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WEAK SOLUTIONS OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

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We consider the Cauchy problem $\dot{x}(t) = f(t,x(t))$, $x(0) = x_0$ in a nonreflexive Banach space X and for $f: T \times X \to X$ a weakly continuous vector field. Using a compactness hypothesis involving a weak measure of noncompactness we prove an existence result that generalizes earlier theorems by Chow-Shur, Kato and Cramer-Lakshmikantham-Mitchell.

1. Introduction

In recent years the study of ordinary differential equations in a Banach space has been developed extensively. However almost all of the work was done using the strong topology (see, for example, Deimling [7], Szufla [12]) while the study of Cauchy problems involving the weak topology is lagging behind. In [11] Szep proved a Peano type theorem for o.d.e. defined in a reflexive Banach space and having a weakly continuous vector field. His main tools were the Eberlein-Smulian theorem and the well known fact that in a reflexive Banach space a set is weakly compact if and only if it is weakly closed and norm bounded (a simple consequence of Alaoglu's theorem and the fact that in a reflexive Banach space weak

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and weak* topologies coincide). The result of Szep was extended to non-reflexive Banach spaces by Boundourides [3] and Cramer-Lakshmikantham-Mitchell [5]. Both papers based their existence result on a compactness type condition, involving the weak measure of noncompactness introduced by DeBlasi [6]. It should be noted however that the result of Cramer-Lakshmikantham-Mitchell [5] is more general than that of Boudourides [3]. Furthermore the proof of the theorem in [3] has a mistake. Specifically, when the author interprets the notion of weak uniform continuity, he claims that the corresponding inequality holds for all elements of the dual space simultaneously (see p. 460). This is not true. The proper way to define weak uniform continuity can be found in [5, p. 170].

The purpose of this note is to prove a more general existence theorem for weak vector fields that includes the above mentioned as well as some earlier ones obtained by Chow-Shur [4] and Kato [9]. We will use a compactness type condition introduced by Pianigiani [10] in connection with the strong (norm) topology.

2. Preliminaries

Let X be a Banach space and X^* its topological dual. By $[0,b]$ we will denote a bounded, closed interval in $\mathbb{R}_+$. To economize in the notation we will write T to denote $[0,b]$.

In [6] DeBlasi introduced the following measure of noncompactness. Let A be a nonempty, bounded subset of X,

$$\beta(A) = \inf\{t > 0: \quad \exists (K \in P_{wk}(X)) \ (A \subseteq K + tB_1)\}$$

where $P_{wk}(X) = \{B \subseteq X: B \neq \phi$ and B weakly compact$\}$ and B_1 is the closed unit ball in X.

The following lemma can be found in [6] and shows that $\beta(\cdot)$ is, according to the terminology of Banas-Goebel [2], a sublinear measure of noncompactness.

LEMMA 2.1. *If A, B are bounded subsets of X, then* 1) $A \subseteq B$ *implies* $\beta(A) \leq \beta(B)$,

2) $\beta(A) = \beta(w\text{-}clA)$ *where w-cl denotes the weak closure,*

3) $\beta(A) = 0$ *if and only if w-clA is weakly compact,*

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4) $\beta(A \cup B) = max[\beta(A), \beta(B)]$,

5) $\beta(A) = \beta(conv\ A)$,

6) $\beta(A + B) \leq \beta(A) + \beta(B)$,

7) $\beta(x + A) = \beta(A)$ *for all* $x \in X$,

8) $\beta(\lambda A) = |\lambda|\beta(A)$ *for all* $\lambda \in \mathbb{R}$,

9) $\beta(\bigcup_{0 \leq t \leq t_0} tA) = t_0\beta(A)$.

The next lemma is a result analogous to the one proved by Ambrosetti [1] for the Kuratowski measure of noncompactness.

LEMMA 2.2. *If* $E \subseteq C_X(T)$ *is bounded and equicontinuous for the strong topology, then* $\beta(E) = \sup_{t \in T} \beta(E(t)) = \beta(E(T))$ *where* $E(t) = \{f(t) : f(\cdot) \in E,\ t \in T\}$.

Proof. The first equality can be found in [3] and its proof is based on the "weak" Arzela-Ascoli theorem [5, Theorem 1.2] .

Now we will show that $\beta(E(T)) \leq \sup_{t \in T} \beta(E(t)) = \lambda$.

Since by hypothesis E is an equicontinuous family, given any $\varepsilon > 0$ we can find a $\delta > 0$ such that if $|t-t'| < \delta$ then $||f(t) - f(t')|| < \varepsilon$ for all $f(\cdot) \in E$. Let $\{t_\ell\}_{\ell=0}^{n}$ be a partition of T such that for all $\ell \in \{0,1,\ldots,n-1\}$ $|t_{\ell+1} - t_\ell| < \delta$. Also let $K_\ell \in P_{wk}(X)$ be such that

$$E(t_\ell) \subseteq K_\ell + (\lambda + \varepsilon)B_1 .$$

The claim is that $E(T) \subseteq \bigcup_{\ell=0}^{n} K_\ell + (\lambda + 2\varepsilon)B_1$. To see this, let $x \in E(T)$. Assume that $x = f(t),\ t \notin \{t_\ell\}_{\ell=0}^{n}$ or otherwise there is nothing to prove. Then $t \in (t_\ell, t_{\ell+1})$ for some $\ell \in \{0,1,\ldots,n-1\}$ and we can write:

$$x \in f(t_\ell) + (f(t) - f(t_\ell)) \in K_\ell + (\lambda + \varepsilon)B_1 + \varepsilon B_1 = K_\ell + (\lambda + 2\varepsilon)B_1$$

from which the claim follows. Then directly from the definition of $\beta(\cdot)$ we get

$$\beta(E(T)) \le \lambda + 2\varepsilon .$$

Let $\varepsilon \downarrow 0$. We have

$$\beta(E(T)) \le \lambda = \sup_{t \in T} \beta(E(t)) . \tag{1}$$

On the other hand using property 1 in Lemma 2.1 we have

$$\beta(E(t)) \le \beta(E(T))$$

for all $t \in T$. Hence

$$\sup_{t \in T} \beta(E(t)) \le \beta(E(T)) . \tag{2}$$

Combining (1) and (2) above, we finally get the lemma.

3. The Main Result.

In the sequel by X_w we will denote the Banach space X with the weak topology. Also by $T_{t,r}$ we will denote the subinterval $[t,t+r]$ of T (provided $t + r \le b$). When t is fixed, we will simply write T_r.

By a Kamke function we mean a function $w: T \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $y \in \mathbb{R}_+$, $t \to w(t,y)$ is measurable and $w(t,y) \le \phi(t)$ with $\phi(\cdot) \in L^1_+$, for all $t \in T$, $y \to w(t,y)$ is continuous, and $y(t) \equiv 0$ is the only solution of $y(t) \le \int_0^t w(s,y(s))ds$, $y(0) = 0$.

Given a weakly continuous vector field $f: T \times X \to X$, we will consider the following Cauchy problem:

$$\left\{ \begin{matrix} \dot{x}(t) = f(t,x(t)) \\ x(0) = x_0 \end{matrix} \right\} . \tag{*}$$

By a solution of (*) we understand a strongly continuous, once weakly differentiable function $x: T \to X$ satisfying (*) on T , with $\dot{x}(\cdot)$ denoting the weak derivative. In this case $x(\cdot)$ is almost everywhere strongly differentiable and satisfies (*) with $\dot{x}(\cdot)$ being the strong derivative.

Our existence result has as follows:

THEOREM 3.1. *If* $f\colon T \times X \to X$ *is a vector field such that*

1) $f(\cdot,\cdot)$ *is continuous from* $T \times X_w$ *into* X_w *(that is* $f(\cdot,\cdot)$ *is weakly continuous),*
2) *for all* (t,x), $||f(t,x)|| \le N$,
3) *for all* $A \subseteq X$ *nonempty and bounded we have*

$$\lim_{r \downarrow 0} \beta(f(T_{t,r} \times A)) \le w(t,\beta(A))$$

where $w(\cdot,\cdot)$ *is a Kamke function,*

then (*) *admits a solution.*

Proof. Consider the nonlinear integral operator $\Phi\colon \mathrm{Lip}_N(T) \to \mathrm{Lip}_N(T)$ where $\mathrm{Lip}_N(T) = \{x(\cdot) \in C_X(T)\colon x(\cdot)$ *is N-Lipschitz*$\})$, defined by

$$(\Phi x)(t) = x_0 + \int_0^t f(s,x(s))ds \quad .$$

We claim that it is weakly-weakly sequentially continuous. So assume that $x_n(\cdot) \xrightarrow{w-C_X} x(\cdot)$. From Dinculeanu [8, p. 380] we know that $[C_X(T)]^* = M_{X^*}(T)$ = bounded, regular, vector measures from T into X^* , which are of bounded variation. Thus for all $m(\cdot) \in M_{X^*}(T)$ we have

$$(m,x_n(\cdot) - x(\cdot)) \to 0 \text{ as } n \to \infty .$$

Let $m = x^*\delta_t$, where $x^* \in X^*$, $t \in T$ and δ_t is the Dirac measure concentrated on t . Then we get that

$$(x^*,x_n(t) - x(t)) \to 0 \text{ as } n \to \infty ,$$

and so

$$x_n(t) \overset{w}{\to} x(t) \text{ as } n \to \infty$$

for all $t \in T$. Then using the weak continuity of the vector field $f(\cdot,\cdot)$ and the Lebesgue dominated convergence theorem we get that

$$\int_0^t f(s,x_n(s))ds \overset{w}{\to} \int_0^t f(s,x(s))ds$$

for all $t \in T$. Now for every $m(\cdot) \in M_{X^*}(T)$ we have that

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$$\langle m, \Phi x_n - \Phi x \rangle = \int_T [\int_0^t (f(s,x_n(s)) - f(s,x(s)))ds] dm(t).$$

Using the fact that for all $t \in T$, $\int_0^t f(s,x_n(s))ds \overset{w}{\to} \int_0^t f(s,x(s))ds$ as $n \to \infty$, and by approximating $m(\cdot)$, uniformly on $\mathrm{Lip}_N(T)$, by linear combinations of Dirac measures, we finally get that

$$\langle m, \Phi x_n - \Phi x \rangle \to 0$$

as $n \to \infty$. Hence $\Phi(\cdot)$ is weakly-weakly sequentially continuous.

Consider the classical Caratheodory approximations

$$x_n(t) = \begin{cases} x_0 & \text{for} \quad 0 \le t \le \frac{1}{n} \\ x_0 + \int_0^{t-\frac{1}{n}} f(s,x_n(s))ds & \text{for} \quad \frac{1}{n} \le t \le T \end{cases}$$

It is easy to see that for all $n \geq 1$, $x_n(\cdot) \in C_X(T)$. Note that:

$$||x_n(t) - \Phi x_n(t)|| = ||x_0 - x_0 - \int_0^t f(s,x_n(s))ds||$$

$$\le ||\int_0^t f(s,x_n(s))ds|| \le Nt \quad \text{for} \quad 0 \le t \le \frac{1}{n}$$

and

$$||x_n(t) - \Phi x_n(t)|| = ||x_0 + \int_0^{t-\frac{1}{n}} f(s,x_n(s))ds - x_0 - \int_0^t f(s,x_n(s))ds||$$

$$\le ||\int_{t-\frac{1}{n}}^t f(s,x_n(s))ds|| \le N\frac{1}{n}$$

$$\text{for} \quad \frac{1}{n} \le t \le T.$$

Thus we see that $||x_n - \Phi x_n||_\infty \to 0$ as $n \to \infty$.

Set $K = \{x_n(\cdot)\}_{n \geq 1}$ and $L = \Phi(K) = \{\Phi x_n(\cdot)\}_{n \geq 1}$. We have just seen that

$$\beta(K - L) = \beta((I - \Phi)(K)) = 0.$$

Observe that for all $t \in T$

$$K(t) \subseteq K(t) - L(t) + L(t) = (I - \Phi)(K(t)) + L(t) ,$$

and so $$\beta(K(t)) \leq \beta((I - \Phi)(K(t))) + \beta(L(t)) .$$

Using the fact that $\beta((I - \Phi)(K)) = 0$ and Lemma 2.2, we see that $\beta((I - \Phi)(K(t))) = 0$. Hence

$$\beta(K(t)) \leq \beta(L(t)) . \qquad (1)$$

On the other hand for all $t \in T$, we have

$$L(t) \subseteq L(t) - K(t) + K(t) = K(t) - (I - \Phi)(K(t)) ,$$

and so $$\beta(L(t)) \leq \beta(K(t)) + \beta((I - \Phi)(K(t))) .$$

For the same reason as before, $\beta((I - \Phi)(K(t))) = 0$. Hence

$$\beta(L(t)) \leq \beta(K(t)) . \qquad (2)$$

From (1) and (2) above we get that

$$\beta(K(t)) = \beta(L(t)) = p(t)$$

Now we claim that $L = \Phi(K)$ is an equicontinuous set. So let $t,t' \in T$, and then

$$||\Phi x(t') - \Phi x(t)|| = ||x_0 + \int_0^{t'} f(s,x(s))ds - x_0 - \int_0^t f(s,x(s))ds||$$

$$\leq || \int_t^{t'} f(s,x(s))ds|| \leq N|t' - t|$$

and this shows that L is equicontinuous.

Then employing Lemma 13.2.1 of Banas-Goebel [2] we have that

$$|p(t') - p(t)| \leq \beta(B_1)m_L(t' - t)$$

where $m_L(\cdot)$ is the modulus of equicontinuity of the family L , that is

$$m_L(r) = \sup\{||y(t')-y(t)||: t,t' \in T, |t' - t| \leq r, y(\cdot) \in L\} .$$

Also recall (see [6]) that $\beta(B_1) \leq 1$. Thus

$$|p(t') - p(t)| \leq m_L(|t' - t|) \leq N|t' - t|$$

which shows that $p(\cdot)$ is absolutely continuous, hence differentiable at all $t \in T\backslash N, \lambda(N) = 0$.

Next fix $t \in T\backslash N$ and let $\varepsilon > 0$ be given. We can find $\delta > 0$ such that

$$|z - p(t)| < \delta \text{ implies } |w(t,z) - w(t,p(t))| < \varepsilon .$$

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This is possible since $w(t,\cdot)$ is continuous, being a Kamke function. Also let $r > 0$ be such that

$$Nr < \delta \quad \text{and} \quad t + r \leq T .$$

Because of hypothesis 3) we can find q such that $0 < q < r$ and

$$\beta(f(T_q \times K_r)) \leq h(t,\beta(K_r)) + \varepsilon$$

where $T_q = [t,t+q]$ and $K_r = \{x(s): x(\cdot) \in K,\ t \leq s \leq t+r\}$.

From Lemma 2.2 we know that

$$\beta(K_r) = \sup_{s \in [t,t+r]} \beta(K(s)) = p(\hat{t})$$

for some $\hat{t} \in [t,t+r]$. Then

$$0 \leq \beta(k_r) - p(t) \leq m_L(|\hat{t} - t|) \leq N|\hat{t} - t| \leq Nr < \delta .$$

But then from the choice of $\delta > 0$ we get that

$$|w(t,\beta(K_r)) - w(t,p(t))| < \varepsilon .$$

Next since $f(\cdot,\cdot)$ is weakly continuous and Pettis integrable, for all $x(\cdot) \in K$ and for $v < q$ we have that

$$\Phi x(t+v) = \Phi x(t) + \int_t^{t+v} f(s,x(s))ds \subseteq \Phi x(t) + v\ \overline{\text{conv}} \bigcup_{s \in T_v} f(s,x(s)).$$

Thus $$L(t+v) \subseteq L(t) + v.\overline{\text{conv}}\ f(T_v \times K_r) ,$$

and so $$\beta(L(t+v)) \leq \beta(L(t)) + \beta[v\cdot\overline{\text{conv}}\ f(T_v \times K_r)] .$$

Using the properties of $\beta(\cdot)$ listed in Lemma 2.1, we have:

$$\beta(v\cdot\overline{\text{conv}}\ f(T_v \times K_r)) = v\cdot\beta(\overline{\text{conv}}\ f(T_v \times K_r)) = v\cdot\beta(f(T_v \times K_r))$$
$$\leq v\cdot w(t,\beta(K_r)) + v\cdot\varepsilon \leq v\cdot w(t,p(t)) + 2v\cdot\varepsilon .$$

Therefore we can write that

$$p(t+v) \leq p(t) + v\cdot w(t,p(t)) + 2v\cdot\varepsilon ,$$

and so $$\frac{p(t+v) - p(t)}{v} \leq w(t,p(t)) + 2\varepsilon .$$

Passing to the limits as $v \to 0^+$ and since $t \in T\backslash N$ we get that

$$\dot{p}(t) \leq w(t,p(t)) + 2\varepsilon .$$

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Let $\varepsilon \downarrow 0$. We finally have that

$$\dot{p}(t) \leq w(t,p(t))$$

and so
$$p(t) \leq \int_0^t w(s,p(s))ds$$

while $p(0) = \beta(K(0)) = \beta(L(0)) = \beta(\{x_0\}) = 0$. Since $w(\cdot,\cdot)$ is by hypothesis a Kamke function we get that for all $t \in T$

$$p(t) = 0$$

and so
$$\beta(K(t)) = 0 .$$

Employing Lemma 2.2 we deduce that

$$\beta(K) = 0$$

which by Lemma 2.1 implies that $w\text{-}cl\ K$ is w-compact in $C_X(T)$. Invoking the Eberlein-Smulian theorem and by passing to a subsequence if necessary, we may assume that

$$x_n(\cdot) \xrightarrow{w-C_X} x(\cdot) \in C_X(T) .$$

We have already seen in the beginning of the proof that $\Phi(\cdot)$ is w-sequentially continuous. Thus $\Phi x_n \xrightarrow{w-C_X} \Phi x$. So we have that

$$x_n - \Phi x_n \xrightarrow{w-C_X} x - \Phi x \quad \text{as} \quad n \to \infty .$$

Recalling that the norm is w-lower semicontinuous we get that

$$\varliminf_{n \to \infty} ||x_n - \Phi x_n||_\infty \geq ||x - \Phi x||_\infty .$$

But we have already seen earlier in the proof that

$$||x_n - \Phi x_n||_\infty \to 0 \quad \text{as} \quad n \to \infty .$$

Therefore we conclude that

$$||x - \Phi x||_\infty = 0 ,$$

and so
$$x(t) = x_0 + \int_0^t f(s,x(s))ds .$$

Because $f(\cdot,\cdot)$ is weakly continous, $x(\cdot)$ is weakly differentiable and the weak derivative $\dot{x}(\cdot)$ satisfies $(^*)$. Thus we conclude that $x(\cdot)$ is the desired weak solution of $(^*)$.

Suppose that $w(t,x) = w(x)$ and that for all $A \subseteq X$ bounded we have that $\beta(f(T \times A)) \le w(\beta(A))$. Then for every $t \in T$ and for every $r > 0$ such that $t+r \le T$, we have

$$f(T_{tr} \times A) \subseteq f(T \times A) ,$$

so
$$\beta(f(T_{tr} \times A)) \le \beta(f(T \times A)) \le w(\beta(A))$$

and
$$\lim_{r \downarrow 0} \beta(f(T_{tr} \times A)) \le w(\beta(A)) .$$

So hypothesis (H_1) of Cramer-Lakshmikantham-Mitchell [5] implies our compactness hypothesis. Thus we can recover from our result, as a corollary, theorem 3.1 of [5].

COROLLARY 3.1. [5]. *If* $f: T \times X \to X$ *is a vector hield such that*

1) $f(\cdot,\cdot)$ *is weakly continuous,*

2) for all $(t,x) \in T \times X$, $||f(t,v)|| \le M$,

3) for all $A \subseteq X$ *bounded,* $\beta(f(T \times A)) \le w(\beta(A))$ *where* $w(\cdot)$ *is a time independent Kamke function,*

then $(^*)$ *admits a solution.*

If X *is a reflexive Banach space every bounded set is relatively weakly compact. So for a weakly continuous vector field* $f(\cdot,\cdot)$ *our compactness hypothesis is trivially satisfied. Hence the results of Chow-Shur* [4] *and Szep* [11] *are special cases of Theorem 3.1. Thus we have for* X *reflexive:*

COROLLARY 3.2. [11]. *If* $f:T \times X \to X$ *is a vector field such that*

1) $f(\cdot,\cdot)$ *is weakly continuous,*

2) for all $(t,x) \in T \times X$, $||f(t,x)|| \le M$,

then $(^*)$ *admits a solution.*

For X *any Banach space:*

COROLLARY 3.3. [9]. *If* $f:T \times X \to X$ *is a vector field such that*

1) $f(\cdot,\cdot)$ *is weakly continuous,*

2) $f(\cdot,\cdot)$ *is compact (for example* $w\text{-}cl\, f(T \times X) \in P_{wk}(X))$,

then $(^*)$ *admits a solution.*

Remark. If the domain of $f(\cdot,\cdot)$ is the set $T \times B_r(x_0)$, where $B_r(x_0) = \{x \in X: ||x-x_0|| < r\}$, then the local version of Theorem 3.1 is valid.

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