A NOTE ON REGULAR LOCAL NOETHER LATTICES

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If R is a local (Noetherian) ring, it is well known that R is regular if and only if its completion is regular. It is the purpose of this note to show that a similar result is true for Noether lattices.

Let $\mathscr{L}$ be a complete lattice on which there is defined a commutative, associative, join distributive (finite and infinite) multiplication such that the unit element of $\mathscr{L}$ is an identity for multiplication. An element b of $\mathscr{L}$ is said to be meet principal if $(c \wedge (d:b))b = cb \wedge d$, for all c, d in $\mathscr{L}$; b is said to be join principal if $d \vee (c:b) = (db \vee c):b$ for all c, d in $\mathscr{L}$; and b is said to be principal if b is both meet and join principal. $\mathscr{L}$ is called a Noether lattice if $\mathscr{L}$ is modular, satisfies the ascending chain condition, and every element of $\mathscr{L}$ is the join of principal elements. A Noether lattice $\mathscr{L}$ is said to be local if it has a unique maximal (proper) prime element m and we denote this by $(\mathscr{L}, m)$. In general we shall adopt the Noether lattice terminology of [**1**].

If q is a m-primary element of a local Noether lattice $(\mathscr{L}, m)$, then, for each positive integer n, we let $D(q, n)$ denote the lattice dimension of $\mathscr{L}/q^n$. We will require the following two known results from [**2**] (Corollary 3.5, p. 135, and Theorem 3.9, p. 136, respectively).

THEOREM 1. *Let q be a m-primary element of $\mathscr{L}$. Then there exists a polynomial $D^*(q, x)$ such that $D^*(q, n) = D(q, n)$, for all sufficiently large integers n.*

THEOREM 2. *Let q be a m-primary element of $\mathscr{L}$. Then the degree of $D^*(q, x)$ is the altitude of $\mathscr{L}$.*

Throughout the remainder of this paper, $\mathscr{L}$ will be a local Noether lattice with unique maximal prime element m. Elements of $\mathscr{L}$ will be denoted by small letters $a, b, c, \ldots$. As in section 2 of [**2**] we let $\mathscr{L}^*$ be the collection of all formal sums $\sum_{i=1}^{\infty} a_i$ of elements of $\mathscr{L}$ such that $a_i = a_{i+1} \vee m^i$, for $i = 1, 2, \ldots$. Elements of $\mathscr{L}^*$ are denoted by $A, B, C, \ldots$, and for A in $\mathscr{L}^*$, let $A = \sum_{i=1}^{\infty} a_i$. On $\mathscr{L}^*$ define

$$A \leqq B \quad \text{if and only if} \quad a_i \leqq b_i,\ i = 1, 2, \ldots,$$

$$AB = \sum_{i=1}^{\infty} (a_i b_i \vee m^i),$$

so that $\mathscr{L}^*$ is a multiplicative lattice which satisfies the ascending chain condition [**3**, Theorem 2.1, p. 330].

Let $\langle e_i \rangle$ $(i = 1, 2, \ldots)$ be a sequence of elements of $\mathscr{L}$ such that, for each positive integer n,

$$e_{i+1} \leqq e_i \vee m^n, \text{ for all positive integers } i \geqq n. \tag{1}$$

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For each positive integer n, set $d_n = \bigwedge_{i \geqq n} (e_i \vee m^n)$. Then $D = \sum_{i=1}^{\infty} d_i$ is an element of $\mathscr{L}^*$, which we will call the element of $\mathscr{L}^*$ derived from the sequence $\langle e_i \rangle$ $(i = 1, 2, \ldots)$. If a is an element of $\mathscr{L}$, set $a^* = \sum_{i=1}^{\infty} (a \vee m^i)$, so that a^* is the element of $\mathscr{L}^*$ derived from the constant sequence $a_i = a$, $i = 1, 2, \ldots$. For each $A = \sum_{i=1}^{\infty} a_i$ in $\mathscr{L}^*$, define $C(A)$ in $\mathscr{L}$ by $C(A) = \bigwedge_{i=1}^{\infty} a_i$.

REMARK 1. For each a in $\mathscr{L}$ note that

$$C(a^*) = C\left(\sum_{i=1}^{\infty} (a \vee m^i)\right) = \bigwedge_{i=1}^{\infty} (a \vee m^i) = a,$$

by [**1**, Corollary 3.2, p. 486].

The following properties of $\mathscr{L}^*$ are known and we collect them here for the convenience of the reader (see [**3**, p. 331] and [**4**]).

(i) If $\langle c_i \rangle$ is a sequence of principal elements of $\mathscr{L}$ satisfying (1), then the derived element in $\mathscr{L}^*$ of $\langle c_i \rangle$ is a principal element of $\mathscr{L}^*$.

(ii) $\mathscr{L}^*$ is a local Noether lattice with maximal element $m^* = \sum_{i=1}^{\infty} m$.

(iii) The map $a \to a^*$ of $\mathscr{L} \to \mathscr{L}^*$ is a multiplicative lattice monomorphism.

(iv) For each natural number n, $\mathscr{L}^*/m^{*n} \cong \mathscr{L}/m^n$.

The height of a prime element p of a Noether lattice $\mathscr{L}$ is defined to be the supremum of all integers n for which there exists a prime chain $p_0 < p_1 < p_2 < \ldots < p_n = p$ in $\mathscr{L}$, and the altitude of $\mathscr{L}$ is defined to be the supremum of the heights of the prime elements of $\mathscr{L}$. A local Noether lattice $\mathscr{L}$ of altitude k is said to be regular in case the unique maximal prime element of $\mathscr{L}$ is the join of k principal elements.

THEOREM 3. *$(\mathscr{L}, m)$ is a regular local Noether lattice if and only if $(\mathscr{L}^*, m^*)$ is a regular local Noether lattice.*

Proof. Since $\mathscr{L}^*/m^{*i} \cong \mathscr{L}/m^i$, for all integers $i \geqq 1$, by (iv), it follows from Theorem 2 that $\mathscr{L}$ and $\mathscr{L}^*$ have the same altitude; so let the altitude of $\mathscr{L}$ and $\mathscr{L}^*$ be d. Assume that $\mathscr{L}^*$ is regular and let $a_1, a_2, \ldots, a_k$ be principal elements of $\mathscr{L}$ that form a basis for m, so that $m = a_1 \vee \ldots \vee a_k$. Since the altitude of $\mathscr{L}$ is d, it follows that m cannot be a join of fewer than d principal elements of $\mathscr{L}$ (cf. [**1**, Theorem 6.5, p. 496]); thus $k \geqq d$. Since

$$m^* = (a_1 \vee \ldots \vee a_k)^* = a_1^* \vee \ldots \vee a_k^*$$

by (iii), and each a_i^* $(1 \leqq i \leqq k)$ is principal by (i), it follows that $\{a_1^*, \ldots, a_k^*\}$ forms a basis for the maximal element m^* of $\mathscr{L}^*$ that can be reduced to a minimal basis that will necessarily have d elements (cf. [**2**, Theorem 1.4, p. 126]). Thus assume that

$$m^* = a_{i_1}^* \vee \ldots \vee a_{i_d}^*,$$

so that

$$m = C(m^*) = C((a_{i_1} \vee \dots \vee a_{i_d})^*) = a_{i_1} \vee \dots \vee a_{i_d},$$

by Remark 1. Consequently, m is the join of d principal elements and therefore $\mathscr{L}$ is regular.

Conversely, if $\mathscr{L}$ is regular, then there are d principal elements $a_1, \dots, a_d$ of $\mathscr{L}$ such that

$$m = a_1 \vee \dots \vee a_d.$$

Hence

$$m^* = (a_1 \vee \dots \vee a_d)^* = a_1^* \vee \dots \vee a_d^*$$

by (iii), where, as above, by (i) each a_i^* $(1 \leqq i \leqq d)$ is principal in $\mathscr{L}^*$, so that $\mathscr{L}^*$ is regular. This completes the proof of the theorem.

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