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Kinematical Theorems.

By WILLIAM HARVEY, B.A.

I.—Motion of a plane figure in its plane.

§ 1. Figure 30. Let AB be any straight line in the plane figure, A′B the position of the same line after a displacement, the point A moving to A′ and B to B′. Since the position of a plane figure in its plane is determined, when the position of a straight line rigidly attached to it is determined, the motion of the plane figure is determined if we determine the motion of the straight line AB.

Now, it is known that the straight line AB may be brought into the position A′B′ by a rotation about a fixed point O, in the general case at a finite distance. It is proved that the point O is the intersection of two straight lines, one drawn through the middle point of AA′ perpendicular to AA′, and the other drawn through the middle point of BB perpendicular to BB′. This follows, very simply, from the equality of the triangles OAB and OA′B′. The point O so determined is unique.

Further, the rotation about the point O is measured by the angle AOA′ or BOB′, each of these angles being equal to the angle between the lines AB and A′B′.

If we define the initial and final positions of any point of the figure as corresponding points, so that A, A′; B, B′ are corresponding points, and if we take any point P on AB and the corresponding point P′ on A′B′, then the rotation about O which brings AB into the position A′B′, will clearly bring the line AP into the position A′P′.

§ 2. Hence, by the theorem referred to, the straight line through Q, the middle point of PP′, perpendicular to PP′, passes through O, and PP′ subtends an angle POP′ at O equal to the angle of rotation of the plane figure about O. And the triangles AOP and A′OP′ are equal in every respect. These propositions hold for any pair of corresponding points.

§ 3. We shall denote the intersection of the lines AB, A′B′ by two letters D or C′, for a reason which will be obvious immediately. On the straight line A′B′, take A′D′ equal to AD; then D and D′ are corresponding points. Again, on AB take AC equal to A′C′; then C and C′ are corresponding points. *And it is obvious that* CC′ is equal to DD′. It follows that the straight line drawn through E, E′, the middle points of CC′ and DD′ perpendicular to these lines respectively, pass through O. It is further evident that the angle EOE′ is equal to the angle of rotation, that E and E′ are corresponding points, and that OD bisects the exterior angle between the lines AB, A′B′.

§ 4. Again, since the angle OPD is equal to the angle OP′D, a circle can be described to pass through the four points O, D, P, P′. But if, from a point O in the circumference of a circle, perpendiculars be let fall on the sides of a triangle inscribed in the circle, the feet of these perpendiculars lie in a straight line, which is equally distant from O and from the orthocentre of the inscribed triangle. This line is called the pedal line of the triangle. Hence, the points E, E′ and Q lie in one straight line, the pedal line of the triangle PDP′.

Further, if we take any other pair of corresponding points on AB and A′B′, R and R′, then the middle point of RR′ lies on the same line. Hence the locus of such points is a straight line.

§ 5. If we regard O as the focus, and the line EE′Q as the tangent at the vertex of a parabola, then the line OQ is drawn from the focus, and meets the tangent at the vertex in Q, and PP′ is drawn at right angles to OQ, and, therefore, PP′ touches this parabola. Further, if any other pair of corresponding points R, R′ be taken on AB, A′B′ respectively, then RR′ will touch the same parabola. Hence, if corresponding points be taken on AB, A′B′, the lines joining them envelope a parabola. The directrix of this parabola is the locus of the orthocentres of triangles, formed by the lines joining corresponding points and the lines AB, A′B′.

When the motion is infinitesimal, this theorem becomes: If any plane figure is in motion in its plane, the trajectories of points situated, at any instant in a straight line, envelope a parabola.

II.—Motion of a solid body in space.

I intend, in this division, to give two theorems for the representation of the finite displacement of a solid body in space, and to illustrate the geometry of displacement by a few deductions from these theorems, analogous to those I have given in two dimensions.

The first of these theorems is given by Chasles in the *Comptes Rendus de l' Académie des Sciences* for 1843 for the case of infinitesimal motion, and the proof he gives is indirect :—

§ 1. Figure 31. The theorem is as follows :—Any finite displacement of a solid body in space may be represented, in an infinite number of ways, by two successive finite rotations about two axes, at right angles to each other, and, in general, not intersecting.

Since the position of a solid body is determined when the position of a plane rigidly attached to it is determined, the displacement of such a body will be determined, when that of the plane is determined.

Let CAB be any such plane, C′A′B′ the position of the same plane after the displacement. Let A′B′ be the intersection of the two positions of the plane, and let AB be the line in the plane CAB corresponding to A′B′ in C′A′B′, so that A, A′ ; B, B′ are corresponding points. It is clear, if we except the case of motion in a plane, that the only points which remain in the plane after the displacement, and whose displacements are entirely in the plane, are the points in the line A′B′. Now AB is the initial position of A′B′, and, from what has been proved of motion in a plane, it is clear that AB may be brought into the position A′B′ by a finite rotation about an axis perpendicular to the plane CAB, and, therefore, to the line A′B′. By this rotation, a line AB of the solid body in its initial position, is brought into co-incidence with a line A′B′ in its final position, and it is evident that the displacement may be completed by a simple rotation about A′B′ regarded as a fixed line. Hence any solid body may be brought from one position in space to another by two rotations, about two axes at right angles to one another. Such rotations are called conjugate rotations, and the axes conjugate rectangular axes. The order of the rotations is evidently important, and there will be a pair of such axes corresponding to every plane in the body.

§ 2. Figure 32. When the motion is represented by two rotations as above, there is a point on each axis whose motion is entirely along that axis.

Let A′B′, C′D′ be the axes, A′X the shortest distance between them, and let C′D′ be the axis perpendicular to the base plane, and A′B′ the axis in it.

At the point A′ in the straight line A′X, make the angles XA′S, XA′S′ on opposite sides of A′X, in the plane A′XD′, each equal to the semi-angle of rotation about A′B′, and let S, S′ lie on C′D′. By the rotation about C′D′, the position of S is unaltered, and by the subsequent rotation about A′B′, S is clearly brought to S′. Hence, the axis C′D′ joins the corresponding points S, S′. Again, if we take T, T′ on A′B′ obtained in a similar way, then by the rotation about C′D′, T is brought to T′, and the subsequent rotation about A′B′ will not effect its position. Hence, the axis A′B′ joins the corresponding points T, T′.

From this, it follows that every line joining two corresponding points is one of a pair of conjugate rectangular axes.

§ 3. Figure 32. The line joining any two corresponding points, is also the intersection of two corresponding planes.

Let C′D′ be a line joining two corresponding points S and S′, and let CD be the initial position of C′D′. Then CD evidently cuts C′D′ in the point S, and, therefore, CD and C′D′ are in the same plane, and therefore, as above, CD may be brought to the position C′D′ by a finite rotation about an axis perpendicular to this plane. (It is to be observed that this axis is not A′B′). Hence the whole displacement of the body may, as before, be accomplished by a rotation about the axis perpendicular to the plane containing CD and C′D′, followed by a rotation about C′D′ regarded as a fixed line. Hence C′D′ is the intersection of two corresponding planes. And we have seen that A′B′ is also the intersection of two corresponding planes, and, hence the proposition follows whether the axis considered is that round which the first or second rotation takes place. That is to say, the lines joining S, S′ and T, T′ are the intersections of corresponding planes, and the same is true for every line joining corresponding points.

§ 4. If we consider any two positions of a solid body and join the corresponding points, the lines which lie in one and the same plane envelope a parabola.

As usual, let us represent the motion by a rotation about an axis perpendicular to the plane, followed by a rotation about an axis in the plane.

Let CD and A′B′ be these axes, and let AB be the initial position of A′B′, then we have shown that the only points whose displacements are entirely in the plane, are the points lying in the line AB. And hence from the last theorem on motion in two dimensions we see that the lines joining the different points of AB to the corresponding points of A′B′ envelope a parabola.

This theorem may also be stated as follows:—The lines of inter-

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section of corresponding planes which lie in one and the same plane, envelope a parabola.

§ 5. Figures 33, 34. If we consider any two positions of a solid body and join the corresponding points, the lines which pass through one and the same point form a cone of the second order, and the points themselves form, on this cone, a curve of the third order.

This theorem is given by Chasles for the case of finite as well as infinitesimal displacements in the *Bulletin Universel des Sciences* for 1830. I have not seen his proof. It may, however, be proved in the same way as the preceding theorems.

Let O be the given point, and let PP′, QQ′ pass through O, where P, P′ and Q, Q′ are corresponding points.

As before, the displacement may be represented by a rotation about a line perpendicular to the plane OPQ, by which PQ is brought into the position P′Q′, followed by a rotation about P′Q′ regarded as a fixed line, and the points on PQ are the only points whose displacement lies in the plane OPQ. Hence, if R be any point in the plane OPQ not lying in the line PQ and R′ the corresponding point, RR′ meets the plane in R, and therefore cannot pass though O. If R lies in PQ, then the corresponding point R′ lies in P′Q′ and PR = P′R′ as also PQ = P′Q′. Hence RR′ cannot pass through O unless P′Q′ is parallel to PQ, in which case the point O would not be at a finite distance. Hence if any plane passes through O, it contains two generators PP′, QQ′ of the locus only, and therefore the locus is a cone of the second order. If the point O is at infinity, the locus reduces to a plane.

The result may also be stated as follows:—The intersections of corresponding planes which pass through one and the same point form a cone of the second order.

To prove the second part of the theorem, let OP, OQ be two generators of the cone, and let O′ be the final position of the point O, then by the hypothesis OO′ lies on the cone. (Fig. 34).

Again, if we join O′ to the points P′, Q′, &c., the final positions of P, Q, &c., we clearly get a cone equal in every respect to the cone OPQ. In fact, the cone O′P′Q′ is simply the cone OPQ moved into a new position by the displacement.

Further, if O″ be the initial position of a point whose final position is O, then OO″ lies on the cone OPQ, and after the displacement OO″ takes up the position O′O. Hence, O′O is the line joining O′ to the final position of O″, and, therefore, by the hypothesis lies on the cone O′P′Q′. Hence the generator OO′ is common to the two cones. Therefore, the curve P′Q′, &c., is the intersection of two cones of the second order, which have a generating line in common, and therefore is of the third order, since the complete intersection of the two cones is of the fourth. The curves PQ &c., and P′Q′ &c., are evidently the same, whence the second part of the theorem follows.

§ 6. The following theorem connects the two preceding. If any number of axes pass through a point, their respective conjugates lie in a plane. Let O be the given point, and let O′ as before be the final position of O. A rotation about any axis through O will not affect the position of O, hence the displacement OO′ is entirely due when rotation about the conjugate of the particular axis chosen. But O can be brought to O′ only by a rotation about an axis lying in a plane bisecting OO′ at right angles, because O and O′ must be equi-distant from every point of the axis. Hence whatever axis through O is chosen, its conjugate must lie in this plane, which is therefore the locus of the conjugates.

§ 7. We may state the three preceding results in one proposition as follows :—If the displacement of a body be represented by two conjugate rotations about axes at right angles to each other, then if any number of axes pass through a point they form a cone of the second order, while their respective conjugates lie in a plane and envelope a parabola.

§ 8. Figures 35, 36. We come now to the second theorem, which is a more general case of the first. In the first case we may choose any line joining two corresponding points as an axis, its conjugate then lies in the plane bisecting the line joining the corresponding points at right angles, and is the intersection of this plane with its final position after the displacement. The axis in the plane is called by Chasles the characteristic of the plane. We see then, that in the first case the choice of the axes is not altogether arbitrary.

The more general theorem is :—Any given finite displacement of a solid body in space may be represented by two finite rotations, one about any given straight line and the other about another straight line which does not generally intersect the first.

A demonstration of this theorem is given by Rodrigues in Liouville's *Journal de Mathématiques* for 1840, but it depends on the laws of composition of finite rotations. Mr Routh, who also gives the theorem, follows Rodrigues. The following is a direct demonstration :—

Let AB be the initial, aud A′B′ the final position of any straight line in the body. We shall show that the straight line may be brought from the position AB to the position A′B′ by a rotation about a certain axis.

Let A, A′, B, B′ be corresponding points. Through the middle point of AA′ draw a plane perpendicular to AA′, and through the middle point of BB′ draw a plane perpendicular to BB′. These planes intersect in the required axis. Through AA′ draw the plane AA′O perpendicular to the axis, and through BB′ draw BB′O′ perpendicular to the axis, and let these planes meet the axis in O, O′ respectively. Then since O lies in the plane bisecting AA′ at right angles, OA = OA′; and since O lies in a plane bisecting BB′ at right angles OB = OB′ similarly O′A = O′A′ and O′B = O′B′, also AB = A′B′. Hence the six edges of the tetrahedron OABO′ are respectively equal to the six edges of the tetrahedron OA′B′O′, and therefore the tetrahedra are equal in every respect. Therefore the angles between their opposite edges are equal. Whence the angle between OA and O′B is equal to the angle between OA′ and O′B′ and the planes OAA′, OBB′ are parallel. It follows from this that the angle AOA′ is equal to angle BOB′, and that the rotation about OO′, which brings A to A′, brings B to B′, and therefore the straight line AB to A′B′.

The axis OO′ is perpendicular to a plane containing two straight lines parallel to AA′, BB′, respectively. This property is included in the following proof. (Fig. 36).

Let AB, A′B′, be the two positions of the line, so that AB = A′B′.

Through AA′ draw a plane parallel to BB′ and let BP, B′P′ be perpendicular to this plane. Then BP = B′P′.

Also $AP = \sqrt{AB^2 - PB^2} = \sqrt{A'B'^2 - P'B'^2} = A'P'$. Therefore by a rotation about an axis perpendicular to the plane APA′ we may bring A to A′, P to P′, and, therefore, B to B′.*

If, then, we rotate the body about the axis so determined, the line AB in the initial position of the body is brought into its final position A′B′, and the displacement may evidently be completed by a second rotation about A′B′ regarded as a fixed line. Here we see one of the axes, A′B′, is arbitrary, but the second is known as soon as the first is determined.

* This proof was suggested to me by a friend.

§ 9. If the straight line AB is parallel to A′B′ then for one of the rotations we may substitute a translation, and the change of position is accomplished by a translation and a rotation. Again if AB and A′B′ coincide in situation, without the corresponding points in each coinciding, then the change of position is accomplished by a translation along and a rotation about the same axis. I here assume Chasles' celebrated results, published in 1830 in the *Bulletin Universel des Sciences*, viz :—If any finite displacement be given to a free solid body in space, there exist in the body a set of straight lines, which, after the displacement, are parallel to their initial positions, and in particular there is one straight line, and one only, which after the displacement remains in its original situation. All that need be pointed out at present is that the representation of the displacement of a solid body by a translation and a rotation, and the unique representation by a screw movement, are particular cases of the more general representation by two conjugate rotations.

§ 10. If we consider any two positions of a solid body in space, the displacement of the different points lying in any straight line lie on an hyperbolic paraboloid.

For let AB be the given line, CD its conjugate, then the displacements of every point in AB pass through two straight lines, viz., AB and its displaced position A′B′, and, further, they are all parallel to a plane perpendicular to CD. Hence they lie in an hyperbolic paraboloid. This proposition is given in Routh's *Rigid Dynamics* for infinitesimal motions, and the proof is exactly the same for finite motions.

A Proof of Lagrange's Theorem.

By T. Hugh Miller, M.A.

If $y = z + xf(y)$ where x and z are independent, Lagrange's series gives the expansion of any function of y in terms of x. The coefficients of the powers of x may be found thus. Let $\phi(y)$ be the function to be expanded. Then by Taylor's theorem

$$\phi(y) = \phi\{z + xf(y)\}$$

$$= \phi(z) + xf(y)\phi'(z) + \frac{x^2[f(y)]^2}{\underline{|2}}\phi''(z) + \&c., \qquad (1)$$