A GENERALIZATION OF SONINE'S FIRST FINITE INTEGRAL

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In this note I show that

$$J_{\mu+\nu+2n+1}(z) = \frac{z^{\nu+1}\Gamma(\mu+n+1)}{2^{\nu}\Gamma(\mu+1)\Gamma(\nu+n+1)}$$

$$\times \int_0^{\frac{1}{2}\pi} J_\mu(z \sin\theta)\,{}_2F_1(-n, \mu+\nu+n+1; \mu+1; \sin^2\theta) \sin^{\mu+1}\theta \cos^{2\nu+1}\theta \, d\theta, \quad (1)$$

where J denotes the Bessel function of the first kind of the orders and arguments indicated, $n = 0, 1, 2, 3, \ldots$ and the real parts of both μ and ν exceed -1. This is a generalization of Sonine's first finite integral [1, p. 373] to which it reduces in the special case $n = 0$.

I start with the Weber-Schafheitlin integral

$$I(\mu, \nu, n, r) = \int_0^\infty z^{-\nu} J_{\mu+\nu+2n+1}(z) J_\mu(rz)\, dz, \quad (2)$$

with the conditions on n, μ and ν as given above. The integral is convergent and [1, p. 401] its value is given by

$$I(\mu, \nu, n, r) = \begin{cases} \dfrac{r^\mu \Gamma(\mu+n+1)}{2^\nu \Gamma(\mu+1)\Gamma(\nu+n+1)}\, {}_2F_1(\mu+n+1, n-\nu; \mu+1; r^2) & (0 < r < 1), \\ 0 \quad (1 < r < \infty), \end{cases} \quad (3)$$

the integral vanishing when $r > 1$ because of a factor $\Gamma(-n)$ in the denominator of the term multiplying the hypergeometric function. Applying Hankel's inversion formula to (2), we obtain

$$z^{-\nu-1} J_{\mu+\nu+2n+1}(z) = \int_0^\infty r I(\mu, \nu, n, r) J_\mu(zr)\, dr,$$

and substitution from (3) gives

$$z^{-\nu-1} J_{\mu+\nu+2n+1}(z) = \frac{\Gamma(\mu+n+1)}{2^\nu \Gamma(\mu+1)\Gamma(\nu+n+1)} \int_0^1 r^{\mu+1}\, {}_2F_1(\mu+n+1, -n-\nu; \mu+1; r^2) J_\mu(zr)\, dr. \quad (4)$$

Using the well-known transformation formula [2, p. 8],

$${}_2F_1(\mu+n+1, -n-\nu; \mu+1; r^2) = (1-r^2)^\nu\, {}_2F_1(-n, \mu+\nu+n+1; \mu+1; r^2),$$

and writing $r = \sin\theta$, we obtain the required result (1) directly from (4).

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As well as Sonine's first finite integral, there are some further interesting special cases of the general formula (1). Thus the two modifications of Bessel's integral [1, pp. 20, 21],

$$J_{2n}(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \cos 2n\theta \cos (z \sin \theta)\, d\theta,$$

$$J_{2n+1}(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \sin (2n+1)\theta \sin (z \sin \theta)\, d\theta,$$

are obtained by writing $\nu = -\frac{1}{2}$ and $\mu = \mp\frac{1}{2}$ respectively in (1). Again, taking $\nu = 0$, $\mu = -\frac{1}{2}$ in (1), expressing the hypergeometric function in terms of a Legendre polynomial [2, p. 50], making a few reductions and writing $x = \sin \theta$, we have

$$J_{2n+\frac{1}{2}}(z) = (-1)^n \sqrt{\left(\frac{2z}{\pi}\right)}\int_0^1 P_{2n}(x) \cos zx\, dx,$$

and this formula gives, in effect, the so-called even Legendre transform of cos zx [3, p. 97]. In a similar way, substitution of $\nu = 0$, $\mu = \frac{1}{2}$ in (1) leads to

$$J_{2n+\frac{3}{2}}(z) = (-1)^n \sqrt{\left(\frac{2z}{\pi}\right)}\int_0^1 P_{2n+1}(x) \sin zx\, dx$$

and hence to the odd Legendre transform of sin zx.

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