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ON THE ANTI-CANONICAL GEOMETRY OF WEAK $\mathbb{Q}$-FANO THREEFOLDS, III

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Abstract. For a terminal weak $\mathbb{Q}$-Fano threefold X, we show that the mth anti-canonical map defined by $|-mK_X|$ is birational for all $m \geq 59$.

§1. Introduction

Throughout this paper, we work over an algebraically closed field of characteristic 0 (e.g., the complex number field $\mathbb{C}$). We adopt standard notation in [14].

A normal projective variety X is called a *weak $\mathbb{Q}$-Fano variety* (resp. *$\mathbb{Q}$-Fano variety*) if $-K_X$ is nef and big (resp. ample). According to the minimal model program, (weak) $\mathbb{Q}$-Fano varieties form a fundamental class in birational geometry. Motivated by the classification theory of three-dimensional algebraic varieties, we are interested in the study of explicit geometry of (weak) $\mathbb{Q}$-Fano varieties with terminal or canonical singularities. In this direction, there are a lot of works in the literature (see, e.g., [2], [4]–[6], [10]–[12], [16]–[19]).

Given a terminal weak $\mathbb{Q}$-Fano threefold X, the *mth anti-canonical map* $\varphi_{-m,X}$ (or simply φ_{-m}) is the rational map induced by the linear system $|-mK_X|$. We are interested in the fundamental question of finding an optimal integer c_3 such that φ_{-m} is birational for all $m \geq c_3$. The existence of such c_3 follows from the boundedness result in [13]. More generally, Birkar [1] showed that, for a positive integer d, there exists a positive integer c_d such that φ_{-m} is birational for all $m \geq c_d$ and for all terminal weak $\mathbb{Q}$-Fano d-folds, which is one important step toward the solution of the Borisov–Alexeev–Borisov conjecture. The following example shows that $c_3 \geq 33$.

Example 1.1 [8, List 16.6, No. 95]. A general weighted hypersurface $X_{66} \subset \mathbb{P}(1,5,6,22,33)$ is a $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano threefold of Picard number 1 with φ_{-m} birational for $m \geq 33$ but φ_{-32} not birational.

In [5], it was showed that for a terminal weak $\mathbb{Q}$-Fano threefold X, φ_{-m} is birational for all $m \geq 97$, which seems far from being optimal comparing to Example 1.1. Later in [6], it was showed that any terminal weak $\mathbb{Q}$-Fano threefold is birational to some terminal weak $\mathbb{Q}$-Fano threefold Y such that $\varphi_{-m,Y}$ is birational for all $m \geq 52$. Moreover, in recent works [10], [11], we can make use of the behavior of the pluri-anti-canonical maps studied in [5] in the classification of terminal $\mathbb{Q}$-Fano threefolds. So we believe that a better understanding of the behavior of the pluri-anti-canonical maps (including new methods developed during the approach) will help us understand the classification of terminal $\mathbb{Q}$-Fano threefolds better.

The main goal of this paper is to give an improvement of [5], [6] without passing to a birational model. The main theorem of this paper is the following.

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THEOREM 1.2. *Let X be a terminal weak $\mathbb{Q}$-Fano threefold. Then the mth anti-canonical map φ_{-m} defined by $|-mK_X|$ is birational for all $m \geq 59$.*

REMARK 1.3. Theorem 1.2 holds for canonical weak $\mathbb{Q}$-Fano threefolds by taking a $\mathbb{Q}$-factorial terminalization by [14, Ths. 6.23 and 6.25].

For terminal $\mathbb{Q}$-Fano threefolds, we have a slightly better bound.

THEOREM 1.4. *Let X be a terminal $\mathbb{Q}$-Fano threefold. Then the mth anti-canonical map φ_{-m} defined by $|-mK_X|$ is birational for all $m \geq 58$.*

To prove the main theorem, we already have several criteria to determine the birationality in [5], [6], which are optimal in many cases (cf. [5, Exam. 5.12]). In order to study the birationality of $|-mK_X|$, as indicated in [4]–[6], it is crucial to study when $|-mK_X|$ is not composed with a pencil. In fact, finding a criterion for $|-mK_X|$ not composed with a pencil is one of the central problems in [5], [6] (see [5, Prob. 1.3], [6, Prob. 1.5]). Comparing to the birationality criteria, the non-pencil criteria in [5], [6] are not satisfactory. As one of the main ingredients of this paper, we give a new criterion for $|-mK_X|$ not composed with a pencil.

THEOREM 1.5 (=Theorem 4.2). *Let X be a terminal weak $\mathbb{Q}$-Fano threefold. If*

$$h^0(X, -mK_X) > 12m+1$$

for some positive integer m, then $|-mK_X|$ is not composed with a pencil.

The following special case is already interesting for the study of anti-canonical systems of terminal weak $\mathbb{Q}$-Fano threefolds, and might have applications on upper bounds of degrees of terminal weak $\mathbb{Q}$-Fano threefolds (cf. [16], [17]).

COROLLARY 1.6. *Let X be a terminal weak $\mathbb{Q}$-Fano threefold. If $h^0(X, -K_X) > 13$, then $|-K_X|$ is not composed with a pencil.*

The paper is organized as follows: in §2, we recall basic knowledge. In §3, we recall the birationality criteria of terminal weak $\mathbb{Q}$-Fano threefolds in [5], [6] with some generalizations. In §4, we prove the new criterion Theorem 4.2 and give an effective method to apply it. In §5, we prove the main results.

Notation

For the convenience of readers, we list here the notation that will be frequently used in this paper. Let X be a terminal weak $\mathbb{Q}$-Fano threefold.

φ_{-m}	The rational map defined by $\|-mK_X\|$
$P_{-m} = h^0(X, -mK_X)$	The mth anti-plurigenus of X
$B_X = \{(b_i, r_i)\}$	The Reid basket of orbifold points of X
$\mathcal{R}_X = \{r_i\}$	The collection of local indices of X
$r_X = \mathrm{lcm}\{r_i \mid r_i \in \mathcal{R}_X\}$	The Cartier index of K_X
$r_{\max} = \max\{r_i \mid r_i \in \mathcal{R}_X\}$	The maximal local index of X
$\sigma(B_X) = \sum_i b_i$	An invariant of B_X contributing to the Riemann–Roch formula
$\sigma'(B_X) = \sum_i \frac{b_i^2}{r_i}$	An invariant of B_X contributing to the Riemann–Roch formula
$\gamma(B_X) = \sum_i \frac{1}{r_i} - \sum_i r_i + 24$	An invariant of B_X from the Miyaoka inequality
$B_X^{(0)} = \{n_{1,r}^0 \times (1,r)\}$	The initial basket of B_X

§2. Preliminaries

Let X be a terminal weak $\mathbb{Q}$-Fano threefold. Denote by r_X the Cartier index of K_X. For any positive integer m, the number $P_{-m} = h^0(X, \mathcal{O}_X(-mK_X))$ is called the *mth anti-plurigenus* of X and φ_{-m} denotes the *mth anti-canonical map* defined by $|-mK_X|$.

2.1 The fibration induced by $|D|$

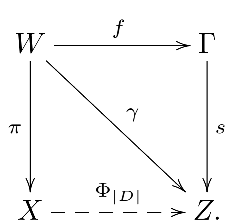
Let X be a terminal weak $\mathbb{Q}$-Fano threefold. Consider a $\mathbb{Q}$-Cartier Weil divisor D on X with $h^0(X,D) \geq 2$. Then there is a rational map defined by $|D|$:

$$\Phi_{|D|} : X \dashrightarrow \mathbb{P}^{h^0(X,D)-1}.$$

By Hironaka's desingularization theorem, we can take a projective birational morphism $\pi : W \to X$ such that:

(i) W is smooth.
(ii) The movable part $|M|$ of $|\lfloor \pi^*(D) \rfloor|$ is base-point-free and, consequently, $\gamma := \Phi_{|D|} \circ \pi$ is a morphism.
(iii) The sum of $\pi_*^{-1}(D)$ and the exceptional divisors of π has simple normal crossing support.

Let $W \xrightarrow{f} \Gamma \xrightarrow{s} Z$ be the Stein factorization of γ with $Z := \gamma(W) \subset \mathbb{P}^{h^0(X,D)-1}$. We have the following commutative diagram:



If $\dim(\Gamma) \geq 2$, then a general member S of $|M|$ is a smooth projective surface by Bertini's theorem. In this case, $|D|$ is said to be *not composed with a pencil of surfaces* (not composed with a pencil, for short).

If $\dim(\Gamma) = 1$, then $\Gamma \cong \mathbb{P}^1$ as $h^1(\Gamma, \mathcal{O}_\Gamma) \leq h^1(W, \mathcal{O}_W) = h^1(X, \mathcal{O}_X) = 0$. Furthermore, a general fiber S of f is a smooth projective surface by Bertini's theorem. In this case, $|D|$ is said to be *composed with a (rational) pencil of surfaces* (composed with a pencil, for short).

In each case, S is called a *generic irreducible element* of $|M|$. We can also define a generic irreducible element of a moving linear system on a surface in the similar way.

Definition 2.1. Keep the same notation as above. Let D' be another $\mathbb{Q}$-Cartier Weil divisor on X with $h^0(X, D') \geq 2$. We say that $|D|$ and $|D'|$ are *composed with the same pencil*, if both of them are composed with pencils and they define the same fibration structure $W \to \mathbb{P}^1$. In particular, $|D|$ and $|D'|$ are not composed with the same pencil if one of them is not composed with a pencil.

2.2 Reid's Riemann–Roch formula and Chen–Chen's method

A basket B is a collection of pairs of coprime integers where a pair is allowed to appear several times, say

$$\{(b_i, r_i) \mid i = 1, \ldots, s; b_i \text{ is coprime to } r_i\}.$$

For simplicity, we will alternatively write a basket as a set of pairs with weights, say, for example,

$$B = \{2\times(1,2),(1,3),(3,7),(5,11)\}.$$

Let X be a terminal weak $\mathbb{Q}$-Fano threefold. According to Reid [20], there is a basket of (virtual) orbifold points

$$B_X = \left\{(b_i,r_i) \mid i=1,\dots,s; 0<b_i\le \frac{r_i}{2}; b_i \text{ is coprime to } r_i\right\}$$

associated with X, where a pair (b_i,r_i) corresponds to an orbifold point Q_i of type $\frac{1}{r_i}(1,-1,b_i)$. Denote by $\mathcal{R}_X$ the collection of r_i (counted with multiplicities) appearing in B_X, and $r_{\max} = \max\{r_i \mid r_i \in \mathcal{R}_X\}$. Note that the Cartier index r_X of K_X is just $\operatorname{lcm}\{r_i \mid r_i \in \mathcal{R}_X\}$.

According to Reid [20], for any positive integer n,

$$P_{-n} = \frac{1}{12}n(n+1)(2n+1)\left(-K_X^3\right)+(2n+1)-l(n+1), \tag{2.1}$$

where $l(n+1)=\sum_i\sum_{j=1}^n \frac{\overline{jb_i}\left(r_i-\overline{jb_i}\right)}{2r_i}$ and the first sum runs over Reid's basket of orbifold points. Here, $\overline{jb_i}$ means the smallest nonnegative residue of jb_i mod r_i.

Set $\sigma(B_X)=\sum_i b_i$ and $\sigma'(B_X)=\sum_i \frac{b_i^2}{r_i}$. From (2.1), for $n=1,2$,

$$-K_X^3 = 2P_{-1}+\sigma(B_X)-\sigma'(B_X)-6, \tag{2.2}$$

$$\sigma(B_X) = 10-5P_{-1}+P_{-2}. \tag{2.3}$$

Denote

$$\gamma(B_X) := \sum_i \frac{1}{r_i} - \sum_i r_i + 24.$$

By [13] and [20, 10.3],

$$\gamma(B_X)\ge 0. \tag{2.4}$$

We recall Chen–Chen's method on basket packing from [2]. Let

$$B = \left\{(b_i,r_i) \mid i=1,\dots,s; 0<b_i\le \frac{r_i}{2}; b_i \text{ is coprime to } r_i\right\}$$

be a basket and assume that $b_1r_2-b_2r_1=1$, then the new basket

$$B' = \{(b_1+b_2,r_1+r_2),(b_3,r_3),\dots,(b_s,r_s)\}$$

is called a *prime packing* of B. We say that a basket B' is *dominated by* B, denoted by $B\succeq B'$, if B' can be achieved from B by a sequence of prime packings (including the case $B=B'$).

By [2, §2.5], there is a unique basket $B_X^{(0)}$, called *the initial basket of* B_X, of the form $B_X^{(0)} = \{n_{1,r}^0\times(1,r) \mid r\ge 2\}$ such that $B_X^{(0)}\succeq B_X$. By [2, §2.7], we have

$$n_{1,2}^0 = 5-6P_{-1}+4P_{-2}-P_{-3}, \tag{2.5}$$

$$n_{1,3}^0 = 4-2P_{-1}-2P_{-2}+3P_{-3}-P_{-4}, \tag{2.6}$$

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$$n^0_{1,4} = 1+3P_{-1} - P_{-2} - 2P_{-3} + P_{-4} - \sigma_5, \tag{2.7}$$

where $\sigma_5 = \sum_{r\geq 5} n^0_{1,r}$. We refer to [2] for more details.

2.3 Auxiliary results

We list here some useful results on terminal weak $\mathbb{Q}$-Fano threefolds.

PROPOSITION 2.2. *Let X be a terminal weak $\mathbb{Q}$-Fano threefold. Then:*

(1) $r_X = 840$ *or* $r_X \leq 660$ *[5, Prop. 2.4].*
(2) $P_{-8} \geq 2$ *[2, Th. 1.1]; moreover, if* $P_{-1} = 0$ *and* $P_{-2} > 0$*, then* $P_{-6} \geq 2$ *[2, Case 1 of Proof of Prop. 3.10].*
(3) $-K_X^3 \geq \frac{1}{330}$ *[2, Th. 1.1]; moreover, if* $P_{-1} = 0$ *and* $P_{-2} > 0$*, then* $-K_X^3 \geq \frac{1}{70}$*, and if in addition* $P_{-4} \geq 2$*, then* $-K_X^3 \geq \frac{1}{30}$ *[2, (4.1), Lem. 4.2, and Case I of Proof of Th. 4.4].*
(4) *If* $P_{-1} = 0$*, then* $2 \in \mathcal{R}_X$ *[5, Proof of Th. 1.8, p. 106].*

LEMMA 2.3. *Suppose that* $\{(b_i, r_i) \mid 1 \leq i \leq k\}$ *is a collection of pairs of integers with* $0 < 2b_i \leq r_i$ *for* $1 \leq i \leq k$*. Then* $\sum_{i=1}^k (r_i - \frac{1}{r_i}) \geq \frac{3}{2}\sum_{i=1}^k b_i$.

Proof. $r_i \geq 2b_i$ implies that $r_i - \frac{1}{r_i} \geq \frac{3}{2} b_i$. □

§3. The criteria for birationality

In this section, we recall the birationality criteria of terminal weak $\mathbb{Q}$-Fano threefolds in [5], [6]. Here, we remark that all birationality criteria in this section are from [5], [6] except for Theorem 3.5 and Corollary 3.7 (which are minor generalizations of [6, Th. 5.9]). Also, we provide Lemma 3.3 in order to apply Corollary 3.7 efficiently. In fact, in [6], [6, Th. 5.9] is only used for very special cases, but in this paper, thanks to Lemma 3.3, we make use of Corollary 3.7 in many cases.

3.1 General settings

We recall numerical invariants needed in the birationality criteria, namely, ν_0, m_0, $a(m_0)$, m_1, μ'_0, and N_0.

NOTATION 3.1. Let X be a terminal weak $\mathbb{Q}$-Fano threefold.

Let ν_0 be a positive integer such that $P_{-\nu_0} > 0$.

Take a positive integer m_0 such that $P_{-m_0} \geq 2$. Set

$$a(m_0) = \begin{cases} 6, & \text{if } m_0 \geq 2, \\ 1, & \text{if } m_0 = 1. \end{cases}$$

Take $m_1 \geq m_0$ to be an integer with $P_{-m_1} \geq 2$ such that $|-m_0K_X|$ and $|-m_1K_X|$ are not composed with the same pencil.

Set $D := -m_0K_X$ and keep the same notation as in §2.1. Denote S to be a generic irreducible element of $|M_{-m_0}| = \mathrm{Mov}|\lfloor \pi^*(-m_0K_X)\rfloor|$. Choose a positive rational number μ'_0 such that

$$\mu'_0\pi^*(-K_X) - S \sim_{\mathbb{Q}} \text{effective } \mathbb{Q}\text{-divisor}.$$

Set $N_0 = r_X(\pi^*(-K_X)^2 \cdot S)$.

REMARK 3.2 [6, Rem. 5.8]. Here, we explain how to choose μ'_0. In general, by assumption, we can always take $\mu'_0 = m_0$. On the other hand, if $|-m_0K_X|$ and $|-kK_X|$

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are composed with the same pencil for some positive integer k, and $\frac{k}{P_{-k}-1} < m_0$, then we can take $\mu_0' = \frac{k}{P_{-k}-1}$ as

$$k\pi^*(-K_X) \sim_{\mathbb{Q}} (P_{-k}-1)S + \text{effective } \mathbb{Q}\text{-divisor}.$$

LEMMA 3.3. *In Notation 3.1, $N_0 \geq \lceil \frac{r_X}{m_1\nu_0 r_{\max}} \rceil$.*

Proof. We may modify π such that $|M_{-m_1}| = \operatorname{Mov}|\lfloor \pi^*(-m_1K_X)\rfloor|$ is base-point-free. Pick a generic irreducible element C of the base-point-free linear system $|M_{-m_1}|_S|$. Since $\pi^*(-m_1K_X) \geq M_{-m_1}$, $\pi^*(-m_1K_X)|_S \geq C$. Set

$$\zeta := (\pi^*(-K_X)\cdot C) = (\pi^*(-K_X)|_S \cdot C)_S.$$

By [5, Prop. 5.7(v)], $\zeta \geq \frac{1}{\nu_0 r_{\max}}$. Since $\pi^*(-K_X)|_S$ is nef,

$$\pi^*(-K_X)^2\cdot S \geq \pi^*(-K_X)|_S \cdot \frac{1}{m_1}C \geq \frac{1}{m_1\nu_0 r_{\max}}.$$

Hence, $N_0 \geq \lceil \frac{r_X}{m_1\nu_0 r_{\max}} \rceil$ as N_0 is an integer by [5, Lem. 4.1]. ∎

3.2 Birationality criteria

We recall the birationality criteria of terminal weak $\mathbb{Q}$-Fano threefolds.

THEOREM 3.4 [5, Th. 5.11]. *Keep the setting in Notation 3.1. Then the mth anti-canonical map φ_{-m} is birational if one of the following conditions holds:*

(1) $m \geq \max\{m_0 + m_1 + a(m_0), \lfloor 3\mu_0' \rfloor + 3m_1\}$.
(2) $m \geq \max\{m_0 + m_1 + a(m_0), \lfloor \frac{5}{3}\mu_0' + \frac{5}{3}m_1 \rfloor, \lfloor \mu_0' \rfloor + m_1 + 2r_{\max}\}$.
(3) $m \geq \max\{m_0 + m_1 + a(m_0), \lfloor \mu_0' \rfloor + m_1 + 2\nu_0 r_{\max}\}$.

As another criterion, we have the following modification of [6, Th. 5.9].

THEOREM 3.5. *Keep the setting in Notation 3.1. Fix a real number $\beta \geq 8$. Then the mth anti-canonical map φ_{-m} is birational if*

$$m \geq \max\left\{ m_0 + a(m_0), \left\lceil \mu_0' + \frac{4\nu_0 r_{\max}}{1+\sqrt{1-\frac{8}{\beta}}} \right\rceil - 1, \lfloor \mu_0' + \sqrt{\beta r_X/N_0} \rfloor \right\}.$$

Proof. The proof is the same as [6, Th. 5.9] by replacing [6, Lem. 5.10] with Lemma 3.6. ∎

LEMMA 3.6 [3, Th. 2.8]. *Let S be a smooth projective surface, and let L be a nef and big $\mathbb{Q}$-divisor on S satisfying the following conditions:*

(1) *$L^2 > \beta$, for some real number $\beta \geq 8$, and*
(2) *$(L\cdot C_P) \geq \frac{4}{1+\sqrt{1-\frac{8}{\beta}}}$ for all irreducible curves C_P passing through any very general point $P \in S$.*

Then the linear system $|K_S + \lceil L \rceil|$ separates two distinct points in very general positions. Consequently, $|K_S + \lceil L \rceil|$ gives a birational map.

We will use the following version of Theorem 3.5.

COROLLARY 3.7. *Keep the setting in Notation 3.1. Then the mth anti-canonical map φ_{-m} is birational if one of the following conditions holds:*

(1) $m \geq \max\{m_0 + a(m_0), \lceil \mu_0' \rceil + 4\nu_0 r_{\max} - 1, \lfloor \mu_0' + \sqrt{8r_X/N_0} \rfloor\}$.
(2) $\nu_0 r_{\max} \geq \sqrt{\frac{r_X}{2N_0}}$ *and*

$$m \geq \max\left\{m_0 + a(m_0), \left\lfloor \mu_0' + 2\nu_0 r_{\max} + \frac{r_X}{N_0\nu_0 r_{\max}} \right\rfloor\right\}.$$

Proof. (1) follows directly from [6, Th. 5.9] or Theorem 3.5 with $\beta = 8$. For (2), take

$$\beta = \frac{N_0}{r_X}\left(2\nu_0 r_{\max} + \frac{r_X}{N_0\nu_0 r_{\max}}\right)^2 \geq 8$$

in Theorem 3.5. Then

$$\begin{aligned}\frac{4\nu_0 r_{\max}}{1+\sqrt{1-\frac{8}{\beta}}} &= \frac{4\nu_0 r_{\max}\sqrt{\beta}}{\sqrt{\beta}+\sqrt{\beta-8}} \\ &= \frac{4\nu_0 r_{\max}\sqrt{\beta}}{\sqrt{\frac{N_0}{r_X}}\left(2\nu_0 r_{\max} + \frac{r_X}{N_0\nu_0 r_{\max}} + \left|2\nu_0 r_{\max} - \frac{r_X}{N_0\nu_0 r_{\max}}\right|\right)} \\ &= \sqrt{\beta r_X/N_0}.\end{aligned}$$

So the conclusion follows from Theorem 3.5. ∎

Finally, we explain the strategy to apply the birationality criteria to assert the birationality. It is clear that in order to apply Theorem 3.4 and Corollary 3.7, we need to control the values of (some of) ν_0, m_0, m_1, μ_0', N_0, and $r_X, r_{\max}$. To be more precise, we need to give upper bounds of ν_0, m_0, m_1, μ_0', r_X, $r_{\max}$ and lower bounds of N_0. Here, m_0 and ν_0 can be controlled by Proposition 2.2 (in particular, we can always take $m_0 = 8$), μ_0' can be controlled by Remark 3.2, N_0 can be controlled by Lemma 3.3 (in most cases, we use the trivial lower bound $N_0 \geq 1$), and r_X and $r_{\max}$ can be controlled by (2.4). So the most important and difficult part is to bound m_1. We will deal with this issue in the next section.

§4. A new criterion for $|-mK|$ not composed with a pencil

In this section, we give a new criterion on when $|-mK_X|$ is not composed with a pencil for a terminal weak $\mathbb{Q}$-Fano threefold X. Such a criterion is essential in order to apply criteria for birationality in §3 (see also [4]–[6]). In [5], the following proposition is used to determine when $|-mK|$ is not composed with a pencil.

Proposition 4.1 [5, Cor. 4.2]. *Let X be a terminal weak $\mathbb{Q}$-Fano threefold. If*

$$P_{-m} > r_X(-K_X^3)m + 1$$

for some positive integer m, then $|-mK_X|$ is not composed with a pencil.

However, Proposition 4.1 is too weak for application, especially when $r_X(-K_X^3)$ is large (see Example 4.10). In [6], there is a modification of this inequality (cf. [6, Lem. 4.2 and Prop. 5.2]), but one has to replace X with a birational model. In this paper, by technique developed recently in [12], we give a new criterion.

Theorem 4.2. *Let X be a terminal weak $\mathbb{Q}$-Fano threefold. If*

$$P_{-m} > 12m + 1$$

for some positive integer m, then $|-mK_X|$ is not composed with a pencil.

4.1 A structure theorem of terminal weak $\mathbb{Q}$-Fano threefolds

We recall the following structure theorem of terminal weak $\mathbb{Q}$-Fano threefolds from [12]. It plays the role of Fano–Mori triples as in [6]. Unlike [6], we do not need to replace by a birational model (cf. [6, Prop. 3.9]).

PROPOSITION 4.3 [12, Prop. 4.1]. *Let X be a terminal weak $\mathbb{Q}$-Fano threefold. Then there exists a normal projective threefold Y birational to X satisfying the following properties:*

(1) *Y is $\mathbb{Q}$-factorial terminal.*
(2) *$-K_Y$ is big.*
(3) *For any sufficiently large and divisible positive integer n, $|-nK_Y|$ is movable.*
(4) *For a general member $M \in |-nK_Y|$, M is irreducible and $(Y, \frac{1}{n}M)$ is canonical.*
(5) *There exists a projective morphism $g: Y \to S$ with connected fibers where F is a general fiber of g, such that one of the following conditions holds:*
   (a) *S is a point and Y is a $\mathbb{Q}$-Fano threefold with $\rho(Y) = 1$.*
   (b) *$S = \mathbb{P}^1$ and F is a smooth weak del Pezzo surface.*
   (c) *S is a del Pezzo surface with at worst Du Val singularities and $\rho(S) = 1$, and $F \simeq \mathbb{P}^1$.*

Here, we remark that in the proof of [12, Prop. 4.1], Y is obtained by running a K-MMP on a $\mathbb{Q}$-factorialization of X, so the induced map $X \dashrightarrow Y$ is a contraction, that is, it does not extract any divisor.

4.2 Bounding coefficients of anti-canonical divisors

In this subsection, we discuss coefficients of certain divisors in the $\mathbb{Q}$-linear system of the anti-canonical divisor in several cases.

LEMMA 4.4. *Let S be a smooth weak del Pezzo surface, and let C be a nonzero effective integral divisor on S which is movable. If $-K_S \sim_{\mathbb{Q}} aC + B$ for some positive rational number a and some effective $\mathbb{Q}$-divisor B, then $a \leq 4$.*

Proof. By classical surface theory, it is well known that there is a birational map from S to $\mathbb{P}^2$ or the Hirzebruch surface $\mathbb{F}_0$ or $\mathbb{F}_2$. So, by taking pushforward, we may replace S by $\mathbb{P}^2$ or $\mathbb{F}_0$ or $\mathbb{F}_2$. Here, C is not contracted by the pushforward as it is movable.

If $S = \mathbb{P}^2$, then intersecting with a general line L, we get $a \leq a(C \cdot L) \leq (-K_S \cdot L) = 3$.

If $S = \mathbb{F}_0$, then we may find a ruling structure $\phi: \mathbb{F}_0 \to \mathbb{P}^1$ such that C is not vertical. Then we get $a \leq 2$ by intersecting with a fiber of ϕ.

If $S = \mathbb{F}_2$, then we consider the natural ruling structure $\phi: \mathbb{F}_2 \to \mathbb{P}^1$. If C is not vertical, then intersecting with a fiber of ϕ, we get $a \leq 2$. If C is vertical, then intersecting with $-K_S$, we get $a(-K_S \cdot C) \leq K_S^2$ which implies that $a \leq 4$. ∎

LEMMA 4.5. *Let Y be a $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano threefold with $\rho(Y) = 1$, and let D be an integral divisor with $h^0(D) \geq 2$. If $-K_Y \sim_{\mathbb{Q}} aD + B$ for some positive rational number a and some effective $\mathbb{Q}$-divisor B, then $a \leq 7$. Moreover, the equality holds if and only if $Y \simeq \mathbb{P}(1,1,2,3)$, $B = 0$, and $\mathcal{O}_Y(D) \simeq \mathcal{O}_Y(1)$.*

Proof. Suppose that $a \geq 7$. As $\rho(Y) = 1$, we have $-K_Y \sim_{\mathbb{Q}} tD$ for some rational number $t \geq a \geq 7$. Recall that the $\mathbb{Q}$-*Fano index* of Y is defined by

$$q\mathbb{Q}(Y) = \max\{q \mid -K_Y \sim_{\mathbb{Q}} qA, A \text{ is a Weil divisor}\}.$$

By [18, Cor. 3.4(ii)], $t = q\mathbb{Q}(Y) \geq 7$. As $h^0(D) \geq 2$, there are two different effective divisors $D_1, D_2 \in |D|$ such that $-K_Y \sim_{\mathbb{Q}} tD_1 \sim_{\mathbb{Q}} tD_2$, which implies that $Y \simeq \mathbb{P}(1,1,2,3)$ by [18, Th. 1.4(vi)]. But, in this case, $t = q\mathbb{Q}(Y) = 7$. Hence, $a = 7$, $B = 0$, and $\mathcal{O}_Y(D) \simeq \mathcal{O}_Y(1)$. ∎

LEMMA 4.6. *Keep the setting in Proposition 4.3, and suppose that $S = \mathbb{P}^1$. If $-K_Y \sim_{\mathbb{Q}} \omega F + E$ for some positive rational number ω and some effective $\mathbb{Q}$-divisor E, then $\omega \leq 12$.*

This lemma is from the proof of [12, Prop. 4.2]. For the reader's convenience, we recall the proof here.

Proof. We may assume that $\omega > 2$. By Proposition 4.3(3)(4), for a sufficiently large and divisible integer n, $|-nK_Y|$ is movable, and there exists an effective $\mathbb{Q}$-divisor $M \sim -nK_Y$ such that $(Y, \frac{1}{n}M)$ is canonical. Since $-K_Y$ is big, we can write $-K_Y \sim_{\mathbb{Q}} A + N$, where A is an ample $\mathbb{Q}$-divisor and N is an effective $\mathbb{Q}$-divisor. Set $B_\epsilon = \frac{1-\epsilon}{n}M + \epsilon N$ for a rational number $0 < \epsilon < 1$. Take two general fibers F_1, F_2 of g. Denote

$$\Delta = (1 - \frac{2}{\omega})B_\epsilon + \frac{2}{\omega}E + F_1 + F_2.$$

Then

$$-(K_Y + \Delta) \sim_{\mathbb{Q}} -\left(1 - \frac{2}{\omega}\right)(K_Y + B_\epsilon) \sim_{\mathbb{Q}} \left(1 - \frac{2}{\omega}\right)\epsilon A$$

is ample as $\omega > 2$. Hence, by the connectedness lemma [12, Lem. 2.6], $\mathrm{Nklt}(Y, \Delta)$ is connected. By construction, $F_1 \cup F_2 \subset \mathrm{Nklt}(Y, \Delta)$, then $\mathrm{Nklt}(Y, \Delta)$ dominates $\mathbb{P}^1$. By the inversion of adjunction [14, Lem. 5.50], $(F, (1 - \frac{2}{\omega})B_\epsilon|_F + \frac{2}{\omega}E|_F)$ is not klt for a general fiber F of g. As being klt is an open condition on the coefficients, by the arbitrariness of ϵ, it follows that $(F, (1 - \frac{2}{\omega})\frac{1}{n}M|_F + \frac{2}{\omega}E|_F)$ is not klt for a very general fiber F of g.

On the other hand, as $(Y, \frac{1}{n}M)$ is canonical, $(F, \frac{1}{n}M|_F)$ is canonical by Bertini's theorem (see [14, Lem. 5.17]). Since M is a general member of a movable linear system by assumption, $M|_F$ is a general member of a movable linear system on F. So each irreducible component of $M|_F$ is nef. Also, we can take M such that $M|_F$ and $E|_F$ have no common irreducible component. By construction, $\frac{1}{n}M|_F \sim_{\mathbb{Q}} E|_F \sim_{\mathbb{Q}} -K_F$. So we can apply [12, Th. 3.3] to $F, \frac{1}{n}M|_F, E|_F$, which implies that $\frac{2}{\omega} \geq \frac{1}{6}$. Hence, $\omega \leq 12$. ∎

LEMMA 4.7. *Keep the setting in Proposition 4.3, and suppose that S is a del Pezzo surface. Suppose that D is a nonzero effective integral divisor on Y which is movable. If $-K_Y \sim_{\mathbb{Q}} \omega D + E$ for some positive rational number ω and some effective $\mathbb{Q}$-divisor E, then $\omega \leq 12$.*

Proof. As S is a del Pezzo surface with at worst Du Val singularities and $\rho(S) = 1$, there are three cases (see [15], [17, Rem. 3.4(ii)]):

(1) $K_S^2 = 9$ and $S \simeq \mathbb{P}^2$.
(2) $K_S^2 = 8$ and $S \simeq \mathbb{P}(1,1,2)$.
(3) $1 \leq K_S^2 \leq 6$.

Consider the linear system $\mathcal{H}$ on S defined by

$$\mathcal{H} := \begin{cases} |\mathcal{O}_{\mathbb{P}^2}(1)|, & \text{if } S \simeq \mathbb{P}^2, \\ |\mathcal{O}_{\mathbb{P}(1,1,2)}(2)|, & \text{if } S \simeq \mathbb{P}(1,1,2), \\ |-K_S|, & \text{if } 2 \leq K_S^2 \leq 6, \\ |-2K_S|, & \text{if } K_S^2 = 1. \end{cases}$$

Then $\mathcal{H}$ is base-point-free and defines a generically finite map (cf. [7, Th. 8.3.2]). By Bertini's theorem, we can take a general element $H \in \mathcal{H}$ such that H and $G = g^{-1}(H) = g^*H$ are smooth. Note that for a general fiber C of $g|_G$, $C \simeq \mathbb{P}^1$, $(-K_G \cdot C) = 2$, and $G|_G \sim (H^2) \cdot C$.

Note that $g|_G$ is factored through by a ruled surface over H, so $K_G^2 \leq 8 - 8g(H)$. Then

$$\begin{aligned} (-K_Y|_G)^2 &= (-K_G + G|_G)^2 \\ &= K_G^2 + 4H^2 \\ &\leq 8 - 8g(H) + 4H^2 \\ &= -4(K_S \cdot H) \leq 24. \end{aligned}$$

By construction, as $|G|$ defines a morphism from Y to a surface and D is movable, $D|_G$ is an effective nonzero integral divisor for a general G. So we may write

$$-K_Y|_G \sim_{\mathbb{Q}} \omega D|_G + E|_G. \tag{4.1}$$

Take a general fiber C of $g|_G$. If $(D|_G \cdot C) \neq 0$, then by (4.1) intersecting with C, $\omega \leq 2$. If $(D|_G \cdot C) = 0$, then $D|_G$ is vertical over H and thus $D|_G$ is numerically equivalent to a multiple of C. By Proposition 4.3(3), $-K_Y|_G$ is nef. Then, by (4.1), intersecting with $-K_Y|_G$,

$$24 \geq (-K_Y|_G)^2 \geq \omega(-K_Y|_G \cdot D|_G) \geq \omega(-K_Y|_G \cdot C) = 2\omega,$$

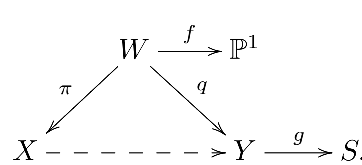
which implies that $\omega \leq 12$. ∎

4.3 A new geometric inequality

Now, we are prepared to prove Theorem 4.2.

Proof of Theorem 4.2. It suffices to show that, if $|-mK_X|$ is composed with a pencil, then $P_{-m} \leq 12m + 1$.

Take $g: Y \to S$ to be the morphism in Proposition 4.3. Take a common resolution $\pi: W \to X$, $q: W \to Y$. We may modify π such that $f: W \to \mathbb{P}^1$ is the fibration induced by $|-mK_X|$ as in §2.1. See the following diagram:



Denote by F_W a general fiber of f. Then

$$\pi^*(-mK_X) \sim (P_{-m} - 1)F_W + E, \tag{4.2}$$

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where E is an effective $\mathbb{Q}$-divisor on W. Set $\omega = \frac{P_{-m}-1}{m}$. Pushing forward (4.2) to Y, we have

$$-K_Y \sim_{\mathbb{Q}} \omega q_* F_W + E_Y, \tag{4.3}$$

where E_Y is an effective $\mathbb{Q}$-divisor on Y. Note that q_*F_W is a general member of a movable linear system.

Case 1. S is a point.

In this case, $\omega \le 7$ by (4.3) and Lemma 4.5.

Case 2. $S = \mathbb{P}^1$.

If $S = \mathbb{P}^1$ and $q_*F_W|_F = 0$, then $q_*F_W \sim F$ and $-K_Y \sim_{\mathbb{Q}} \omega F + E_Y$. By Lemma 4.6, $\omega \le 12$.

If $S = \mathbb{P}^1$ and $q_*F_W|_F \neq 0$, then $q_*F_W|_F$ is a movable effective nonzero integral divisor on F. Restricting (4.3) on F, we have $-K_F \sim_{\mathbb{Q}} \omega(q_*F_W|_F) + E_Y|_F$. By Lemma 4.4, $\omega \le 4$.

Case 3. S is a del Pezzo surface.

In this case, $\omega \le 12$ by (4.3) and Lemma 4.7.

Combining all above cases, we proved that $\frac{P_{-m}-1}{m} = \omega \le 12$ as long as $|-mK_X|$ is composed with a pencil. ∎

Applying Proposition 4.1 and Theorem 4.2, we have the following criteria for $|-mK|$ not composed with a pencil (cf. [6, Prop. 5.4]).

PROPOSITION 4.8. *Let X be a terminal weak $\mathbb{Q}$-Fano threefold. Let t be a positive real number, and let m be a positive integer. If $m \ge t, m \ge \frac{r_{\max} t}{3}$, and one of the following conditions holds:*

(1) $m > -\frac{3}{4} + \sqrt{\frac{12}{t\cdot(-K_X^3)} + 6r_X + \frac{1}{16}}$,

(2) $m > -\frac{3}{4} + \sqrt{\frac{12}{t\cdot(-K_X^3)} + \frac{72}{-K_X^3} + \frac{1}{16}}$,

then $|-mK_X|$ is not composed with a pencil.

Proof. By [6, Prop. 5.3],

$$P_{-m} \ge \frac{1}{12} m(m+1)(2m+1)(-K_X^3) + 1 - \frac{2m}{t}.$$

The assumption implies that either $P_{-m} > r_X(-K_X^3)m + 1$ or $P_{-m} > 12m + 1$. Hence, $|-mK_X|$ is not composed with a pencil by Proposition 4.1 and Theorem 4.2. ∎

By the same method, we have the following corollary.

COROLLARY 4.9. *Let X be a terminal weak $\mathbb{Q}$-Fano threefold. Let t be a positive real number, and let m be a positive integer. If $m \ge t, m \ge \frac{r_{\max} t}{3}$, and $m > -\frac{3}{4} + \sqrt{\frac{12}{t\cdot(-K_X^3)} + \frac{6}{l\cdot(-K_X^3)} + \frac{1}{16}}$ for some positive real number l, then $P_{-m} - 1 > \frac{m}{l}$.*

We illustrate in the following example on how efficient Proposition 4.8 is comparing to [5, Cor. 4.2].

EXAMPLE 4.10. Suppose that X is a terminal weak $\mathbb{Q}$-Fano threefold with $P_{-1} = 0$, $B_X = \{2 \times (1,2), (2,5), (3,7), (4,9)\}$, and $-K_X^3 = \frac{43}{315}$. Then [5, Cor. 4.2] implies that $|-mK_X|$ is not composed with a pencil for all $m \ge 61$ (see the last paragraph of [5, p. 106]). On the other hand, by Proposition 4.8, $|-mK_X|$ is not composed with a pencil for all $m \ge 23$ (see Case 4 of Proof of Theorem 5.6), which significantly improves the previous

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result. Also, it can be computed directly by (2.1) to get $P_{-22} = 260 < 12 \times 22 + 1$, which tells that the estimates in Proposition 4.8 are efficient enough comparing to directly using the Riemann–Roch formula.

4.4 A remark on [5, Cor. 4.2]

In this subsection, we discuss the equality case of [5, Cor. 4.2] for terminal $\mathbb{Q}$-Fano threefolds.

PROPOSITION 4.11. *Let X be a terminal $\mathbb{Q}$-Fano threefold, and let m be a positive integer. If*

$$P_{-m} = r_X(-K_X^3)m + 1$$

and $|-mK_X|$ is composed with a pencil, then:

(1) $r_X(-K_X^3) = 1$.
(2) *If, moreover, the Weil divisor class group of X has no m-torsion element, then $h^0(X, -kK_X) = k+1$ for all $1 \le k \le m$.*

Proof. We recall the proof of [5, Cor. 4.2]. As $|-mK_X|$ is composed with a pencil, take $D = -mK_X$ and keep the notation in §2.1, we have

$$\pi^*(-mK_X) \sim (P_{-m} - 1)S + F, \tag{4.4}$$

where S is a generic irreducible element of $\mathrm{Mov}|\lfloor \pi^*(-mK_X) \rfloor|$ and F is an effective $\mathbb{Q}$-divisor. Then

$$m(-K_X^3) \ge (P_{-m} - 1)(\pi^*(-K_X)^2 \cdot S) \ge \frac{1}{r_X}(P_{-m} - 1)$$

by [5, Lem. 4.1].

Now, by assumption, the equality holds. So $(\pi^*(-K_X)^2 \cdot F) = 0$. This implies that F is π-exceptional as $-K_X$ is ample. So (4.4) implies that

$$-mK_X \sim (P_{-m} - 1)\pi_* S.$$

Then

$$-K_X \sim_{\mathbb{Q}} r_X(-K_X^3)\pi_* S. \tag{4.5}$$

By [9, Lem. 2.3], $(\pi_* S)^3 \ge \frac{1}{r_X}$. Then (4.5) implies that

$$-K_X^3 \ge \frac{(r_X(-K_X^3))^3}{r_X},$$

which implies that $r_X(-K_X^3) = 1$ as it is a positive integer. Under the assumption that the Weil divisor class group of X has no m-torsion element, (4.5) implies that $-K_X \sim \pi_* S$. So the conclusion follows as $-kK_X \sim k\pi_* S$ is composed with a pencil for any $1 \le k \le m$ (see [5, p. 63, Case (f_p)]). □

The following example shows that Proposition 4.11 is nonempty.

EXAMPLE 4.12 [8, List 16.6, No. 88]. A general weighted hypersurface $X_{42} \subset \mathbb{P}(1,1,6,14,21)$ is a terminal $\mathbb{Q}$-Fano threefold with $r_X(-K_X^3) = 1$ and $B_X = \{(1,2),(1,3),(1,7)\}$. By [18, Prop. 2.9], the Weil divisor class group of X is torsion-free. Certainly, $P_{-k} = k+1$ and $|-kK_X|$ is composed with a pencil for $1 \le k \le 5$.

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§5. Proofs of main results

In this section, we apply the birationality criteria (Theorem 3.4 and Corollary 3.7) and the non-pencil criteria (Proposition 4.8) to prove the main theorem. The proof will be divided into several cases:

1. $r_X = 840$.
2. $P_{-2} = 0$.
3. $P_{-2} > 0$, $P_{-1} = 0$, and $r_{\max} \geq 14$.
4. $P_{-2} > 0$, $P_{-1} = 0$, and $r_{\max} \leq 13$.
5. $P_{-1} > 0$ and $r_{\max} \geq 14$.
6. $P_{-1} > 0$ and $r_{\max} \leq 13$.

Here, recall that $r_X = \mathrm{lcm}\{r_i \mid r_i \in \mathcal{R}_X\}$ is the Cartier index of K_X, and $r_{\max} = \max\{r_i \mid r_i \in \mathcal{R}_X\}$ is the maximal local index.

5.1 The case $r_X = 840$

THEOREM 5.1. *Let X be a terminal weak $\mathbb{Q}$-Fano threefold with $r_X = 840$. Then φ_{-m} is birational for all $m \geq 48$.*

Proof. Keep the setting in Notation 3.1. By [6, Lem. 6.5] and the first line of its proof, we know that $r_{\max} = 8$, $P_{-1} \geq 1$, and $-K_X^3 \geq \frac{47}{840}$. Take $m_0 = 8$ and $\nu_0 = 1$. By Corollary 4.9 (with $l = 1$, $t = 4.5$, and $-K_X^3 \geq \frac{47}{840}$), we have $P_{-12} - 1 > 12$.

If $|-12K_X|$ and $|-8K_X|$ are composed with the same pencil, then take $\mu_0' = \frac{12}{P_{-12}-1} < 1$ by Remark 3.2. By Proposition 4.8(2) (with $t = 13.5$ and $-K_X^3 \geq \frac{47}{840}$), we can take $m_1 = 36$. By Lemma 3.3, $N_0 \geq \lceil \frac{840}{8m_1} \rceil = 3$. Then, by Corollary 3.7(1), φ_{-m} is birational for all $m \geq 48$.

If $|-12K_X|$ and $|-8K_X|$ are not composed with the same pencil, then take $m_1 = 12$ and $\mu_0' = m_0 = 8$. Then, by Theorem 3.4(3), φ_{-m} is birational for all $m \geq 36$. ∎

5.2 The case $P_{-2} = 0$

THEOREM 5.2. *Let X be a terminal weak $\mathbb{Q}$-Fano threefold. If $P_{-2} = 0$, then φ_{-m} is birational for all $m \geq 51$.*

Proof. Keep the setting in Notation 3.1. In this case, the possible baskets are classified in [2, Th. 3.5] with 23 cases in total (see Table A.1 in the Appendix). Here, we refer to the numbering in Table A.1.

For Nos. 1–5 of Table A.1, $r_X \leq 84, -K_X^3 \geq \frac{1}{84}$, $P_{-8} \geq 2$, and $r_{\max} \leq 11$. So we can take $m_0 = 8$. By Corollary 4.9 (with $l = 2$ and $t = 5.7$), $P_{-21} - 1 > \frac{21}{2}$. If $|-21K_X|$ and $|-8K_X|$ are composed with the same pencil, then take $\mu_0' = \frac{21}{P_{-21}-1} < 2$ by Remark 3.2. By Proposition 4.8(1) (with $t = 6.6$), we can take $m_1 = 25$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 48$. If $|-21K_X|$ and $|-8K_X|$ are not composed with the same pencil, then take $m_1 = 21$ and $\mu_0' = m_0 = 8$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 51$.

For Nos. 6–13 and Nos. 16–23 of Table A.1, $r_X \leq 78, -K_X^3 \geq \frac{1}{30}$, $P_{-6} \geq 2$, and $r_{\max} \leq 14$. So we can take $m_0 = 6$. By Corollary 4.9 (with $l = 1$ and $t = 3.6$), $P_{-17} - 1 > 17$. If $|-17K_X|$ and $|-6K_X|$ are composed with the same pencil, then take $\mu_0' = \frac{17}{P_{-17}-1} < 1$ by Remark 3.2. By Proposition 4.8(1) (with $t = 4.9$), we can take $m_1 = 23$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 51$. If $|-17K_X|$ and $|-6K_X|$ are not composed with the same pencil, then take $m_1 = 17$ and $\mu_0' = m_0 = 6$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 51$.

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For No. 14 of Table A.1, $r_X = 210, -K_X^3 = \frac{17}{210}$, $P_{-5} = 2$, and $r_{\max} = 7$. So we can take $m_0 = 5$. By Corollary 4.9 (with $l = 1$ and $t = 4.2$), $P_{-10} - 1 > 10$. If $|-10K_X|$ and $|-5K_X|$ are composed with the same pencil, then take $\mu_0' = \frac{10}{P_{-10}-1} < 1$ by Remark 3.2. By Proposition 4.8(2) (with $t = 12$), we can take $m_1 = 30$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 51$. If $|-10K_X|$ and $|-5K_X|$ are not composed with the same pencil, then take $m_1 = 10$ and $\mu_0' = m_0 = 5$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 29$.

For No. 15 of Table A.1, $r_X = 120, -K_X^3 = \frac{3}{40}$, $P_{-5} = 2$, and $r_{\max} = 8$. So we can take $m_0 = 5$. By Corollary 4.9 (with $l = 1$ and $t = 4$), $P_{-11} - 1 > 11$. If $|-11K_X|$ and $|-5K_X|$ are composed with the same pencil, then take $\mu_0' = \frac{11}{P_{-11}-1} < 1$ by Remark 3.2. By Proposition 4.8(1) (with $t = 10$), we can take $m_1 = 27$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 46$. If $|-11K_X|$ and $|-5K_X|$ are not composed with the same pencil, then take $m_1 = 11$ and $\mu_0' = m_0 = 5$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 32$. ∎

5.3 The case $P_{-2} > 0$ and $P_{-1} = 0$

LEMMA 5.3. *Let X be a terminal weak $\mathbb{Q}$-Fano threefold. If $P_{-1} = 0$ and $P_{-2} > 0$, then*

$$\gamma(B_X) \geq 0, \quad 2 \in \mathcal{R}_X, \quad \sigma(B_X) \geq 11. \tag{5.1}$$

Proof. By (2.3), we have $\sigma(B_X) = 10 - 5P_{-1} + P_{-2} = 10 + P_{-2} \geq 11$. Other statements follow from (2.4) and Proposition 2.2(4). ∎

THEOREM 5.4. *Let X be a terminal weak $\mathbb{Q}$-Fano threefold. If $P_{-2} > 0$, $P_{-1} = 0$, and $r_{\max} \geq 14$, then φ_{-m} is birational for all $m \geq 59$. Moreover, φ_{-58} may not be birational only if $B_X = \{(1,2), 2 \times (1,3), (8,17)\}$ and $|-24K_X|$ is composed with a pencil.*

Proof. Keep the setting in Notation 3.1. By [5, Case II of Proof of Th. 3.12] (especially the last paragraph of Subsubcase II-3-iii) or the second paragraph of [5, Case IV of Proof of Th. 1.8] (see Table A.2 in the Appendix), we can see that $r_{\max} \leq 13$ provided $P_{-4} = 1$. Hence, by assumption, $P_{-4} \geq 2$, and we can always take $m_0 = 4$.

Case 1. $r_{\max} \geq 16$.

It is not hard to search by hands or with the help of a computer program to get all possible B_X satisfying (5.1) and $r_{\max} \geq 16$. Here, note that $\sigma(B_X) \geq 11$ implies that $\sum_i r_i > 2\sigma(B_X) \geq 22$.

If $22 \leq r_{\max} \leq 24$, then there is no B_X satisfying (5.1). If $16 \leq r_{\max} \leq 21$, then all possible B_X satisfying (5.1) are listed in Table 1.

Here, we explain briefly how to get Table 1. The algorithm is the following: first, we can list all possible $\mathcal{R}_X$ satisfying $2 \in \mathcal{R}_X$ and $\gamma(B_X) \geq 0$; then we find all possible b_i for those $\mathcal{R}_X$ such that $\sigma(B_X) \geq 11$. For example, let us consider the case $r_{\max} = 17$. As $2 \in \mathcal{R}_X$, $\{2, 17\} \subset \mathcal{R}_X$. So we can list all possible $\mathcal{R}_X$ with $\gamma(B_X) \geq 0$ by enumeration method by considering the second largest r_i:

{2, 5, 17};	{2, 2, 4, 17};	{2, 4, 17};
{2, 3, 3, 17};	{2, 2, 3, 17};	{2, 3, 17};
{2, 2, 2, 2, 17};	{2, 2, 2, 17};	{2, 2, 17};
{2, 17}.		

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Table 1. Baskets satisfying Lemma 5.3 with $r_{\max} \geq 16$.

No.	B_X	$-K^3$
1	$\{(1,2),(10,21)\}$	<0
2	$\{2\times(1,2),(10,21)\}$	$5/21$
3	$\{2\times(1,2),(9,20)\}$	<0
4	$\{2\times(1,2),(9,19)\}$	<0
5	$\{(1,2),(1,3),(9,19)\}$	<0
6	$\{3\times(1,2),(9,19)\}$	$9/38$
7	$\{3\times(1,2),(8,19)\}$	$5/38$
8	$\{4\times(1,2),(7,18)\}$	$5/18$
9	$\{(1,2),(2,5),(8,17)\}$	<0
10	$\{3\times(1,2),(8,17)\}$	<0
11	$\{2\times(1,2),(1,3),(8,17)\}$	<0
12	$\{2\times(1,2),(1,4),(8,17)\}$	<0
13	$\{(1,2),2\times(1,3),(8,17)\}$	$7/102$
14	$\{4\times(1,2),(8,17)\}$	$4/17$
15	$\{4\times(1,2),(7,17)\}$	$2/17$
16	$\{2\times(1,2),(2,5),(7,16)\}$	$11/80$
17	$\{4\times(1,2),(7,16)\}$	<0
18	$\{3\times(1,2),(1,3),(7,16)\}$	$5/48$
19	$\{5\times(1,2),(7,16)\}$	$7/16$

Then all possible B_X with $\sigma(B_X) \geq 11$ are listed in Table 1; for instance, there is no such basket B_X with $\mathcal{R}_X = \{2,4,17\}$ because in this case $\sigma(B_X) \leq 1+1+8 = 10$.

For No. 2 of Table 1, $-K_X^3 = \frac{5}{21}$, and in this case, $P_{-2} = 2$, $r_X = 42$, and $r_{\max} = 21$. We can take $\mu_0' = m_0 = 2$. By Proposition 4.8(1) (with $t = 2.28$), we can take $m_1 = 16$. Then, by Theorem 3.4(1), φ_{-m} is birational for all $m \geq 54$.

For other cases with $-K_X^3 > 0$, we have $-K_X^3 \geq \frac{7}{102}$ and $r_{\max} \leq 19$. By Corollary 4.9 (with $l = 1, t = 2$), $P_{-13} - 1 > 13$. If $|-13K_X|$ and $|-4K_X|$ are not composed with the same pencil, then take $m_1 = 13$ and $\mu_0' = m_0 = 4$. Then, by Theorem 3.4(1), φ_{-m} is birational for all $m \geq 51$.

So we may assume that $|-13K_X|$ and $|-4K_X|$ are composed with the same pencil. Then, in the following, we can take $\mu_0' = \frac{13}{P_{-13}-1} < 1$ by Remark 3.2 and $m_0 = 4$.

For Nos. 6–8 of Table 1, $-K_X^3 \geq \frac{5}{38}$, $r_X \leq 38$, and $r_{\max} \leq 19$. By Proposition 4.8(1) (with $t = 2.4$), we can take $m_1 = 16$. Then, by Theorem 3.4(1), φ_{-m} is birational for all $m \geq 50$.

For Nos. 14–16 and Nos. 18 and 19 of Table 1, $-K_X^3 \geq \frac{5}{48}$, $r_X \leq 80$, and $r_{\max} \leq 17$. By Proposition 4.8(1) (with $t = 3.88$), we can take $m_1 = 22$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 56$.

For No. 13 of Table 1, $-K_X^3 = \frac{7}{102}$, $r_X = 102$, and $r_{\max} = 17$. By Proposition 4.8(1) (with $t = 3.6$), we can take $m_1 = 25$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 59$. Moreover, if $|-24K_X|$ is not composed with a pencil, then take $m_1 = 24$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 58$.

Case 2. $14 \leq r_{\max} \leq 15$.

For the remaining cases $14 \leq r_{\max} \leq 15$, we have $-K_X^3 \geq \frac{1}{30}$ by Proposition 2.2(3) as $P_{-4} \geq 2$. By Corollary 4.9 (with $l = 1, t = 3.4$), $P_{-17} - 1 > 17$.

If $|-17K_X|$ and $|-4K_X|$ are not composed with the same pencil, then take $m_1 = 17$ and $\mu_0' = m_0 = 4$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 51$.

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If $|-17K_X|$ and $|-4K_X|$ are composed with the same pencil, then take $\mu_0' = \frac{17}{P_{-17}-1} < 1$ by Remark 3.2 and $m_0 = 4$.

If $r_{\max} = 15$, then we claim that $r_X \leq 60$. In fact, as $\{15,2\} \subset \mathcal{R}_X$, by (2.4), $s \notin \mathcal{R}_X$ for all $s > 7$. If $7 \notin \mathcal{R}_X$, then r_X divides 60. If $7 \in \mathcal{R}_X$, then $\mathcal{R}_X = \{15,7,2\}$ by (2.4), and moreover $B_X = \{(1,2),(3,7),(7,15)\}$ by $\sigma(B_X) \geq 11$. But this basket has $-K_X^3 < 0$ by (2.2), which is absurd. Hence, $r_X \leq 60$. By Proposition 4.8(1) (with $t = 4$, $-K_X^3 \geq \frac{1}{30}$), we can take $m_1 = 21$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 51$.

If $r_{\max} = 14$, then we claim that $B_X = \{6 \times (1,2),(5,14)\}$. Suppose that $B_X = \{(b_1,r_1),\dots,(b_k,r_k),(b,14)\}$ with $b \in \{1,3,5\}$. Then $\sigma(B_X) \geq 11$ implies that $\sum_{i=1}^k b_i \geq 6$. If there exists some $r_i > 2$, then $\sum_{i=1}^k r_i > 2\sum_{i=1}^k b_i \geq 12$, that is, $\sum_{i=1}^k r_i \geq 13$. So (2.4) implies that $\sum_{i=1}^k \frac{1}{r_i} \geq 3 - \frac{1}{14}$. On the other hand, (2.4) implies that $\frac{3}{2}k + 14 - \frac{1}{14} \leq 24$, which says that $k \leq 6$. So $\sum_{i=1}^k \frac{1}{r_i} \leq 5 \times \frac{1}{2} + \frac{1}{3} < 3 - \frac{1}{14}$, a contradiction. So all $r_i = 2$ and $k \geq 6$. Then $\gamma(B_X) \geq 0$ implies that $k = 6$, and $\sigma(B_X) \geq 11$ implies that $b = 5$. We conclude that $B_X = \{6 \times (1,2),(5,14)\}$. In this case, $-K_X^3 = \frac{3}{14}$ and $r_X = r_{\max} = 14$. By Proposition 4.8(1) (with $t = 2$), we can take $m_1 = 10$. Then, by Theorem 3.4(1), φ_{-m} is birational for all $m \geq 32$.

Combining all above cases, we have proved the theorem. ∎

LEMMA 5.5. (cf. [2, Case I of Proof of Th. 4.4]). *Let X be a terminal weak $\mathbb{Q}$-Fano threefold with $P_{-1} = 0$ and $P_{-2} > 0$. If $P_{-4} \geq 2$ and $-K_X^3 < \frac{1}{12}$, then B_X is dominated by one of the following initial baskets:*

$$\{8 \times (1,2), 3 \times (1,3)\},$$
$$\{9 \times (1,2),(1,4),(1,5)\},$$
$$\{9 \times (1,2),(1,4),(1,6)\},$$
$$\{9 \times (1,2), 2 \times (1,5)\}.$$

Note that in the latter three cases, all possible packings have $r_{\max} \leq 9$.

Proof. Following [2, Case I of Proof of Th. 4.4], we only need to consider the cases $(P_{-3},P_{-4}) = (1,2)$ or $(0,2)$ in [2, Subcase I-3 of Proof of Th. 4.4].

If $(P_{-3},P_{-4}) = (1,2)$, then [2, Subcase I-3 of Proof of Th. 4.4] shows that B_X is dominated by $\{8 \times (1,2), 3 \times (1,3)\}$. (Actually, it shows moreover that B_X is dominated by $\{7 \times (1,2),(2,5),2 \times (1,3)\}$.)

If $(P_{-3},P_{-4}) = (0,2)$, then $P_{-1} = 0$ and $P_{-2} = 1$. Then, by (2.5)–(2.7), $n^0_{1,2} = 9$, $n^0_{1,3} = 0$, and $n^0_{1,4} + \sigma_5 = 2$. So B_X is dominated by $\{9 \times (1,2),(1,s_1),(1,s_2)\}$ for some $s_2 \geq s_1 \geq 4$. The case $(s_1,s_2) = (4,4)$ is ruled out by [2, Subcase I-3 of Proof of Th. 4.4]. Hence, we get the conclusion by (2.4) and [2, Lem. 3.1]. ∎

THEOREM 5.6. *Let X be a terminal weak $\mathbb{Q}$-Fano threefold. If $P_{-1} = 0, P_{-2} > 0$, and $r_{\max} \leq 13$, then φ_{-m} is birational for all $m \geq 56$.*

Proof. Keep the setting in Notation 3.1. By Proposition 2.2, $P_{-6} \geq 2$ and $-K_X^3 \geq \frac{1}{70}$. We always take $\nu_0 = 2$.

If $P_{-4} = 1$, then $P_{-2} = 1$. Following the second paragraph of [5, Case IV of Proof of Th. 1.8] (see Table A.2 in the Appendix), we have $r_X \leq 130$. By Corollary 4.9 (with $l = 1, t = 5.5$, and $-K_X^3 \geq \frac{1}{70}$), $P_{-24} - 1 > 24$. If $|-24K_X|$ and $|-6K_X|$ are not composed with the same pencil, then take $m_1 = 24$ and $\mu_0' = m_0 = 6$. Then, by Theorem 3.4(2), φ_{-m} is birational

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for all $m \geq 56$. If $|-24K_X|$ and $|-6K_X|$ are composed with the same pencil, then take $m_0 = 6$ and $\mu_0' = \frac{24}{P_{-24}-1} < 1$ by Remark 3.2. By Proposition 4.8(1) (with $t = 6.9$), we can take $m_1 = 30$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 56$.

From now on, we assume that $P_{-4} \geq 2$. Note that $-K_X^3 \geq \frac{1}{30}$ by Proposition 2.2(3). By Corollary 4.9 (with $l = 1$ and $t = 3.6$), $P_{-16} - 1 > 16$. If $|-16K_X|$ and $|-4K_X|$ are not composed with the same pencil, then take $m_1 = 16$ and $\mu_0' = m_0 = 4$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 46$.

In the following discussions, we assume that $|-16K_X|$ and $|-4K_X|$ are composed with the same pencil. We can always take $m_0 = 4$ and $\mu_0' = \frac{16}{P_{-16}-1} < 1$ by Remark 3.2.

Case 1. $r_{\max} \leq 6$ or $r_{\max} \in \{10, 12\}$.

If $r_{\max} \leq 6$, then $r_X \leq 60$. If $r_{\max} = 10$ (resp. $r_{\max} = 12$), then $r_X \leq 210$ (resp. $r_X \leq 84$) by [5, p. 107]. Then, by Corollary 3.7(2), φ_{-m} is birational for all $m \geq 52$.

Case 2. $r_{\max} = 7$.

If $r_{\max} = 7$, then r_X divides $\mathrm{lcm}(2,3,4,5,6,7) = 420$. Hence, either $r_X = 420$ or $r_X \leq 210$. If $r_X = 420$, then $\{4,5,7\} \subset \mathcal{R}_X$ and one element of $\{3,6\}$ is in $\mathcal{R}_X$. Suppose that

$$B_X = \{(b_1, r_1), \ldots, (b_k, r_k), (1, r), (1, 4), (a_5, 5), (a_7, 7)\},$$

where $r \in \{3,6\}$, $a_5 \leq 2$, and $a_7 \leq 3$. Then $\sigma(B_X) \geq 11$ implies that $\sum_{i=1}^k b_i \geq 4$. Lemma 2.3 implies that

$$\gamma(B_X) \leq 24 - \left(7 - \frac{1}{7} + 5 - \frac{1}{5} + 4 - \frac{1}{4} + 3 - \frac{1}{3} + 4 \times \frac{3}{2}\right) < 0, \tag{5.2}$$

a contradiction. Hence, $r_X \leq 210$. Then, by Corollary 3.7(2), φ_{-m} is birational for all $m \geq 43$.

Case 3. $r_{\max} = 8$.

If $r_{\max} = 8$, then we claim that $r_X \leq 168$. In fact, if $r_X > 168$, then $\{5,7\} \subset \mathcal{R}_X$. Suppose that

$$B_X = \{(b_1, r_1), \ldots, (b_k, r_k), (a_5, 5), (a_7, 7), (a_8, 8)\},$$

where $a_5 \leq 2$, $a_7 \leq 3$, and $a_8 \leq 3$. Then $\sigma(B_X) \geq 11$ implies that $\sum_{i=1}^k b_i \geq 3$. Similar to (5.2), Lemma 2.3 implies that $\gamma(B_X) < 0$, a contradiction. Hence, $r_X \leq 168$.

Then, by Corollary 3.7(2), φ_{-m} is birational for all $m \geq 43$.

Case 4. $r_{\max} = 9$.

If $r_{\max} = 9$, then we claim that $r_X \leq 252$ or $B_X = \{2 \times (1,2), (2,5), (3,7), (4,9)\}$.

If 7 and 8 are not in $\mathcal{R}_X$, then $r_X \leq 180$.

If $8 \in \mathcal{R}_X$, then as $\{2,8,9\} \subset \mathcal{R}_X$, we know that 6 and 7 are not in $\mathcal{R}_X$ by $\gamma(B_X) \geq 0$. If $5 \notin \mathcal{R}_X$, then $r_X = 72$. If $5 \in \mathcal{R}_X$, then $\mathcal{R}_X = \{2,5,8,9\}$ as $\gamma(B_X) \geq 0$, but in this case, $\sigma(B_X) \leq 10$, a contradiction.

If $7 \in \mathcal{R}_X$, then as $\{2,7,9\} \subset \mathcal{R}_X$, we know that at most one element of $\{4,5,6\}$ is in $\mathcal{R}_X$ by $\gamma(B_X) \geq 0$. If $5 \notin \mathcal{R}_X$, then $r_X \leq 252$. If $5 \in \mathcal{R}_X$, then by $\sigma(B_X) \geq 11$ and $\gamma(B_X) \geq 0$, it is not hard to check that the only possible basket is $B_X = \{2 \times (1,2), (2,5), (3,7), (4,9)\}$. This concludes the claim.

If $r_X \leq 252$, then by Corollary 3.7(2), φ_{-m} is birational for all $m \geq 50$.

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If $B_X = \{2\times(1,2),(2,5),(3,7),(4,9)\}$, then $-K_X^3 = \frac{43}{315}$. By Proposition 4.8(2) (with $t = 7.6$), we can take $m_1 = 23$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 41$.

Case 5. $r_{\max} = 11$.

If $r_{\max} = 11$, then we claim that $r_X \leq 264$ or $B_X = \{3\times(1,2),(1,3),(2,5), (5,11)\}$ or $B_X = \{2\times(1,2),(1,3),(3,7),(5,11)\}$.

As $\{2,11\} \subset \mathcal{R}_X$, we know that at most one element of $\{6,7,8,9,10\}$ is in $\mathcal{R}_X$ by $\gamma(B_X) \geq 0$. If $10 \in \mathcal{R}_X$, then $r_X = 110$ by [5, p. 107]. If $9 \in \mathcal{R}_X$ or $8 \in \mathcal{R}_X$, then $r_X \leq 264$ by [5, p. 107]. If $7 \in \mathcal{R}_X$, then $5 \notin \mathcal{R}_X$ by $\gamma(B_X) \geq 0$. So either $r_X = 154$ or at least one element of $\{3,4\}$ is in $\mathcal{R}_X$. For the latter case, it is not hard to check that the only basket satisfying $\sigma(B_X) \geq 11$ and $\gamma(B_X) \geq 0$ is $B_X = \{2\times(1,2),(1,3),(3,7),(5,11)\}$. If $6 \in \mathcal{R}_X$, then we get a contradiction by Lemma 2.3 as (5.2).

If none element of $\{6,7,8,9,10\}$ is in $\mathcal{R}_X$, then r_X divides 660 and $r_X < 660$ by [5, p. 107]. So either $r_X \leq 220$ or $r_X = 330$. Moreover, if $r_X = 330$, then $\{2,3,5,11\} \subset \mathcal{R}_X$ and $4 \notin \mathcal{R}_X$, and it is not hard to check that the only basket satisfying $\sigma(B_X) \geq 11$ and $\gamma(B_X) \geq 0$ is $B_X = \{3\times(1,2),(1,3),(2,5),(5,11)\}$. This concludes the claim.

If $r_X \leq 264$, then by Corollary 3.7(2), φ_{-m} is birational for all $m \geq 56$.

If $B_X = \{3\times(1,2),(1,3),(2,5),(5,11)\}$, then $-K_X^3 = \frac{31}{330}$. By Proposition 4.8(2) (with $t = 7.6$), we can take $m_1 = 28$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 50$.

If $B_X = \{2\times(1,2),(1,3),(3,7),(5,11)\}$, then $-K_X^3 = \frac{50}{462}$. By Proposition 4.8(2) (with $t = 7$), we can take $m_1 = 26$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 48$.

Case 6. $r_{\max} = 13$.

If $r_{\max} = 13$, then $r_X \leq 390$ or $r_X = 546$ by [5, p. 107].

If $r_X = 546$, then again by [5, p. 107], $B_X = \{(1,2),(1,3),(3,7),(6,13)\}$ and $-K_X^3 = \frac{61}{546}$. By Proposition 4.8(2) (with $t = 6$), we can take $m_1 = 26$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 52$.

If $r_X \leq 390$ and $-K_X^3 \geq \frac{1}{12}$, then by Proposition 4.8(2) (with $t = 6.9$), we can take $m_1 = 30$. Then, by Theorem 3.4(2), φ_{-m} is birational for all $m \geq 56$.

If $r_X \leq 390$ and $-K_X^3 < \frac{1}{12}$, then by Lemma 5.5, B_X is dominated by $\{8\times(1,2),3\times(1,3)\}$. As $r_{\max} = 13$, this implies that B_X is dominated by either $\{3\times(1,2),(6,13),2\times(1,3)\}$ or $\{6\times(1,2),(5,13)\}$. So we get the following possibilities of B_X by $\gamma(B_X) \geq 0$:

$$
\begin{aligned}
&\{3\times(1,2),(6,13),2\times(1,3)\} && -K^3 = 5/78,\\
&\{2\times(1,2),(6,13),(2,5),(1,3)\} && -K^3 = 19/195,\\
&\{2\times(1,2),(6,13),(3,8)\} && -K^3 = 11/104,\\
&\{(1,2),(6,13),(3,7),(1,3)\} && -K^3 = 61/546,\\
&\{6\times(1,2),(5,13)\} && -K^3 = 1/13.
\end{aligned}
$$

In the above list, only the first and the last have $-K_X^3 < \frac{1}{12}$. In particular, in these cases, $r_X \leq 78$. Then, by Corollary 3.7(2), φ_{-m} is birational for all $m \geq 55$.

Combining all above cases, we have proved the theorem. ∎

5.4 The case $P_{-1} > 0$

LEMMA 5.7. *Let X be a terminal weak $\mathbb{Q}$-Fano threefold. If $P_{-1} > 0$ and $r_{\max} \geq 16$, then $P_{-4} \geq 2$.*

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Proof. If $P_{-4}=1$, then $P_{-2}=P_{-3}=1$. Since $r_{\max}\geq 16$, by the classification in [2, Subsubcase II-4f of Proof of Th. 4.4], B_X is dominated by $\{2\times(1,2),2\times(1,3),(1,s_1),(1,s_2)\}$ with $s_2\geq s_1\geq 5$, which means that $s_1+s_2=r_{\max}\geq 16$. But in this case $2\times\frac{3}{2}+2\times\frac{8}{3}+16-\frac{1}{16}>24$, contradicting (2.4) and [2, Lem. 3.1]. So $P_{-4}\geq 2$. ∎

THEOREM 5.8. *Let X be a terminal weak $\mathbb{Q}$-Fano threefold. If $P_{-1}>0$ and $r_{\max}\geq 14$, then φ_{-m} is birational for all $m\geq 52$.*

Proof. Keep the setting in Notation 3.1. We always take $\nu_0=1$.

If $14\leq r_{\max}\leq 15$, then $r_X\leq 210$ by [5, p. 104]. By Proposition 2.2(2), we can take $\mu_0'=m_0=8$. Then, by Corollary 3.7(2), φ_{-m} is birational for all $m\geq 52$.

If $r_{\max}=24$, then $B_X=\{(b,24)\}$ with $b\in\{1,5,7,11\}$. If $P_{-1}=1$, then $b=\sigma(B_X)=5+P_{-2}\geq 6$ by (2.3); hence, $b\geq 7$ and $P_{-2}\geq 2$. By (2.2), we have $-K_X^3\geq\frac{23}{24}$. Similarly, if $P_{-1}=2$, then $P_{-2}\geq 2P_{-1}-1=3$, and thus $b\geq 5$. Hence, by (2.2), $-K_X^3\geq\frac{47}{24}$. If $P_{-1}\geq 3$, then by (2.2), $-K_X^3\geq b-\frac{b^2}{24}\geq\frac{23}{24}$. In summary, $-K_X^3\geq\frac{23}{24}$ and $P_{-2}\geq 2$. We can take $\mu_0'=m_0=2$. By Proposition 4.8(2) (with $t=1$), we can take $m_1=9$. Then, by Theorem 3.4(1), φ_{-m} is birational for all $m\geq 33$.

In the following, we consider $16\leq r_{\max}\leq 23$. By Lemma 5.7, we always take $\mu_0'=m_0=4$ and $\nu_0=1$.

If $r_{\max}=23$, then $B_X=\{(b,23)\}$ with $1\leq b\leq 11$. If $P_{-1}=1$, then $b=5+P_{-2}\geq 6$ and thus by (2.2) $-K_X^3\geq\frac{10}{23}$. If $P_{-1}=2$, then $b=P_{-2}\geq 2P_{-1}-1=3$; hence, by (2.2) $-K_X^3\geq\frac{14}{23}$. If $P_{-1}\geq 3$, then $-K_X^3\geq b-\frac{b^2}{23}\geq\frac{22}{23}$. In summary, $-K_X^3\geq\frac{10}{23}$. By Proposition 4.8(1) (with $t=\frac{36}{23}$), we can take $m_1=12$. Then, by Theorem 3.4(1), φ_{-m} is birational for all $m\geq 48$.

If $20\leq r_{\max}\leq 22$, then by (2.4), we have $r_X\leq 60$. Then, by Corollary 3.7(2), φ_{-m} is birational for all $m\geq 50$.

If $18\leq r_{\max}\leq 19$, then $r_X\leq 190$ by [5, p. 104]. Then, by Corollary 3.7(2), φ_{-m} is birational for all $m\geq 52$.

If $16\leq r_{\max}\leq 17$, then $r_X\leq 240$ by [5, p. 104]. Then, by Corollary 3.7(2), φ_{-m} is birational for all $m\geq 52$. ∎

THEOREM 5.9. *Let X be a terminal weak $\mathbb{Q}$-Fano threefold. If $P_{-1}>0$ and $r_{\max}\leq 13$, then φ_{-m} is birational for all $m\geq 58$.*

Proof. Keep the setting in Notation 3.1. By Theorem 5.1 and Proposition 2.2(1), we may assume that $r_X\leq 660$. By Proposition 2.2(2), we can take $m_0=8$ and $\nu_0=1$. We take $\mu_0'=8$ unless stated otherwise.

Case 1. $r_{\max}\leq 8$.

If $r_{\max}\leq 8$, then r_X divides $\mathrm{lcm}(8,7,6,5)=840$. As $r_X\leq 660$, $r_X=420$ or $r_X\leq 280$.

If $r_X=420$, then by Proposition 4.8(1) (with $t=19.5$, $-K_X^3\geq\frac{1}{330}$), we can take $m_1=52$. By Lemma 3.3, $N_0\geq\lceil\frac{420}{8m_1}\rceil=2$. Then, by Corollary 3.7(1), φ_{-m} is birational for all $m\geq 48$.

If $r_X\leq 280$, then by Corollary 3.7(1), φ_{-m} is birational for all $m\geq 55$.

Case 2. $r_{\max}=9$.

If $r_{\max}=9$, then r_X divides 2,520. As $r_X\leq 660$ and 9 divides r_X, we have $r_X\leq 360$ or $r_X\in\{504,630\}$.

If $r_X\leq 360$, then by Corollary 4.9 (with $l=4,t=10$, and $-K_X^3\geq\frac{1}{330}$), we have $P_{-30}-1>\frac{30}{4}$. If $|-30K_X|$ and $|-8K_X|$ are composed with the same pencil, then take $\mu_0'=\frac{30}{P_{-30}-1}<4$ by Remark 3.2. Then, by Corollary 3.7(1), φ_{-m} is birational for all $m\geq 57$. If $|-30K_X|$

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and $|-8K_X|$ are not composed with the same pencil, then take $m_1 = 30$ and $\mu_0' = m_0 = 8$. Then, by Theorem 3.4(3), φ_{-m} is birational for all $m \geq 56$.

If $r_X = 630$, then $-K_X^3 \geq \frac{1}{315}$ (note that $-K_X^3 \geq \frac{1}{330}$ and $r_X(-K_X^3)$ is an integer). By Corollary 4.9 (with $l = 2.8$ and $t = 11$), we have $P_{-33} - 1 > \frac{33}{2.8}$. If $|-33K_X|$ and $|-8K_X|$ are composed with the same pencil, then take $\mu_0' = \frac{33}{P_{-33}-1} < 2.8$ by Remark 3.2. By Proposition 4.8(1) (with $t = 20$ and $-K_X^3 \geq \frac{1}{315}$), we can take $m_1 = 63$. By Lemma 3.3, $N_0 \geq \lceil \frac{630}{9m_1} \rceil = 2$. Then, by Corollary 3.7(1), φ_{-m} is birational for all $m \geq 52$. If $|-33K_X|$ and $|-8K_X|$ are not composed with the same pencil, then take $m_1 = 33$ and $\mu_0' = m_0 = 8$. By Lemma 3.3, $N_0 \geq \lceil \frac{630}{9m_1} \rceil = 3$. Then, by Corollary 3.7(1), φ_{-m} is birational for all $m \geq 48$.

If $r_X = 504$, then $\mathcal{R}_X = \{9,8,7\}$ by (2.4). Write $B_X = \{(a,7),(b,8),(c,9)\}$, where $a \leq 3, b \in \{1,3\}$ and $c \in \{1,2,4\}$. If $P_{-1} \geq 2$, then by (2.2),

$$\begin{aligned} -K_X^3 &= 2P_{-1} + \frac{a(7-a)}{7} + \frac{b(8-b)}{8} + \frac{c(9-c)}{9} - 6 \\ &\geq \frac{6}{7} + \frac{7}{8} + \frac{8}{9} - 2 > 0.6. \end{aligned} \tag{5.3}$$

If $P_{-1} = 1$, then by (2.2) and (2.3),

$$\begin{aligned} -K_X^3 &= \frac{a(7-a)}{7} + \frac{b(8-b)}{8} + \frac{c(9-c)}{9} - 4 > 0, \\ \sigma(B_X) &= a + b + c = 5 + P_{-2} \geq 6. \end{aligned} \tag{5.4}$$

So it is easy to check that $-K_X^3 \geq \frac{73}{504}$ by considering all possible values of (a,b,c). By Proposition 4.8(2) (with $t = 7.3$ and $-K_X^3 \geq \frac{73}{504}$), we can take $m_1 = 22$. By Lemma 3.3, $N_0 \geq \lceil \frac{504}{9m_1} \rceil = 3$. Then, by Corollary 3.7(1), φ_{-m} is birational for all $m \geq 44$.

Case 3. $r_{\max} = 10$.

If $r_{\max} = 10$, then we claim that $r_X \leq 210$ or $r_X = 420$.

By (2.4), at most one element of $\{7,8,9\}$ is in $\mathcal{R}_X$. If $7 \notin \mathcal{R}_X$, then r_X divides either $120 = \operatorname{lcm}(10,8,60)$ or $180 = \operatorname{lcm}(10,9,60)$. If $7 \in \mathcal{R}_X$, then r_X divides $420 = \operatorname{lcm}(10,7,60)$, so either $r_X \leq 210$ or $r_X = 420$. This concludes the claim.

If $r_X \leq 210$, then by Corollary 3.7(1), φ_{-m} is birational for all $m \geq 48$.

If $r_X = 420$, then by (2.4), $\mathcal{R}_X = \{10,7,4,3\}$. Write $B_X = \{(1,3),(1,4),\ (a,7),(b,10)\}$, where $a \leq 3$ and $b \in \{1,3\}$. If $P_{-1} \geq 2$, then by (2.2),

$$\begin{aligned} -K_X^3 &= 2P_{-1} + \frac{2}{3} + \frac{3}{4} + \frac{a(7-a)}{7} + \frac{b(10-b)}{10} - 6 \\ &\geq \frac{2}{3} + \frac{3}{4} + \frac{6}{7} + \frac{9}{10} - 2 > 1. \end{aligned} \tag{5.5}$$

If $P_{-1} = 1$, then by (2.2) and (2.3),

$$\begin{aligned} -K_X^3 &= \frac{2}{3} + \frac{3}{4} + \frac{a(7-a)}{7} + \frac{b(10-b)}{10} - 4 > 0, \\ \sigma(B_X) &= 2 + a + b = 5 + P_{-2} \geq 6. \end{aligned} \tag{5.6}$$

So it is easy to check that $-K_X^3 \geq \frac{13}{420}$ by considering all possible values of (a,b). By Corollary 4.9 (with $l = 1$ and $t = 4.8$), we have $P_{-16} - 1 > 16$. If $|-16K_X|$ and $|-8K_X|$ are composed with the same pencil, then take $\mu_0' = \frac{16}{P_{-16}-1} < 1$ by Remark 3.2. Then, by Corollary 3.7(1), φ_{-m} is birational for all $m \geq 58$. If $|-16K_X|$ and $|-8K_X|$ are not composed with the same

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pencil, then take $m_1 = 16$ and $\mu_0' = m_0 = 8$. Then, by Theorem 3.4(3), φ_{-m} is birational for all $m \geq 44$.

Case 4. $r_{\max} = 11$.

If $r_{\max} = 11$, then we claim that $r_X \leq 330$ or $r_X \in \{385, 396, 440, 462, 660\}$.

By (2.4), at most one element of $\{7, 8, 9, 10\}$ is in $\mathcal{R}_X$. So r_X divides one element of $\{1,980, 1,320, 4,620\}$. As $r_X \leq 660$ and 11 divides r_X, it is clear that $r_X \leq 330$ or $r_X \in \{385, 396, 440, 462, 495, 660\}$. Moreover, if $r_X = 495$, then $\{11, 9, 5\} \subset \mathcal{R}_X$, which contradicts (2.4). This concludes the claim.

If $r_X \leq 330$, then by Corollary 4.9 (with $l = 7.6$, $t = 7.6$, and $-K_X^3 \geq \frac{1}{330}$), $P_{-28} - 1 > \frac{28}{7.6}$. If $|-28K_X|$ and $|-8K_X|$ are composed with the same pencil, then take $\mu_0' = \frac{28}{P_{-28}-1} < 7.6$ by Remark 3.2. Then, by Corollary 3.7(1), φ_{-m} is birational for all $m \geq 58$. If $|-28K_X|$ and $|-8K_X|$ are not composed with the same pencil, then take $m_1 = 28$ and $\mu_0' = m_0 = 8$. Then, by Theorem 3.4(3), φ_{-m} is birational for all $m \geq 58$.

If $r_X = 385$ (resp. 396), then $-K_X^3 \geq \frac{2}{385}$ (resp. $\geq \frac{2}{396}$). By Corollary 4.9 (with $l = 1.5$ and $t = 9$), $P_{-33} - 1 > \frac{33}{1.5}$. If $|-33K_X|$ and $|-8K_X|$ are composed with the same pencil, then take $\mu_0' = \frac{33}{P_{-33}-1} < 1.5$ by Remark 3.2. Then, by Corollary 3.7(1), φ_{-m} is birational for all $m \geq 56$ (resp. ≥ 57). If $|-33K_X|$ and $|-8K_X|$ are not composed with the same pencil, then take $m_1 = 33$ and $\mu_0' = m_0 = 8$. By Lemma 3.3, $N_0 \geq \lceil \frac{385}{11m_1} \rceil = 2$. Then, by Corollary 3.7(2), φ_{-m} is birational for all $m \geq 47$ (resp. ≥ 48).

We claim that if $r_X \in \{440, 462, 660\}$, then $-K_X^3 \geq \frac{74}{462}$ or $B_X = \{(1,2), (2,5), (1,3), (1,4), (1,11)\}$ with $-K_X^3 = \frac{17}{660}$.

If $r_X = 440$, then by (2.4), $\mathcal{R}_X = \{11, 8, 5\}$. Arguing similarly as (5.3), we get $-K_X^3 > 0.5$ when $P_{-1} \geq 2$. Arguing similarly as (5.4), we get constrains for B_X when $P_{-1} = 1$. By (2.2) and considering all the possible baskets when $P_{-1} = 1$, we can see that $-K_X^3 \geq \frac{97}{440}$.

If $r_X = 462$, then by (2.4), $\mathcal{R}_X = \{11, 7, 6\}$ or $\{11, 7, 3, 2\}$ or $\{11, 7, 3, 2, 2\}$. Arguing similarly as (5.5), we get $-K_X^3 > 0.5$ or $-K_X^3 > 0.9$ or $-K_X^3 > 1$ when $P_{-1} \geq 2$. Arguing similarly as (5.6), we get constrains for B_X when $P_{-1} = 1$. By (2.2) and considering all the possible baskets when $P_{-1} = 1$, we can see that $-K_X^3 \geq \frac{85}{462}$ or $-K_X^3 \geq \frac{95}{462}$ or $-K_X^3 \geq \frac{74}{462}$ unless $B_X = \{2 \times (1,2), (1,3), (2,7), (1,11)\}$ with $-K_X^3 = \frac{1}{231}$. But the last basket has $P_{-5} = 0$, which is absurd.

If $r_X = 660$, then by (2.4), $\mathcal{R}_X = \{11, 5, 4, 3\}$ or $\{11, 5, 4, 3, 2\}$. Arguing similarly as (5.5), we get $-K_X^3 > 1$ or $-K_X^3 > 1.5$ when $P_{-1} \geq 2$. Arguing similarly as (5.6), we get constrains for B_X when $P_{-1} = 1$. By (2.2) and considering all the possible baskets when $P_{-1} = 1$, we can see that $-K_X^3 \geq \frac{167}{660}$ or $-K_X^3 \geq \frac{233}{660}$ unless $B_X = \{(1,2), (2,5), (1,3), (1,4), (1,11)\}$ with $-K_X^3 = \frac{17}{660}$.

To summarize, we conclude the claim that $-K_X^3 \geq \frac{74}{462}$ or $B_X = \{(1,2), (2,\ 5), (1,3), (1,4), (1,11)\}$ with $-K_X^3 = \frac{17}{660}$.

If $-K_X^3 \geq \frac{74}{462}$, then by Proposition 4.8(2) (with $t = 5.7$), we can take $m_1 = 21$. Then, by Theorem 3.4(3), φ_{-m} is birational for all $m \geq 51$.

Now, we consider the case $B_X = \{(1,2), (2,5), (1,3), (1,4), (1,11)\}$ with $-K_X^3 = \frac{17}{660}$. By Corollary 4.9 (with $l = 1$ and $t = 4.8$), $P_{-18} - 1 > 18$. If $|-18K_X|$ and $|-8K_X|$ are composed with the same pencil, then take $\mu_0' = \frac{18}{P_{-18}-1} < 1$ by Remark 3.2. By Proposition 4.8(2) (with $t = 14.4$), we can take $m_1 = 53$. By Lemma 3.3, $N_0 \geq \lceil \frac{660}{11m_1} \rceil = 2$. Then, by Corollary 3.7(1), φ_{-m} is birational for all $m \geq 52$. If $|-18K_X|$ and $|-8K_X|$ are not composed with the same pencil, then take $m_1 = 18$ and $\mu_0' = m_0 = 8$. By Lemma 3.3, $N_0 \geq \lceil \frac{660}{11m_1} \rceil = 4$. Then, by Corollary 3.7(2), φ_{-m} is birational for all $m \geq 45$.

Case 5. $r_{\max} = 12$.

If $r_{\max} = 12$, then we claim that $r_X \leq 132$ or $r_X = 420$.

By (2.4), at most one element of $\{6,7,8,9,10,11\}$ is in $\mathcal{R}_X$. So r_X divides one element of $\{120,180,420,660\}$. Recalling that 12 divides r_X, it is clear that $r_X \leq 132$ or $r_X \in \{180,420,660\}$. Moreover, if $r_X \in \{180,660\}$, then $\{12,9,5\} \subset \mathcal{R}_X$ or $\{12,11,5\} \subset \mathcal{R}_X$, which contradicts (2.4). This concludes the claim.

If $r_X \leq 132$, then by Corollary 3.7(2), φ_{-m} is birational for all $m \geq 43$.

If $r_X = 420$, then by (2.4), $\mathcal{R}_X = \{12,7,5\}$. Arguing similarly as (5.3), we get $-K_X^3 \geq \frac{241}{420}$ when $P_{-1} \geq 2$. Arguing similarly as (5.4), we get constrains for B_X when $P_{-1} = 1$. By (2.2) and considering all the possible baskets when $P_{-1} = 1$, we can see that $-K_X^3 \geq \frac{241}{420}$. By Proposition 4.8(2) (with $t = 2.75$), we can take $m_1 = 11$. Hence, by Lemma 3.3, $N_0 \geq \lceil \frac{420}{12m_1} \rceil = 4$. Then, by Corollary 3.7(2), φ_{-m} is birational for all $m \geq 40$.

Case 6. $r_{\max} = 13$.

If $r_{\max} = 13$, then we claim that $r_X \leq 364$ or $r_X = 390$ or $r_X = 546$.

By (2.4), at most one element of $\{6,7,8,9,10,11,12\}$ is in $\mathcal{R}_X$. So r_X divides one element of $\{5,460,1,560,2,340,8,580\}$. Recalling that 13 divides r_X and $r_X \leq 660$, it is clear that $r_X \leq 364$ or $r_X \in \{390,429,455,468, 520,546,572, 585\}$. Moreover, if $r_X \in \{429,455,468, 520,572,585\}$, then we can see that $\mathcal{R}_X$ violates (2.4) by discussing the factors. For example, if $r_X = 455$, then $\{13,7,5\} \subset \mathcal{R}_X$ which violates (2.4). This concludes the claim.

If $P_{-4} = 1$, then $P_{-k} = 1$ for $1 \leq k \leq 4$. By [2, Subsubcase II-4f of Proof of Th. 4.4], B_X is dominated by

$$\{2\times(1,2), 2\times(1,3), (1,s_1), (1,s_2)\}$$

for some $s_2 \geq s_1 \geq 4$. As $r_{\max} = 13$, $(s_1,s_2) = (6,7)$. Considering all possible packings, we get the following possibilities of B_X:

$$\begin{aligned}
&\{2\times(1,2), 2\times(1,3), (2,13)\},\\
&\{(1,2),(2,5),(1,3),(2,13)\},\\
&\{(3,7),(1,3),(2,13)\},\\
&\{(1,2),(3,8),(2,13)\},\\
&\{2\times(2,5),(2,13)\}.
\end{aligned}$$

Then $r_X \leq 273$ or $B_X = \{(1,2),(2,5),(1,3),(2,13)\}$.

If $r_X \leq 273$, then by Corollary 3.7(2), φ_{-m} is birational for all $m \geq 55$.

If $B_X = \{(1,2),(2,5),(1,3),(2,13)\}$, then $-K_X^3 = \frac{23}{390}$. By Corollary 4.9 (with $l = 1$ and $t = 3$), $P_{-13} - 1 > 13$. If $|-13K_X|$ and $|-8K_X|$ are composed with the same pencil, then take $\mu_0' = \frac{13}{P_{-13}-1} < 1$ by Remark 3.2. Then, by Corollary 3.7(1), φ_{-m} is birational for all $m \geq 56$. If $|-13K_X|$ and $|-8K_X|$ are not composed with the same pencil, then take $m_1 = 13$ and $\mu_0' = m_0 = 8$. Then, by Lemma 3.3, $N_0 \geq \lceil \frac{390}{13m_1} \rceil = 3$. Then, by Corollary 3.7(2), φ_{-m} is birational for all $m \geq 44$.

So, from now on, we assume that $P_{-4} \geq 2$ and take $m_0 = 4$. We take $\mu_0' = 4$ unless stated otherwise.

If $r_X \leq 364$, then by Corollary 3.7(1), φ_{-m} is birational for all $m \geq 57$.

If $r_X = 546$, then by (2.4), $\mathcal{R}_X = \{13,7,3,2\}$. Arguing similarly as (5.5), we get $-K_X^3 > 0.9$ when $P_{-1} \geq 2$. Arguing similarly as (5.6), we get constrains for B_X when $P_{-1} = 1$. By

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(2.2) and considering all the possible baskets when $P_{-1}=1$, we can see that $-K_X^3 \geq \frac{157}{546}$. By Proposition 4.8(2) (with $t=3.6$), we can take $m_1=16$. Hence, by Lemma 3.3, $N_0 \geq \lceil \frac{546}{13m_1} \rceil = 3$. Then, by Corollary 3.7(2), φ_{-m} is birational for all $m \geq 44$.

If $r_X=390$, then by (2.4), $\mathcal{R}_X=\{13,6,5\}$ or $\{13,5,3,2\}$ or $\{13,5,3,2,2\}$. Arguing similarly as (5.5), we get $-K_X^3>0.5$ or $-K_X^3>0.8$ or $-K_X^3>1$ when $P_{-1}\geq 2$. Arguing similarly as (5.6), we get constrains for B_X when $P_{-1}=1$. By (2.2) and considering all the possible baskets when $P_{-1}=1$, we can see that $-K_X^3 \geq \frac{133}{390}$ or $-K_X^3 \geq \frac{23}{390}$ or $-K_X^3 \geq \frac{62}{390}$. By Corollary 4.9 (with $l=1, t=3$, and $-K_X^3 \geq \frac{23}{390}$), $P_{-13}-1>13$. If $|-13K_X|$ and $|-4K_X|$ are composed with the same pencil, then take $\mu_0' = \frac{13}{P_{-13}-1} < 1$ by Remark 3.2. Then, by Corollary 3.7(1), φ_{-m} is birational for all $m \geq 56$. If $|-13K_X|$ and $|-4K_X|$ are not composed with the same pencil, then take $m_1=13$ and $\mu_0'=m_0=4$. Then, by Lemma 3.3, $N_0 \geq \lceil \frac{390}{13m_1} \rceil = 3$. Then, by Corollary 3.7(2), φ_{-m} is birational for all $m \geq 40$.

Combining all above cases, we have proved the theorem. ∎

5.5 Proofs of Theorem 1.2 and Theorem 1.4

Proof of Theorem 1.2. It follows from Theorems 5.2, 5.4, 5.6, 5.8, and 5.9. ∎

Proof of Theorem 1.4. From the proof of Theorem 1.2, φ_{-58} may not be birational only if $B_X=\{(1,2), 2\times(1,3), (8,17)\}$, $P_{-1}=0$, and $|-24K_X|$ is composed with a pencil. In this case, $r_X(-K_X^3)=7$ and $P_{-24}=169=7\times 24+1$ by (2.1). But this contradicts Proposition 4.11. ∎

Appendix

The possible baskets with $P_{-2}=0$ are the following (cf. [2, Th. 3.5]).

Table A.1. Baskets with $P_{-2}=0$.

No.	B_X	$-K^3$	P_{-3}	P_{-4}	P_{-5}	P_{-6}	P_{-7}	P_{-8}
1	$\{2\times(1,2), 3\times(2,5), (1,3), (1,4)\}$	1/60	0	0	1	1	1	2
2	$\{5\times(1,2), 2\times(1,3), (2,7), (1,4)\}$	1/84	0	1	0	1	1	2
3	$\{5\times(1,2), 2\times(1,3), (3,11)\}$	1/66	0	1	0	1	1	2
4	$\{5\times(1,2), (1,3), (3,10), (1,4)\}$	1/60	0	1	0	1	1	2
5	$\{5\times(1,2), (1,3), 2\times(2,7)\}$	1/42	0	1	0	1	2	3
6	$\{4\times(1,2), (2,5), 2\times(1,3), 2\times(1,4)\}$	1/30	0	1	1	2	2	4
7	$\{3\times(1,2), (2,5), 5\times(1,3)\}$	1/30	1	1	1	3	3	4
8	$\{2\times(1,2), (3,7), 5\times(1,3)\}$	1/21	1	1	1	3	4	5
9	$\{(1,2), (4,9), 5\times(1,3)\}$	1/18	1	1	1	3	4	5
10	$\{3\times(1,2), (3,8), 4\times(1,3)\}$	1/24	1	1	1	3	3	5
11	$\{3\times(1,2), (4,11), 3\times(1,3)\}$	1/22	1	1	1	3	3	5
12	$\{3\times(1,2), (5,14), 2\times(1,3)\}$	1/21	1	1	1	3	3	5
13	$\{2\times(1,2), 2\times(2,5), 4\times(1,3)\}$	1/15	1	1	2	4	5	7
14	$\{(1,2), (3,7), (2,5), 4\times(1,3)\}$	17/210	1	1	2	4	6	8
15	$\{2\times(1,2), (2,5), (3,8), 3\times(1,3)\}$	3/40	1	1	2	4	5	8
16	$\{2\times(1,2), (5,13), 3\times(1,3)\}$	1/13	1	1	2	4	5	8
17	$\{(1,2), 3\times(2,5), 3\times(1,3)\}$	1/10	1	1	3	5	7	10
18	$\{4\times(1,2), 5\times(1,3), (1,4)\}$	1/12	1	2	2	5	6	9
19	$\{4\times(1,2), 4\times(1,3), (2,7)\}$	2/21	1	2	2	5	7	10
20	$\{4\times(1,2), 3\times(1,3), (3,10)\}$	1/10	1	2	2	5	7	10
21	$\{3\times(1,2), (2,5), 4\times(1,3), (1,4)\}$	7/60	1	2	3	6	8	12
22	$\{3\times(1,2), 7\times(1,3)\}$	1/6	2	3	4	9	12	17
23	$\{2\times(1,2), (2,5), 6\times(1,3)\}$	1/5	2	3	5	10	14	20

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The possible baskets with $P_{-1} = 0$ and $P_{-2} = P_{-4} = 1$ are the following (cf. the second paragraph of [5, Case IV of Proof of Th. 1.8]).

Table A.2. Baskets with $P_{-1} = 0$ and $P_{-2} = P_{-4} = 1$.

No.	B_X	r_X
1	$\{9 \times (1,2), (1,3), (1,7)\}$	42
2	$\{8 \times (1,2), (2,5), (1,7)\}$	70
3	$\{8 \times (1,2), (2,5), (1,6)\}$	30
4	$\{7 \times (1,2), (3,7), (1,6)\}$	42
5	$\{6 \times (1,2), (4,9), (1,6)\}$	18
6	$\{7 \times (1,2), (3,7), (1,5)\}$	70
7	$\{6 \times (1,2), (4,9), (1,5)\}$	90
8	$\{5 \times (1,2), (5,11), (1,5)\}$	110
9	$\{4 \times (1,2), (6,13), (1,5)\}$	130

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