$ ([H, 5.3.1]) and let $i\colon Z \to X$ be the inclusion. Then we have an exact sequence*

$$\begin{aligned} \cdots &\to H_c^p(U, (F_n|U)_n) \to H_c^p(X, (F_n)_n) \to H_c^p(Z, i^* R^+\pi_*(F_n)_n) \\ &\to H_c^{p+1}(U, (F_n|U)_n) \to \cdots \end{aligned}$$

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Proof. (i) follows from the exact sequence of $R_{.}$-modules on $X_{\text{ét}}$

$$0 \to (j_! j^* F_n)_n \to (F_n)_n \to (i_* i^* F_n)_n \to 0.$$

(iii) We have the exact sequence

$$\begin{aligned}\cdots &\to H_c^p(U, (R^+\pi_* F)|U) \to H_c^p(X, R^+\pi_* F) \to H_c^p(Z, i^* R^+\pi_* F)\\ &\to H_c^{p+1}(U, (R^+\pi_* F)|U) \to \cdots\end{aligned}$$

and by Lemma (2.3.i) we have $H_c^p(X, R^+\pi_* F) = H_c^p(X, F)$ and $H_c^p(U, (R^+\pi_* F)|U) = H_c^p(U, F|U)$. □

EXAMPLE 2.7. We apply Proposition (2.6) to compute some cohomology groups. Let K be a local field and let k be the algebraic closure of K. With the scheme $\mathbb{P}^1$ over Spec$\mathbb{Z}$ we can associate the adic space $(\mathbb{P}^1)^{\text{ad}} := \mathbb{P}^1 \times_{\text{Spec}\mathbb{Z}} \text{Spa}(k, k^\circ)$ over $\text{Spa}(k, k^\circ)$ ([H$_1$, 3.8]). The set $\mathbb{P}^1(K)$ of K-rational points of $\mathbb{P}^1$ can be considered as a subset of $(\mathbb{P}^1)^{\text{ad}}$. This subset is closed in $(\mathbb{P}^1)^{\text{ad}}$ and the subspace topology of $(\mathbb{P}^1)^{\text{ad}}$ on $\mathbb{P}^1(K)$ agrees with the topology on $\mathbb{P}^1(K)$ induced by the absolute value of K. The open subspace $\Omega := (\mathbb{P}^1)^{\text{ad}} - \mathbb{P}^1(K)$ equals the adic space associated with Drinfeld's upper half plane $(\mathbb{P}^1_k)^{\text{rig}} - \mathbb{P}^1(K)$. $(\mathbb{P}^1)^{\text{ad}}$ is complete and Ω is partially complete.

Let $c(R)$ denote the R-module of all constant mappings from $\mathbb{P}^1(K)$ to R, let $\ell c(R)$ denote the R-module of all locally constant mappings from $\mathbb{P}^1(K)$ to R where $\mathbb{P}^1(K)$ is equipped with the subspace topology of $(\mathbb{P}^1)^{\text{ad}}$, and let $s(R)$ denote the R-module of all continuous mappings from $\mathbb{P}^1(K)$ to R where $\mathbb{P}^1(K)$ is equipped with the subspace topology of $(\mathbb{P}^1)^{\text{ad}}$ and R is equipped with the $\mathfrak{m}$-adic topology.

Let $j\colon \Omega \to (\mathbb{P}^1)^{\text{ad}}$ be the inclusion. We consider the $R_{.}$-module $F := (F_n)_n := (R/\mathfrak{m}^n)_n$ on $\Omega_{\text{ét}}$. Assume that $\text{char}(R/\mathfrak{m}) \neq \text{char}(K^\circ/K^{\circ\circ})$. Then we have

(i) The natural mapping $H_c^1((\mathbb{P}^1)^{\text{ad}}, (j_! F_n)_n) \to \varprojlim_n H_c^1(\Omega, F_n)$ is bijective.

(ii) There is a natural isomorphism $s(R)/c(R) \xrightarrow{\sim} \varprojlim_n H_c^1(\Omega, F_n)$.

(iii) There is a natural isomorphism $\ell c(R)/c(R) \xrightarrow{\sim} H_c^1(\Omega, (F_n)_n)$.

In particular, we obtain

(iv) The natural mapping $H_c^1(\Omega, (F_n)_n) \to \varprojlim_n H_c^1(\Omega, F_n)$ is not bijective.

(v) The natural mapping $H_c^1(\Omega, (F_n)_n) \to H_c^1((\mathbb{P}^1)^{\text{ad}}, (j_! F_n)_n)$ is not bijective.

(vi) The R-module $H_c^1(\Omega, (F_n)_n)$ is not finitely generated.

Proof. First we remark that by [H, 5.7.2], for every torsion group L and every $p \in \mathbb{N}_0$, we have $H^p((\mathbb{P}^1)^{\text{ad}}, L) = H^p(\mathbb{P}^1_k, L)$, and thus $H^1((\mathbb{P}^1)^{\text{ad}}, L) = 0$.

(i) Since $H_c^0((\mathbb{P}^1)^{\mathrm{ad}}, j_! F_n) = \Gamma_c(\Omega, F_n) = 0$, we obtain from Corollary (2.4.i), $H_c^1((\mathbb{P}^1)^{\mathrm{ad}}, (j_! F_n)_n) \xrightarrow{\sim} \varprojlim_n H_c^1((\mathbb{P}^1)^{\mathrm{ad}}, j_! F_n) = \varprojlim_n H_c^1(\Omega, F_n)$.

(ii) Since $H_c^1((\mathbb{P}^1)^{\mathrm{ad}}, R/\mathfrak{m}^n) = 0$, we have the exact sequence

$$H^0((\mathbb{P}^1)^{\mathrm{ad}}, R/\mathfrak{m}^n) \to H^0(\mathbb{P}^1(K), R/\mathfrak{m}^n) \to H_c^1(\Omega, F_n) \to 0,$$

i.e. $\ell c(R/\mathfrak{m}^n)/c(R/\mathfrak{m}^n) \xrightarrow{\sim} H_c^1(\Omega, F_n)$. Then $s(R)/c(R) = \varprojlim_n [\ell c(R/\mathfrak{m}^n)/c(R/\mathfrak{m}^n)] \xrightarrow{\sim} \varprojlim_n H_c^1(\Omega, F_n)$.

(iii) Let $\pi_*\colon \mathrm{mod}((\mathbb{P}^1)^{\mathrm{ad}}_{\text{ét}} - R.) \to \mathrm{mod}((\mathbb{P}^1)^{\mathrm{ad}}_{\text{ét}} - R)$ be the functor from (2.6.ii) and let $G = (G_n)_n$ denote the $R.$-module $(R/\mathfrak{m}^n)_n$ on $(\mathbb{P}^1)^{\mathrm{ad}}_{\text{ét}}$. By [E], $R^q\pi_* G$ is the sheaf on $(\mathbb{P}^1)^{\mathrm{ad}}_{\text{ét}}$ associated with the presheaf $U \mapsto H^q(\pi^* U, G)$, and there is an exact sequence

$$0 \to \varprojlim_n{}^{(1)} H^{q-1}(U, G_n) \to H^q(\pi^* U, G) \to \varprojlim_n H^q(U, G_n) \to 0.$$

For every $x \in \mathbb{P}^1(K) \subseteq (\mathbb{P}^1)^{\mathrm{ad}}$, the set of all open disks of $(\mathbb{P}^1)^{\mathrm{ad}}$ containing x is a fundamental system of étale neighbourhoods of x. For every disk U we have $H^0(U, G_n) = R/\mathfrak{m}^n$ and $H^q(U, G_n) = 0$ for $q > 0$ ([H, 3.2.4]). Hence the natural morphism $\pi_* G \to R^+\pi_* G$ induces an isomorphism $i^*\pi_* G \to i^* R^+\pi_* G$ where i denotes the inclusion $\mathbb{P}^1(K) \to (\mathbb{P}^1)^{\mathrm{ad}}$. Since $H^1((\mathbb{P}^1)^{\mathrm{ad}}, G_n) = 0$, we obtain from Corollary (2.4.i)

$$H_c^1((\mathbb{P}^1)^{\mathrm{ad}}, G) = 0.$$

Hence Proposition (2.6.ii) gives the exact sequence

$$H_c^0((\mathbb{P}^1)^{\mathrm{ad}}, G) \to H_c^0(\mathbb{P}^1(K), i^*\pi_* G) \to H_c^1(\Omega, F) \to 0.$$

Since $\pi_* G$ is the constant sheaf on $(\mathbb{P}^1)^{\mathrm{ad}}_{\text{ét}}$ to the group R, we obtain

$$\ell c(R)/c(R) \xrightarrow{\sim} H_c^1(\Omega, F). \qquad \square$$

3. Finiteness

First we introduce constructible sheaves.

Let X be an analytic adic space, let A be a noetherian ring and let F be an A-module on $X_{\text{ét}}$. We call F constructible if for every $x \in X$ there is a locally closed locally constructible subset L of X such that $x \in L$ and the restriction of F to L is locally constant of finite type ([H, 2.7]). We call F quasi-constructible if for every $x \in X$ there exist an étale morphism of adic spaces $g\colon Y \to X$

and a locally closed constructible subset L of Y and a decreasing sequence $Y = Y_0 \supseteq Y_1 \supseteq \ldots \supseteq Y_\ell = \emptyset$ of closed adic subspaces of Y such that $x \in g(L)$ and, for every $i \in \{0, \ldots, \ell - 1\}$, the restriction of F to $L \cap (Y_i - Y_{i+1})$ is locally constant of finite type ([H$_2$]).

We call an R-module $(F_n)_n$ on the étale site of an analytic adic space X constructible (resp. quasi-constructible) if the following three conditions are satisfied

(1) For every $n \in \mathbb{N}$, the morphism $F_{n+1} \to F_n$ is surjective with kernel $\mathfrak{m}^n F_{n+1}$.
(2) For every $n \in \mathbb{N}$, the $R/\mathfrak{m}^n$-module F_n on $X_{\text{ét}}$ is constructible (resp. quasi-constructible).
(3) The ascending chain $(K_n, n \in \mathbb{N})$ of sub-R-modules of F_1 is locally stationary (i.e., there is an open covering $(X_i)_{i \in I}$ of X such that, for every $i \in I$, the chain $(K_n | X_i, n \in \mathbb{N})$ is stationary), where K_n is defined as follows: Let a be a generating element of the maximal ideal of R. By (1) the morphism $\psi_n \colon F_n \to F_n, x \mapsto a^{n-1}x$ factors through a morphism $\varphi_n \colon F_1 \to F_n$. Then $K_n := \ker(\varphi_n)$.

We call an adic space X over Spa(k, k°) locally algebraic if for every $x \in X$ there exist an open neighbourhood U of x in X and a scheme Y of finite type over k such that U is over Spa(k, k°) isomorphic to an open subspace of the adic space $Y \times_{\operatorname{Spec} k} \operatorname{Spa}(k, k^\circ)$ associated with Y.

THEOREM 3.1. *Let X be a separated adic space of finite type over Spa(k, k°) and let $(F_n)_n$ be an R-module on $X_{\text{ét}}$. Suppose that one of the following conditions (a),(b) is satisfied*

(a) *X is locally algebraic,* $\operatorname{char}(R/\mathfrak{m}) \neq \operatorname{char}(k^\circ/k^{\circ\circ})$ *and $(F_n)_n$ is constructible*
(b) $\operatorname{char}(k) = 0$, $\operatorname{char}(R/\mathfrak{m}) \neq \operatorname{char}(k^\circ/k^{\circ\circ})$, *$R/\mathfrak{m}$ is finite and $(F_n)_n$ is quasi-constructible.*

Then, for every $p \in \mathbb{N}_0$, the natural mapping

$$H_c^p(X, (F_n)_n) \longrightarrow \varprojlim_n H_c^p(X, F_n)$$

is bijective. Furthermore, the projective system of R-modules $(H_c^p(X, F_n))_{n \in \mathbb{N}}$ is $AR - \mathfrak{m}$-adic and, for every $n \in \mathbb{N}$, the R-module $H_c^p(X, F_n)$ is finitely generated. (Hence also the R-module $H_c^p(X, (F_n)_n)$ is finitely generated.)

Proof. For every $n \in \mathbb{N}$ and every $p \in \mathbb{N}_0$ the R-module $H_c^p(X, F_n)$ is finitely generated (in case (a) see [H, 6.2.1], in case (b) see [H$_2$, 2.3]). Hence $H_c^p(X, (F_n)_n) \xrightarrow{\sim} \varprojlim_n H_c^p(X, F_n)$ by Corollary (2.4.i). It remains to show that $(H_c^p(X, F_n))_n$ is $AR - \mathfrak{m}$-adic. For this we follow the proof of [FK, 12.15].

By (1) above, the image of the morphism ψ_{n+1} in (3) equals the kernel of the morphism $F_{n+1} \to F_n$. Hence we have an exact sequence

$$0 \longrightarrow F_1/K_{n+1} \longrightarrow F_{n+1} \longrightarrow F_n \longrightarrow 0.$$

Since the sequence $(K_n, n \in \mathbb{N})$ is stationary, we obtain that there exist locally closed constructible subsets $S_1, \ldots, S_m$ of X such that $X = \bigcup_{i=1}^m S_i$ and such that, for every $i \in \{1, \ldots, m\}$, there exist a quasi-compact quasi-separated étale morphism of adic spaces $f\colon Y \to X$ and a decreasing sequence $Y = Y_0 \supseteq Y_1 \supseteq \ldots \supseteq Y_\ell = \emptyset$ of closed adic subspaces of Y such that $S_i \subseteq f(Y)$ and, for every $j \in \{0, \ldots, \ell - 1\}$ and every $n \in \mathbb{N}$, the restriction of F_n to $f^{-1}(S_i) \cap (Y_j - Y_{j+1})$ is a locally constant R-module of finite type. In case that $(F_n)_n$ is a constructible R.-module we may assume that $Y_1 = \emptyset$.

Let $\mathcal{C}$ denote the full subcategory of the category of R-modules on $X_{\text{ét}}$ consisting of those R-modules F on $X_{\text{ét}}$ such that $\mathfrak{m}^p F = 0$ for some $p \in \mathbb{N}$ and, for every $i \in \{1, \ldots, m\}$ and every $j \in \{0, \ldots, \ell - 1\}$, the restriction of F to $f^{-1}(S_i) \cap (Y_j - Y_{j+1})$ is locally constant of finite type. $\mathcal{C}$ is an abelian category. By [H$_2$, 3.5], $f^{-1}(S_i) \cap (Y_j - Y_{j+1})$ has finitely many connected components. Therefore every object of the category $\mathcal{C}$ is noetherian.

Considering the ascending chain of sub-R.-modules $(P_k, k \in \mathbb{N})$ of $(F_n)_n$ with $P_k := \ker\left((F_n)_n \xrightarrow{a^k\cdot} (F_n)_n\right)$ where a is a generating element of the maximal ideal of R, we can conclude from [SGA 5, V. 5.2.1 and 5.2.2] that there exists an exact sequence of R.-modules on $X_{\text{ét}}$

$$0 \longrightarrow (L_n)_n \longrightarrow (F_n)_n \longrightarrow (G_n)_n \longrightarrow 0$$

with the following properties: L_n and G_n are objects of $\mathcal{C}$ for every $n \in \mathbb{N}$, the projective systems $(L_n)_n$ and $(G_n)_n$ in $\mathcal{C}$ are $AR-\mathfrak{m}$-adic ([SGA 5, V. 3.2.2]), there exists a $k \in \mathbb{N}$ such that $\mathfrak{m}^k L_n = 0$ for every $n \in \mathbb{N}$, the kernel of $((G_n)_n \xrightarrow{a\cdot} (G_n)_n)$ is AR-null.

It suffices to show that, for every $p \in \mathbb{N}_0$, the projective systems of R-modules $(H_c^p(X, L_n))_n$ and $(H_c^p(X, G_n))_n$ are $AR - \mathfrak{m}$-adic ([FK, 12.4]). Since $(L_n)_n$ and $(G_n)_n$ are AR-isomorphic to $\mathfrak{m}$-adic projective systems in $\mathcal{C}$, we may assume that $(L_n)_n$ and $(G_n)_n$ are $\mathfrak{m}$-adic.

Since $\mathfrak{m}^k L_n = 0$ for every $n \in \mathbb{N}$, we have $L_{n+1} \xrightarrow{\sim} L_n$ for every $n \geqslant k$. Then obviously the projective system $(H_c^p(X, L_n))_n$ is $AR - \mathfrak{m}$-adic.

We show that $(H_c^p(X, G_n))_n$ is $AR - \mathfrak{m}$-adic. Let $j\colon X \hookrightarrow \bar{X}$ be a compactification of X with $\dim.\mathrm{tr}(\bar{X}/k) < \infty$ ([H, 5.1.14]). Then $H_c^p(X, G_n) = H^p(\bar{X}, j_! G_n)$. For every $n \in \mathbb{N}$, let $C(j_! G_n)$ be the Godement resolution of $j_! G_n$ on $\bar{X}$ ([SGA 4, XVII. 4.2.2]). We consider the truncation $\tau_{\leqslant d}(C(j_! G_n))$ with $d := 2 \cdot \dim.\mathrm{tr}(\bar{X}/k) + 2$. The morphisms $G_{n+1} \to G_n$ induce morphisms of complexes of R-modules $\Gamma(\bar{X}, \tau_{\leqslant d}(C(j_! G_{n+1}))) \to \Gamma(\bar{X}, \tau_{\leqslant d}(C(j_! G_n)))$. By construction of $(G_n)_n$ above, the kernel of $(G_n)_n \xrightarrow{a\cdot} (G_n)_n$ is AR-null. This implies that, for every $n \in \mathbb{N}$, G_n is a flat $R/\mathfrak{m}^n$-module. Then also $j_!G_n$ is a flat $R/\mathfrak{m}^n$-module. From this together with the fact that $H^p(\bar{X}, F) = 0$ for every $p > 2\cdot\dim.\mathrm{tr}\,(\bar{X}/k)$ and every torsion sheaf F on $\bar{X}_{\text{ét}}$ ([H, 2.8.3]), one can conclude that every component of the complex $\Gamma(\bar{X}, \tau_{\leqslant d}(C(j_!G_n)))$ is a flat $R/\mathfrak{m}^n$-module and that $\Gamma(\bar{X}, \tau_{\leqslant d}(C(j_!G_{n+1}))) \otimes_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n \to \Gamma(\bar{X}, \tau_{\leqslant d}(C(j_!G_n)))$ is an isomorphism. Now by [FK, 12.5], for every $p \in \mathbb{N}_0$, the projective system of R-modules $(H^p_c(X, G_n))_n$ is $AR - \mathfrak{m}$-adic. $\square$

In the following we will consider closed constructible subsets L of adic spaces X and their interiors L°. For example, if $f_1, \ldots, f_n \in \mathcal{O}_X(X)$ then the set

$$L = \{x \in X \,|\, |f_1(x)| < 1, \ldots, |f_n(x)| < 1\}$$

is closed and constructible in X, and for the interior L° of L in X we have (here we assume that X is an adic space over $\mathrm{Spa}(k, k^\circ)$)

$$L^\circ = \bigcup_{\substack{a \in k^* \\ |a| < 1}} \{x \in X \,|\, |f_1(x)| \leqslant |a(x)|, \ldots, |f_n(x)| \leqslant |a(x)|\}$$

([H_2, 1.3]).

3.2. Let X be a separated adic space of finite type over $\mathrm{Spa}(k, k^\circ)$, let U be a taut open subset of X, let $j\colon U \hookrightarrow X$ be the inclusion and let $(F_n)_n$ be an $R.$-module on $U_{\text{ét}}$ such that there exists a quasi-constructible $R.$-module $(G_n)_n$ on $X_{\text{ét}}$ with $(F_n)_n = (G_n|U)_n$. Assume that $\mathrm{char}(k) = 0$, $\mathrm{char}(R/\mathfrak{m}) \neq \mathrm{char}(k^\circ/k^{\circ\circ})$ and $R/\mathfrak{m}$ is finite. We are interested in the following two statements

(a) For every $p \in \mathbb{N}_0$, the natural mapping $H^p_c(U, (F_n)_n) \to \varprojlim_n H^p_c(U, F_n)$ is bijective and the projective system $(H^p_c(U, F_n))_{n\in\mathbb{N}}$ is $AR - \mathfrak{m}$-adic and every $H^p_c(U, F_n)$ is a finitely generated R-module.

(b) For every $p \in \mathbb{N}_0$, the natural mapping $H^p_c(U, (F_n)_n) \to H^p_c(X, (j_!F_n)_n)$ is bijective.

For example, (a) and (b) hold if U is quasi-compact ((a) follows from (3.1) and (b) follows immediately from the definition of compactly supported cohomology for $R.$-modules). But if U is not quasi-compact then in general neither (a) nor (b) holds (see Example (2.7.iv,v)). In the following theorem we describe two types of open subsets U of X which in general are not quasi-compact and for which (a) and (b) hold.

THEOREM 3.3. *In the situation of (3.2) assume that one of the following conditions is satisfied*

(i) *There is a closed adic subspace Z of X with $U = X - Z$.*

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(ii) *There is a locally closed constructible subset L of X with $U = L^\circ$.*

Then (a) *and* (b) *hold.*

(*Remark.* In Theorem (3.3), statement (b) also holds if X is only locally of finite type over Spa(k, k°). Indeed, if $(X_i)_{i\in I}$ is the family of all quasi-compact open subsets of X then by Proposition (2.1.iv), $H_c^p(U, F) = \varinjlim_{i\in I} H_c^p(U \cap X_i, F|U \cap X_i)$ and $H_c^p(X, j_!F) = \varinjlim_{i\in I} H_c^p(X_i, j_!F|X_i)$).

Proof. We need a result of [H_2] which says

(I) Let X be a separated adic space of finite type over Spa(k, k°) and let U be an open subset of X which satisfies (i) or (ii). Let A be a finite ring whose order is prime to char $(k^\circ/k^{\circ\circ})$. Assume that char$(k) = 0$. Then there is a sequence $(U_i | i \in \mathbb{N})$ of open subsets of X such that the following holds

(α) Every U_i is quasi-compact and $U_i \subseteq U_{i+1}$ and $U = \bigcup_{i\in\mathbb{N}} U_i$.
(β) For every quasi-constructible A-module Q on $X_{\text{ét}}$ there is a $i_0 \in \mathbb{N}$ such that, for every $i \geqslant i_0$ and every $p \in \mathbb{N}_0$, the natural mapping $H_c^p(U_i, Q) \to H_c^p(U, Q)$ is bijective. ([H_2, 2.7 and 2.9])

Proposition (2.1.iv) and Theorem (3.1) imply that in order to show that under the assumptions of Theorem (3.3) the statement (a) holds it suffices to show the following

(II) There is an open covering $(U_i | i \in \mathbb{N})$ of U such that every U_i is quasi-compact, $U_i \subseteq U_{i+1}$ for every $i \in \mathbb{N}$, and for every $i, n \in \mathbb{N}$ and $p \in \mathbb{N}_0$ the natural mapping $H_c^p(U_i, F_n) \to H_c^p(U, F_n)$ is bijective.

We show (II): Let $(K_n, n \in \mathbb{N})$ be the ascending chain of sub-R-modules of F_1 which occurs in condition (3) of the definition of quasi-constructible $R_.$-modules. Then, for every $n \in \mathbb{N}$, we have an exact sequence on $U_{\text{ét}}$

$$0 \longrightarrow F_1/K_{n+1} \longrightarrow F_{n+1} \longrightarrow F_n \longrightarrow 0.$$

Since $(K_n, n \in \mathbb{N})$ is stationary, there is a $m \in \mathbb{N}$ with $K_n = K_m$ for every $n \geqslant m$. It suffices to show that there is an open covering $(U_i | i \in \mathbb{N})$ of U such that every U_i is quasi-compact, $U_i \subseteq U_{i+1}$ for every $i \in \mathbb{N}$, and for every $n \in \{1, \ldots, m-1\}, i \in \mathbb{N}$ and $p \in \mathbb{N}_0$ the mappings $H_c^p(U_i, F_n) \to H_c^p(U, F_n)$ and $H_c^p(U_i, F_1/K_m) \to H_c^p(U, F_1/K_m)$ are bijective. But this follows from (I) above.

Thus we have proved that (a) holds. By the following Lemma (3.4), (a) implies (b). □

LEMMA 3.4. *Let X be a separated adic space of $^+$weakly finite type over Spa(k, k°), let U be a taut open subset of X with inclusion $j: U \hookrightarrow X$ and let $(F_n)_n$ be an*

$R_{\cdot}$-module on $U_{\text{ét}}$. Let p be an element of $\mathbb{N}_0$ such that $\varprojlim_n^{(1)} H_c^{p-1}(U, F_n) = 0$ (for example, this is satisfied if for every $n \in \mathbb{N}$ the R-module $H_c^{p-1}(U, F_n)$ is finitely generated). Then the following two statements are equivalent

(i) *The natural mapping $H_c^p(U, (F_n)_n) \to \varprojlim_n H_c^p(U, F_n)$ is bijective.*

(ii) *The natural mapping $H_c^p(U, (F_n)_n) \to H_c^p(X, (j_! F_n)_n)$ is bijective.*

Proof. We consider the commutative diagram

$$\begin{array}{ccc} H_c^p(U, (F_n)_n) & \xrightarrow{\alpha} & H_c^p(X, (j_! F_n)_n) \\ \Big\downarrow{\scriptstyle\delta} & & \Big\downarrow{\scriptstyle\beta} \\ \varprojlim_n H_c^p(U, F_n) & \xrightarrow[\gamma]{} & \varprojlim_n H_c^p(X, j_! F_n). \end{array}$$

The mapping γ is bijective. Since $\varprojlim_n^{(1)} H_c^{p-1}(X, j_! F_n) = \varprojlim_n^{(1)} H_c^{p-1}(U, F_n) = 0$, we obtain from Corollary (2.4.i) that the mapping β is bijective. Hence α is bijective if and only if δ is bijective. $\square$

4. A comparison theorem

With every scheme X locally of finite type over k one can associate an adic space X^{ad} over $\text{Spa}(k, k^\circ)$ ([H$_1$, 3.8]),

$$X^{\text{ad}} := X \times_{\text{Spec}\, k} \text{Spa}(k, k^\circ).$$

The functor $\text{Ét}/X \to \text{Ét}/X^{\text{ad}}$, $Y/X \mapsto Y^{\text{ad}}/X^{\text{ad}}$ is a morphism of sites

$$\varrho\colon (X^{\text{ad}})_{\text{ét}} \longrightarrow X_{\text{ét}}.$$

For every sheaf F on $X_{\text{ét}}$ put

$$F^{\text{ad}} := \varrho^* F.$$

If X is separated and quasi-compact then, for every $p \in \mathbb{N}_0$ and every A-module F on $X_{\text{ét}}$ where A is a torsion ring, there is a natural isomorphism ([H, 5.7.2])

$$H_c^p(X, F) \xrightarrow{\sim} H_c^p(X^{\text{ad}}, F^{\text{ad}}). \tag{4.1}$$

In the following theorem we extend this isomorphism to $R_{\cdot}$-modules F.

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THEOREM 4.2. *Assume that* $\operatorname{char}(k) = 0$, $\operatorname{char}(R/\mathfrak{m}) \neq \operatorname{char}(k^\circ/k^{\circ\circ})$ *and* $R/\mathfrak{m}$ *is finite. Then, for every separated scheme* X *of finite type over* k *and every constructible* R.*-module* $(F_n)_n$ *on* $X_{\text{ét}}$ *and every* $p \in \mathbb{N}_0$, *there is a natural isomorphism*

$$H_c^p(X, (F_n)_n) \xrightarrow{\sim} H_c^p(X^{\text{ad}}, (F_n^{\text{ad}})_n).$$

(*Remark.* The main ingredient of the proof of Thoerem (4.2) is the result [H_2, 2.7] (cf. proof of Theorem (3.3)). It was already remarked by Berkovich [B_1] that one gets the comparison theorem (4.2) once one has a result like [H_2, 2.7]).

Proof. Let $X \subseteq \bar{X}$ be a compactification of X. There is a constructible R.-module $(G_n)_n$ on $\bar{X}_{\text{ét}}$ with $(F_n)_n = (G_n|X)_n$. The associated R.-module $(G_n^{\text{ad}})_n$ on $(\bar{X}^{\text{ad}})_{\text{ét}}$ is quasi-constructible. (Indeed, condition (3) in the definition of quasi-constructible R.-modules is satisfied, since the R-module G_1 on $\bar{X}_{\text{ét}}$ is noetherian). The adic space $\bar{X}^{\text{ad}}$ is of finite type over $\operatorname{Spa}(k, k^\circ)$ and X^{ad} is a Zariski-open subset of $\bar{X}^{\text{ad}}$. Hence by Theorem (3.3) we have

$$\text{(I)}\quad H_c^p(X^{\text{ad}}, (F_n^{\text{ad}})_n) \xrightarrow{\sim} \varprojlim_n H_c^p(X^{\text{ad}}, F_n^{\text{ad}}).$$

We also have

$$\text{(II)}\quad H_c^p(X, (F_n)_n) \xrightarrow{\sim} \varprojlim_n H_c^p(X, F_n).$$

The isomorphisms (I) and (II) together with the comparison isomorphisms (4.1) induce an isomorphism $H_c^p(X, (F_n)_n) \xrightarrow{\sim} H_c^p(X^{\text{ad}}, (F_n^{\text{ad}})_n)$. $\square$

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