FUNDAMENTAL SOLUTIONS FOR A CLASS OF EQUATIONS WITH SEVERAL SINGULAR COEFFICIENTS

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(Received 3 January 1967)

1. Introduction

This paper is concerned with a class of singular elliptic partial differential equations related to the operator of Weinstein's generalized axially symmetric potential theory (GASPT) [1, 2] which has numerous applications.

Let B_i denote the operator

$$B_i[u] \equiv \frac{\partial^2 u}{\partial x_i^2} + \frac{\alpha_i}{x_i}\frac{\partial u}{\partial x_i}, \qquad i = 1, 2, \cdots, n, \ (n \geqq 2)$$

sometimes referred to as Bessel's operator. The α_i are arbitrary real constants. Our main result consists of explicit formulas for fundamental solutions *in the large* for arbitrary iterates $(L_\alpha+\lambda^2)^s$, $s = 1, 2, \cdots$, of the operator $L_\alpha+\lambda^2$ where

$$L_\alpha \equiv \sum_{i=1}^{n} B_i \tag{1.1}$$

with arbitrary real constant $\lambda \geqq 0$. These fundamental solutions represent the potential of a "ring of sources" in Weinstein's spaces of "fractional dimension" [1] and are related to hypergeometric functions of several variables (see Section 3).

If the first m $(m \geqq 0)$ components α_i of the n-vector α are zero then the operator L_α may be written

$$\begin{aligned} L_k &\equiv \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{l}\left(\frac{\partial^2}{\partial x_{m+i}^2} + \frac{k_i}{x_{m+i}}\frac{\partial}{\partial x_{m+i}}\right) \\ &= \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{l} \frac{k_i}{x_{m+i}}\frac{\partial}{\partial x_{m+i}} \end{aligned} \tag{1.2}$$

[1] Research sponsored in part by the U. S. Atomic Energy Commission under contract with the Union Carbide Corporation and in part by a National Science Foundation Post-Doctoral Fellowship.

where the integers l and m satisfy $l+m = n \geqq 2$, $1 \leqq l \leqq n$, $0 \leqq m < n$, and the components k_i of the l-vector k are arbitrary real numbers. For $l = 1$, L_k is precisely Weinstein's GASPT operator. By renumbering of the independent variables it is seen that (1.1) and (1.2) are equivalent and for certain conveniences in the statement of results we deal henceforth with (1.2).

The operator L_k is elliptic except for $x_{m+i} = 0$, $1 \leqq i \leqq l$ if the corresponding k_i is non-zero. We will restrict our attention to the hyper-octant where $x_{m+1} > 0, \cdots, x_n > 0$.

The results presented here generalize those of Weinacht [3] to which they reduce by putting $l = 1$. For information concerning the many applications of these operators (primarily for $l = 1$), see Weinstein [4, 5] and the references given there. For the less studied case $l > 1$, see also Kapilevich [6, 7] for mean value theorems for $s = 1$ and Gilbert [8] and Gilbert and Howard [9] who apply Bergman's integral operator method to the equation $L_k[u]+\lambda^2 u = 0$ $(s = 1)$ for $n = l = 2$, $k_1 > 0$, $k_2 > 0$ with $\lambda = 0$ in [8] and $\lambda > 0$ in [9]. Fundamental solutions have been constructed for the hyperbolic counterpart of (1.1) (put $t = ix_n$ as a new variable) by Stellmacher [10] (see also Lagnese [11]) but these are not valid in the large. Fox [12] considered the singular Cauchy problem for the hyperbolic counterpart of $L_\alpha[u] = 0$ (see also Copson and Erdelyi [13]).

The results presented here were announced in Weinacht [19].

2. Preliminaries

Let φ be any C^{2s} function (i.e. possessing continuous partial derivatives of order $2s$) in some neighborhood of the point b with $b_{m+1} > 0, \cdots, b_n > 0$. The residue (at the point b with respect to the operator $L \equiv (L_k+\lambda^2)^s$) of a function u is the functional defined on the set of such functions φ by

$$\lim_{\rho\to 0}\int_{|x-b|=\rho} M[u, \varphi]\,ds \tag{2.1}$$

where M is a bilinear differential operator of order $2s-1$ arising from the Green's Identity

$$\int_G (\varphi L[u]-uL^*[\varphi])dy = \int_{\partial G} M[u, \varphi]\,dS$$

in which L^* is the formal adjoint to L. By a fundamental solution in the large of $(L_k+\lambda^2)^s$ with pole b is meant a function $E = E(x, b)$ such that (as a function of x) on the positive hyper-octant

$$P = \{(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) : x_{m+1} > 0, \cdots, x_n > 0\}:$$

1) E is a classical solution of $(L_k+\lambda^2)^s u = 0$ except at b and

2) E has residue *cp{b)*. In addition our fundamental solutions remain defined on the boundary of P.

It is easily seen that a fundamental -solution E as defined here is necessarily locally integrable (absolutely integrable over compact subsets of *P)* and is equivalent to the usual definition (John [14; p. 274]) so that *L[E] = d, 6* denoting Dirac's delta functional defined on testing functions with support contained in P. i

We will use in a later section the following simple lemma (see Wemacht [3]) about fundamental solutions:

LEMMA: *Let A be a linear elliptic operator te/ith* C^oo *coefficients in the hyper-octant P such that A is a product* $A = A_t A_2$ *of operators* A_1 *and* A_2 *satisfying the sathe conditions^as A and of orders* $2t_x$ *and* 2£2<) *respectively. Suppose that for all points in P, except for the single point b, a function E is* C°° *and A%[E] = E^ where* E_x *is a fundamental splution in the large jof* A_x *with pole b. Suppose further that for every derivative* D^2*^-^{1} *E of order %*t_2— 1

$$\lim_{x\to b} Ix\text{-}b]"\text{-}^{1} \; D^{2t}\wedge[E] = 0.$$

Then E is a fundamental solution in the large of A with pole b.

3. The fundamental solutions

In (3.1) through (3.(5) we give explicit formulas for fundamental solutions in the large of $(L_k\text{-}\backslash\text{-}k^2)^3$ with pole *b*. By renumbering of the independent variables all possible Values of the vector parameter *k* are covered. Verifications! are given in subsequent section^.

In addition to the obviqus use of the usual vector notations we make the following conventions. It will be convenient to consider two subsets of the / "singular" variables $x_{m+1}, \cdots, x_n$: (1) the #(0' ≤J <f ^ /) J'positive singular" variables $x_{m+1}, \cdots, x_{m+g}$ with corresponding positive parameters $k_x, \cdots, k_a$ and (2) the remaining *r (0 i^,r f=^ I)* "singular',' variables $ar_{m+8+1}, \cdots, x_n$ with corresponding parameters $k_{g+1}, \cdots, k_l$ (k_t 5S 1 for <7-fl ^LikLl). Either of these subsets could be empty but not both (!≤:/ = *q+r).* Furthermore, let

$$11{*}11{=}2{*}.' \quad 2£ = n\text{-}H|4||\text{-}2s, \quad 2p^* = n + \backslash\backslash k^*\backslash\backslash\text{~}2s>$$

where $k^* = (k_{lt}, \cdots, k_Q, 2\text{-}k_{q+1}, \cdots, 2\text{-}*,),$

$$A(k, s_y\, n, I) = \frac{(-1)^s \Gamma\left(\dfrac{n+||k||-2s}{2}\right) b_{m+1}^{k_1} \cdots b_n^{k_l}}{2^{2s-l} \pi^{n/2} \Gamma\left(\dfrac{k_1}{2}\right) \cdots \Gamma\left(\dfrac{k_l}{2}\right) \Gamma(s)}$$

https://doi.org/10.1017/S1446788700006248 Published online by Cambridge University Press

$$B(k, s, n, l) = \frac{(-1)^{(n+\|k\|-2)/2} b_{m+1}^{k_1} \cdots b_n^{k_l}}{2^{2s-l-1} \pi^{n/2} \Gamma\left(\frac{2s+2-n-\|k\|}{2}\right) \Gamma\left(\frac{k_1}{2}\right) \cdots \Gamma\left(\frac{k_l}{2}\right) \Gamma(s)}$$

$$C(k, s, n, l) = \frac{(-1)^{s-1} (\lambda/2)^p b_{m+1}^{k_1} \cdots b_n^{k_l}}{2^{2s-l} \pi^{(n-2)/2} \Gamma\left(\frac{k_1}{2}\right) \cdots \Gamma\left(\frac{k_l}{2}\right) \Gamma(s)}$$

$$H(k, s, n, l) = \int_0^\pi \cdots \int_0^\pi \sigma^{-2p} \sin^{k_1-1}\theta_1 \cdots \sin^{k_l-1}\theta_l \, d\theta_1 \cdots d\theta_l,$$

$$I(k, s, n, l) = \int_0^\pi \cdots \int_0^\pi \sigma^{-2p} \log \sigma \sin^{k_1-1}\theta_1 \cdots \sin^{k_l-1}\theta_l d\theta_1 \cdots d\theta_l,$$

$$J(k, s, n, l) = \int_0^\pi \cdots \int_0^\pi N_p(\lambda\sigma)\sigma^{-p} \sin^{k_1-1}\theta_1 \cdots \sin^{k_l-1}\theta_l d\theta_1 \cdots d\theta_l,$$

$$\sigma = \left[\sum_{i=1}^m (x_i - b_i)^2 + \sum_{i=1}^l (x_{m+i}^2 + b_{m+i}^2 - 2x_{m+i} b_{m+i} \cos\theta_i)\right]^{\frac{1}{2}}.$$

By N_p is denoted the Neumann function of order p [15; p. 16] and Γ denotes the gamma function.

The formulas are

CASE A$_1$: $\lambda = 0,\ k_1 > 0, \cdots, k_l > 0;\ p \neq 0, -1, -2, \cdots$.

(3.1) $$E_1(k, s, n, l) = A(k, s, n, l) H(k, s, n, l).$$

CASE B$_1$: $\lambda = 0,\ k_1 > 0, \cdots, k_l > 0;\ p = 0, -1, -2, \cdots$.

(3.2) $$E_2(k, s, n, l) = B(k, s, n, l) I(k, s, n, l).$$

CASE C$_1$: $\lambda > 0,\ k_1 > 0, \cdots, k_l > 0$.

(3.3) $$F(k, s, n, l) = C(k, s, n, l) J(k, s, n, l)$$

CASE A$_2$: $\lambda = 0,\ k_1 > 0, \cdots, k_q > 0,\ k_{q+1} \leqq 1, \cdots, k_l \leqq 1;$
$p^* \neq 0, -1, -2, \cdots$.

(3.4) $$E_3(k, s, n, l) = \left(\frac{x_{m+q+1}}{b_{m+q+1}}\right)^{1-k_{q+1}} \cdots \left(\frac{x_n}{b_n}\right)^{1-k_l} E_1(k^*, s, n, l)$$

CASE B$_2$: $\lambda = 0,\ k_1 > 0, \cdots, k_q > 0,\ k_{q+1} \leqq 1, \cdots k_l \leqq 1;$
$p^* = 0, -1, -2, \cdots$.

(3.5) $$E_4(k, s, n, l) = \left(\frac{x_{m+q+1}}{b_{m+q+1}}\right)^{1-k_{q+1}} \cdots \left(\frac{x_n}{b_n}\right)^{1-k_l} E_2(k^*, s, n, l)$$

CASE C$_2$: $\lambda > 0,\ k_1 > 0, \cdots, k_q > 0,\ k_{q+1} \leqq 1, \cdots, k_l \leqq 1$.

(3.6) $$F^*(k, s, n, l) = \left(\frac{x_{m+q+1}}{b_{m+q+1}}\right)^{1-k_{q+1}} \cdots \left(\frac{x_n}{b_n}\right)^{1-k_l} F(k^*, s, n, l).$$

The formulas (3.1) through (3.3) represent the potential of a "poly-ring of sources" in a "fractional dimensional" space (see Weinstein [1]) of $(n+||k||)$-dimensions. Specifically consider the Case A_1 with each k_i a positive integer and $n+||k|| > 2s$. Let ξ and β be $(n+||k||)$-dimensional vectors, $\xi = (x_1, x_2, \cdots, x_m, \xi^1, \cdots, \xi^l)$ and $\beta = (b_1, b_2, \cdots, b_m, \beta^1, \cdots, \beta^l)$, where the ξ^i and β^i are vectors of dimension k_i+1 and $x_{m+i} = |\xi^i|$, $i = 1, 2, \cdots, l$.

One obtains the potential of the poly-ring

$$R = \{\beta : |\beta^1| = b_{m+1}, \cdots, |\beta^l| = b_n\}$$

from the fundamental solution of the iterated Laplace operator with pole β in $(n+||k||)$-space (see e.g. [17; p. 44])

$$E(\xi, \beta) = \frac{(-1)^s \Gamma\left(\dfrac{n+||k||-2s}{2}\right)}{2^{2s}\pi^{n+||k||)/2}\Gamma(s)} \; |\xi-\beta|^{2s-n-||k||}$$

by integration over R

$$\int_R E(\xi, \beta)\, dS$$

which upon simplification yields precisely (3.1) now meaningful also for non-integral k_i. The Cases B_1 and C_1 are similar. Formulas (3.4) through (3.6) then follow from the Correspondence Principle (see Section 6). It is observed that the Cases A and B may be obtained by letting λ tend to zero in the singular part of the functions in case C.

For $l = 1$ these fundamental solutions reduce to the corresponding ones given in Weinacht [3] which in turn were generalizations of the fundamental solutions of Weinstein [1] and Diaz and Weinstein [16] in the case of GASPT.

It is emphasized that the formulas given here apply when all variables are "singular" variables ($m = 0$, $l = n$) as in

$$u_{xx}+u_{yy}+\frac{\alpha}{x}\,u_x+\frac{\beta}{y}\,u_y+\lambda^2 u = 0. \tag{3.7}$$

Moreover the subcases 1 and 2 are not mutually exclusive. For example, in (3.7) if $\lambda > 0$, $0 < \alpha \leqq 1$ and $\beta \leqq 0$ then Case C_2 applies with $q = 0$ or $q = 1$. If $\alpha = 0$ and $0 < \beta < 1$ then one can obtain fundamental solutions in four ways: (1) $m = l = q = 1$, $r = 0$; (2) $m = l = r = 1$, $q = 0$; (3) $m = 0$, $q = r = 1$, $l = 2$; (4) $m = q = 0$, $l = r = 2$. Of course the various formulas need agree only in their singular parts (see John [14]).

The fundamental solution (3.1) can be expressed in terms of hyper-geometric functions of several variables. Put $\tau_i = \sin^2(\theta_i/2)$ in (3.1) and let

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$$(3.8)\qquad \varepsilon_i = (4b_{m+i}x_{m+i})^{-\frac{1}{2}}\,|x-b|, \qquad 1 \leqq i \leqq l$$

to obtain

$$2^{l-\|k\|}\,|x-b|^{2p} H(k, s, n, l)$$
$$= \int_0^1 \cdots \int_0^1 [\tau_1(1-\tau_1)]^{(k_1-2)/2} \cdots [\tau_l(1-\tau_l)]^{(k_l-2)/2} \left[1+\sum_{i=1}^{l} \tau_i/\varepsilon_i^2\right]^{-p} d\tau_1 \cdots d\tau_l$$

so that in terms of the function F_A of Lauricella [18; p. 115] we have

$$(3.9)\qquad \begin{aligned} E_1(k, s, n, l) &= A'(k, s, n, l)\,|x-b|^{-2p} \\ &\quad \cdot F_A\left(p, \frac{k_1}{2}, \cdots, \frac{k_l}{2}, k_1, \cdots, k_l, -\varepsilon_1^{-2}, \cdots, -\varepsilon_l^{-2}\right) \end{aligned}$$

where

$$A'(k, s, n, l) = 2^{\|k\|-l} A(k, s, n, l)\, \frac{\Gamma\left(\frac{k_1}{2}\right)\Gamma\left(\frac{k_1}{2}\right)}{\Gamma(k_1)} \cdots \frac{\Gamma\left(\frac{k_l}{2}\right)\Gamma\left(\frac{k_l}{2}\right)}{\Gamma(k_l)}.$$

In case $n = 2$, the function F_A of Lauricella becomes the function F_2 of Appell for hypergeometric functions of two variables [18].

The behavior of F_A in (3.9) for x near b (i.e. for each ε_i near zero) does not seem to be available in the literature and so our verification in Section 4 investigates the corresponding integral directly. Similar remarks apply to (3.3), (3.4) and (3.6).

4. The verification for $E_1(k, l, n, l)$

By differentiation and a subsequent integration by parts one sees that for $s = 1$, (3.1) is a solution of $L_k[u] = 0$ at all points in the hyper-octant P except at the pole b. Introduction of the variables $t_i = \varepsilon_i^{-1} \sin(\theta_i/2)$ in (3.1) with ε_i as in (3.8), $i = 1, 2, \cdots, l$ yields for $j = 1, 2, \cdots, n$ as x tends to b

(4.1)

$$\begin{aligned} \frac{\partial}{\partial x_j} E_1(k, l, n, l) &= (2-n-\|k\|)(x_j-b_j)|x-b|^{-n}(b_{m+1}x_{m+1})^{-(k_1/2)} \\ &\quad \cdots (b_n x_n)^{-(k_l/2)} A(k, 1, n, l) \int_0^{1/\varepsilon_1} \cdots \int_0^{1/\varepsilon_l} [1+|t|^2]^{-p-1} t_1^{k_1-1} \\ &\quad \cdots t_l^{k_l-1}(1-\varepsilon_1^2 t_1^2)^{(k_1-2)/2} \cdots (1-\varepsilon_l^2 t_l^2)^{(k_l-2)/2} dt_1 \cdots dt_l + o(|x-b|^{1-n}) \end{aligned}$$

and in fact $o(|x-b|^{1-n})$ is identically zero for $1 \leqq j \leqq m$. As x tends to b the integral in (4.1) tends to

$$\frac{2^{-l}\Gamma\left(\frac{k_1}{2}\right)\cdots\Gamma\left(\frac{k_l}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+\|k\|}{2}\right)}$$

as is easily verified by transforming to polar coordinates and evaluating the resulting integrals of beta-function type. Now the residue is readily evaluated to be

$$\lim_{\rho\to 0}\int_{|x-b|=\rho}\varphi\,\frac{\partial E_1}{\partial n}\,ds = \varphi(b).$$

With this the verification in this case is complete.

5. The verification for $F(k, l, n, l)$

Again by direct computation it is seen that for $s = 1$, (3.3) is a solution of $L_k[u]+\lambda_u^2 = 0$ at all points in the hyper-octant P except at the pole. By use of the series expansion of the Neumann function it is seen that the highest order singularity of $F(k, 1, n, l)$ is $E_1(k, 1, n, l)$. More precisely (see [3; equa. (5.3), (5.4)]) as x tends to b

$$F(k, 1, n, l) = E_1(k_1, 1, n, l)+o(E_1(k, 1, n, l))$$

and the only contribution to the residue (with respect to $L_k+\lambda^2$) is due to $E_1(k, 1, n, l)$. But the residues with respect to L_k and $L_k+\lambda^2$ are identical and so our verification is complete.

6. The verification in the general case

The proof for $F(k, s, n, l)$ in (3.3) is by induction in s, the order of the iteration. By Section 5 the assertion is true for $s = 1$.

Direct computation with an integration by parts shows that, except at the pole,

$$(L_k+\lambda^2)F(k, s+1, n, l) = F(k, s, n, l)$$

for every positive integer s. Moreover

$$\begin{aligned}(6.1)\quad \left|\frac{\partial}{\partial x_i}F(k, s+1, n, l)\right| &= |\lambda C(k, s+1, n, l)| \\ &\quad\cdot\left|\int_0^\pi\cdots\int_0^\pi N_p(\lambda\sigma)\sigma^{-p}(x_i-\delta_i b_i)\sin^{k_1-1}\theta_1\cdots\sin^{k_l-1}\theta_l d\theta_1\cdots d\theta_l\right| \\ &\leqq \text{const. } [F(k, s, n, l)+1]\end{aligned}$$

where "const." is independent of x in a sufficiently small neighborhood

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of the pole. Here $\delta_i = 1$ for $1 \leqq i \leqq m$ and $\delta_i = \cos\theta_i$ for $m+1 \leqq i \leqq n$. Because of the behavior of the fundamental solution $F(k, s, n, l)$ near the pole [14; p. 288] we have from (6.1)

$$\lim_{x\to b} |x-b|^{n-1} \frac{\partial}{\partial x_i} F(k, s+1, n, l) = 0.$$

Now the induction proof is completed by taking $A_1 = (L_k+\lambda^2)^s$ and $A_2 = L_k+\lambda^2$ in the Lemma of Section 2.

The proof for (3.1) and (3.2) is similar and involves two cases depending on the value of p. An alternate proof for $s > 1$ (now that the verification is complete for (3.3)) can be based on the fact that as λ tends to zero the singular part of $F(k, s, n, l)$ tends to $E_1(k, s, n, l)$ for $p \neq 0, -1, -2, \cdots$ and $E_2(k, s, n, l)$ for $p = 0, -1, -2, \cdots$. In fact, for $s = 1$, this procedure in reverse was used in the previous section.

The subcases A_2, B_2 and C_2 are now verified by use of the following Correspondence Principle:

A function u is a solution of $(L_{k^*}+\lambda^2)^s[u] = 0$ if and only if $v = x_{m+q+1}^{1-k_{q+1}} \cdots x_m^{1-k_l}\, u$ is a solution of $(L_k+\lambda^2)^s v = 0$.

This principle is an immediate consequence of the identity

$$(L_k+\lambda^2)^s[x_{m+q+1}^{1-k_{q+1}} \cdots x_n^{1-k_l} w] = x_{m+q+1}^{1-k_{q+1}} \cdots x_n^{1-k_l} (L_{k^*}+\lambda^2)^s[w]$$

which is obtained by a repeated application of the identity upon which the Correspondence Principle of GASPT (Weinstein [1]) is based.

By what we have already proved $F(k^*, s, n, l)$ is a solution of $(L_{k^*}+\lambda^2)^s[u] = 0$ except at the pole. Then from the Correspondence Principle $F^*(k, s, n, l)$ is a solution of $(L_k+\lambda^2)^s[u] = 0$ except at the pole. Moreover, the residue of $F^*(k, s, n, l)$ (with respect to $(L_k+\lambda^2)^s$) is identical to the residue of $F(k^*, s, n, l)$ (with respect to $(L_{k^*}+\lambda^2)^s$). From this the assertion follows. The cases of E_2 and E_4 are similar. Note the restriction $k_i \leqq 1$ rather than $k_i < 2$, $q+1 \leqq i \leqq l$, is made solely to insure that the fundamental solutions remain defined on the singular hyperplanes.

7. Concluding remarks

It is implied by some writers that the fundamental solution given in [1] could be obtained by a Fourier transform. However, as the existence of a fundamental solution in the large is not known a priori, it seems a verification of the resulting formula is necessary. The same remark applies to the present paper where a complete verification is given.

Finally let us note that equations such as those considered here are also referred to as degenerate elliptic equations (see, e.g., Keldysh [20]).

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