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COMPLEX VECTOR BUNDLES ON REAL ALGEBRAIC VARIETIES OF SMALL DIMENSION

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Let X be an affine real algebraic variety. In this paper, assuming that $\dim X \leqslant 7$ and that X satisfies some other reasonable conditions, we give a characterisation of those continuous complex vector bundles on X which are topologically isomorphic to algebraic complex vector bundles on X.

1. INTRODUCTION

Let F denote one of the fields $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the reals, complexes or quaternions). Let X be an affine real algebraic variety (that is, X is biregularly isomorphic to an algebraic subset of $\mathbb{R}^n$ for some n; for definitions and notions of real algebraic geometry we refer to the book [**2**]). Denote by A the ring of $\mathbb{R}$-valued regular functions on X and set $A(F) = A \otimes_{\mathbb{R}} F$. We shall consider $A(F)$ as a subring of the ring $B(F)$ of continuous F-valued functions on X. A continuous F-vector bundle ξ on X is said to *admit an algebraic structure* if there exists a finitely generated projective $A(F)$-module P such that the F-vector bundle on X associated, in the usual way (see [**17**]), with $P \otimes_{A(F)} B(F)$ is topologically isomorphic to ξ (an equivalent, more geometric, definition is given in [**2**] and [**1**]).

The following problem has attracted the attention of several mathematicians.

Problem. Characterise continuous F-vector bundles on X admitting an algebraic structure.

Until very recently, despite considerable effort, the situation was well understood, only in a few special cases (see [**8, 10, 11**] and [**16**] for a short survey). For $\dim X \leqslant 3$ and $F = \mathbb{R}$ a very satisfactory solution is given in [**4**] (see also [**3, 12, 13**] for earlier results). In [**1**] (see also [**7**]) most results are first obtained for $\mathbb{C}$-vector bundles and then many of them are extended on to F-vector bundles, $F = \mathbb{R}$ or $\mathbb{H}$, by using the realification and quaternionification. The main tool of [**1**], which will also be used here, is the functor $H^{\text{even}}_{\mathbb{C}-\text{alg}}(\cdot, \mathbb{Z})$ from affine real algebraic varieties to graded rings (we recall the definition of $H^{\text{even}}_{\mathbb{C}-\text{alg}}(\cdot, \mathbb{Z})$ in the next section). If X is an affine real algebraic

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variety, then $H^{\text{even}}_{\mathbb{C}-\text{alg}}(X, \mathbb{Z})$ is a subring of the cohomology ring $H^{\text{even}}(X, \mathbb{Z})$. It is known that the total Chern class $c(\xi)$ of a given continuous $\mathbb{C}$-vector bundle ξ on X belongs to $H^{\text{even}}_{\mathbb{C}-\text{alg}}(X, \mathbb{Z})$ if ξ admits an algebraic structure [1] (see also [7]). In this paper we show that if $c(\xi)$ is in $H^{\text{even}}_{\mathbb{C}-\text{alg}}(X, \mathbb{Z})$, $\dim X \leqslant 7$ and X satisfies some reasonable extra conditions, then ξ admits an algebraic structure. This result has been announced in [7], for $\dim X \leqslant 5$, but no proof is given in the detailed version [1] of [7]. It should be mentiioned that the ring $H^{\text{even}}_{\mathbb{C}-\text{alg}}(X, \mathbb{Z})$ is computed in [1] (see also [7]) for a large class of varieties X. It turns out that, in many cases, $H^{\text{even}}_{\mathbb{C}-\text{alg}}(X, \mathbb{Z})$ is small as compared with $H^{\text{even}}(X, \mathbb{Z})$. This imposes strong restrictions on continuous $\mathbb{C}$-vector bundles on X admitting an algebraic structure (see also [5] for other applications of $H^{\text{even}}_{\mathbb{C}-\text{alg}}(X, \mathbb{Z})$).

2. The result

For simplicity we shall recall the definition of $H^{\text{even}}_{\mathbb{C}-\text{alg}}(X, \mathbb{Z})$ only for *nonsingular* affine real algebraic varieties X (see [1] for the general case), which will be sufficient for our purposes.

Let V be a quasi-projective nonsingular n-dimensional complex algebraic variety. One defines the natural homomorphism

$$cl : A^*(V) \to H^*(V, \mathbb{Z}),$$

where $A^*(V)$ is the Chow ring of V and $H^*(V, \mathbb{Z})$ is the Čech cohomology of V, as follows. Let $Y \subseteq V$ be a closed irreducible subvariety of dimension k. Let $\{Y\}$ be the element of $A^{n-k}(V)$ represented by Y and let $[Y]$ be the fundamental class of Y in the Borel-Moore homology group $H^{BM}_{2k}(Y, \mathbb{Z})$ (see [6] or [9, Chapter 19]). Then $cl(\{Y\})$ is the element of $H^{2n-2k}(V, \mathbb{Z})$ which corresponds, via Poincaré duality, to the image of $[Y]$ in $H^{BM}_{2k}(V, \mathbb{Z})$ under the homomorphism $H^{BM}_{2k}(Y, \mathbb{Z}) \to H^{BM}_{2k}(V, \mathbb{Z})$ induced by the inclusion $Y \subseteq V$. Extending by linearity, cl defines a natural homomorphism $cl\colon A^*(V) \to H^*(V, \mathbb{Z})$. Clearly, the image of cl is contained in $H^{\text{even}}(V, \mathbb{Z})$. We set

$$H^{\text{even}}_{\text{alg}}(V, \mathbb{Z}) = cl(A^*(V)).$$

Now let X be an affine real algebraic variety (any such variety can be embedded as a locally closed algebraic subvariety in some real projective space $\mathbb{R}P^n$). Consider $\mathbb{R}P^n$ as a subset of the complex projective space $\mathbb{C}P^n$ and suppose for a moment that X is embedded in $\mathbb{R}P^n$ as a locally closed subvariety. Moreover, assume that X is nonsingular. Let U be a Zariski neighbourhood of X in the set of nonsingular points of the Zariski (complex) closure of X in $\mathbb{C}P^n$. We define $H^{\text{even}}_{\mathbb{C}-\text{alg}}(X, \mathbb{Z})$ by

$$H^{\text{even}}_{\mathbb{C}-\text{alg}}(X, \mathbb{Z}) = H^*(i_U)\Big(H^{\text{even}}_{\text{alg}}(U, \mathbb{Z})\Big),$$

where $i_U\colon X \to U$ is the inclusion mapping. One easily sees that $H^{\mathrm{even}}_{\mathbb{C}-\mathrm{alg}}(X, \mathbb{Z})$ does not depend on the choice of the embedding of X in $\mathbb{R}P^n$ and the choice of U (see [**1**, Section 3]).

THEOREM. *Let X be an affine nonsingular real algebraic variety and let ξ be a continuous $\mathbb{C}$-vector bundle of constant rank on X. Assume that X is compact, $\dim X \leqslant 7$, and the groups $H^6(X, \mathbb{Z})$ and $H^6(X, \mathbb{Z})/H^6_{\mathbb{C}-\mathrm{alg}}(X, \mathbb{Z})$ have no 2-torsion. Then the following conditions are equivalent:*

(a) *ξ admits an algebraic structure;*
(b) *the total Chern class $c(\xi)$ of ξ belongs to $H^{\mathrm{even}}_{\mathbb{C}-\mathrm{alg}}(X, \mathbb{Z})$.*

PROOF: The implication (a) $\Longrightarrow$ (b) is proved in [**1**] (see also [**7**]) for all affine real algebraic varieties X without any additional restrictions. ∎

Before beginning the proof of (b) $\Longrightarrow$ (a), it will be convenient to collect a few facts.

LEMMA. *Let X be a locally closed real algebraic subvariety of $\mathbb{R}P^n$ and let V be a Zariski neighbourhood of X in the Zariski (complex) closure of X in $\mathbb{C}P^n$. Let η be an algebraic vector bundle on V. Then:*

(i) *there exists an affine open complex subvariety U of V containing X;*
(ii) *the restriction $\eta \mid X$ of η to X, considered as a continuous $\mathbb{C}$-vector bundle on X, admits an algebraic structure;*
(iii) *if V is nonsingular, then $cl(C(\eta)) = c(\eta)$, where $C(\eta)$ and $c(\eta)$ are the total Chern classes of η with values in $A^*(V)$ and $H^{\mathrm{even}}(V, \mathbb{Z})$, respectively.*

PROOF: (i) and (ii) are completely elementary (see for example [**1**, Proposition 5.1]), while (iii) is proved in [**6**]. ∎

Now we can return to the proof of (b) $\Longrightarrow$ (a). We may assume that X is a locally closed subvariety of $\mathbb{R}P^n$. Let U be an affine Zariski neighbourhood of X in the set of nonsingular points of the Zariski closure of X in $\mathbb{C}P^n$ (see (i) of the Lemma) and let

$$r\colon H^*(U, \mathbb{Z}) \to H^*(X, \mathbb{Z})$$

be the homomorphism induced by the inclusion $X \subseteq U$. We may assume that for each $i = 1, 2, 3$, there exists an element a_i in $A^i(U)$ such that

$$r(cl(a_i)) = c_i(\xi). \tag{1}$$

Let η_1 and η_2 be algebraic vector bundles on U satisfying rank $\eta_1 = 1$, $C_1(\eta_1) = a_1$, $C_1(\eta_2) = 0$ and $C_2(\eta_2) = a_2$ (the existence of η_1 is obvious, while the existence of η_2 follows at once from the Grothendieck formula [**9**, Example 15.3.6]).

Let

$$\rho : H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{Z}/2)$$

be the reduction (modulo 2) homomorphism. It follows from the Wu formula [**14**, p. 94], applied to ξ and $\eta_2 \mid X$, that

$$(2) \qquad \begin{aligned} Sq^2(\rho(c_2(\xi))) &= \rho(c_1(\xi)c_2(\xi) - c_3(\xi)) \\ Sq^2(\rho(c_2(\eta_2 \mid X))) &= \rho(c_3(\eta_2 \mid X)), \end{aligned}$$

where $Sq^2 : H^4(X, \mathbb{Z}/2) \to H^6(X, \mathbb{Z}/2)$ is the Steenrod square (to obtain the second equality, one uses $C_1(\eta_2) = 0$ and condition (iii) of the Lemma, which guarantees that $c_1(\eta_2) = 0$ and hence $c_1(\eta_2 \mid X) = 0$).

Let $a = a_1a_2 - a_3 + C_3(\eta_2)$. Then, by (1), (2) and condition (iii) of the Lemma,

$$\begin{aligned} \rho(r(cl(a))) &= \rho(r(cl(a_1a_2 - a_3))) + \rho(r(cl(C_3(\eta_2)))) \\ &= \rho(c_1(\xi)c_2(\xi) - c_3(\xi)) + \rho(c_3(\eta_2 \mid X)) \\ &= Sq^2(\rho(c_2(\xi))) + Sq^2(\rho(c_2(\eta_2 \mid X))) \\ &= Sq^2(\rho(c_2(\xi))) + Sq^2(\rho(c_2(\xi))) \\ &= 0. \end{aligned}$$

Hence $r(cl(a)) = -2\nu$ for some ν in $H^6(X, \mathbb{Z})$. Since 2ν is in $H^6_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$ and the group $H^6(X, \mathbb{Z})/H^6_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$ has no 2-torsion, it follows that ν is in $H^6_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$. Shrinking U, if necessary, we may assume that $\nu = r(cl(b))$ for some b in $A^3(U)$. By the Grothendieck formula [**9**, Example 15.3.6], there exists a vector bundle η_3 on U such that $C_i(\eta_3) = 0$ for $i = 1, 2$ and $C_3(\eta_3) = 2b$. Let $\eta = \eta_1 \oplus \eta_2 \oplus \eta_3$. Then

$$\begin{aligned} C_i(\eta) &= a_i \text{ for } i = 1, 2 \\ C_3(\eta) &= a_1a_2 + C_3(\eta_2) + 2b. \end{aligned}$$

Hence, using (1), we obtain

$$\begin{aligned} c_i(\eta \mid X) &= r(cl(a_i)) = C_i(\xi) \text{ for } i = 1, 2 \\ c_3(\eta \mid X) &= r(cl(a_1a_2 + C_3(\eta_2) + 2b)) \\ &= r(cl(a_1a_2 + C_3(\eta_2)) + 2r(cl(b))) \\ &= r(cl(a + a_3)) - r(cl(a)) \\ &= r(cl(a_3)) \\ &= c_3(\xi). \end{aligned}$$

Thus $c(\xi) = c(\eta \mid X)$ and, by Peterson's theorem [**15**], ξ and $\eta \mid X$ are stably equivalent (here we use the assumptions that ξ is of constant rank and $H^6(X, \mathbb{Z})$ has no

2-torsion). Moreover, by condition (ii) of the Lemma, $\eta \mid X$ admits an algebraic structure. It is well-known (see [**18**, Theorem 2.2 (a)] or [**2**, Chapter 12]) that a continuous vector bundle on a compact affine real algebraic variety admits an algebraic structure if and only if it is stably equivalent to a vector bundle admitting an algebraic structure. Thus ξ admits an algebraic structure.

References

[1] J. Bochnak, M. Buchner and W. Kucharz, 'Vector bundles over real algebraic varieties', (preprint) (University of New Mexico).

[2] J. Bochnak, M. Coste and M.-F. Roy, *Géométrie Algébrique Réelle* (Erge. Math. Grenzgeb (3) 12 Springer, 1987).

[3] J. Bochnak and W. Kucharz, 'Sur les classes d'homologie représentables par des hypersurfaces algébriques réelles', *C. R. Acad. Sci. Paris Sér. I Math.* **305** (1987), 609–611.

[4] J. Bochnak and W. Kucharz, 'On homology classes represented by real algebraic hypersurfaces', (preprint) (University of New Mexico).

[5] J. Bochnak and W. Kucharz, 'Morphismes algébriques réelles à valeurs dans S^{2k} et K-theorie algébrique', *C. R. Acad. Sci. Paris Sér. I Math.* **305** (1987), 545–547.

[6] A. Borel and A. Haefliger, 'La classe d'homologie fondamental d'un espace analytique', *Bull. Soc. Math. France* **89** (1961), 461–513.

[7] M. Buchner and W. Kucharz, 'Algebraic vector bundles over real algebraic varieties', *Bull. Amer. Math. Soc.* **17** (1987), 279–282.

[8] R. Fossum, 'Vector bundles over spheres are algebraic', *Invent. Math.* **8** (1969), 222–225.

[9] W. Fulton, *Intersection Theory* **2** (Erge. Math. Grenzgeb (3) Springer, 1984).

[10] A.V. Geramita and L. G. Roberts, 'Algebraic vector bundles on projective spaces', *Invent. Math.* **10** (1970), 298–304.

[11] J.-P. Jouanolou, 'Comparison des K-théories algebrique et topologique de quelque variétés algébrique', *C. R. Acad. Sci. Paris, Sér A, Ser. A* **272** (1971), 1373–1375.

[12] W. Kucharz, 'Vector bundles over real algebraic surfaces and threefolds', *Compositio Math.* **60** (1986), 209–225.

[13] W. Kucharz, 'Topology of real algebraic threefolds', *Duke Math. J.* **53** (1986), 1073–1079.

[14] J. Milnor and J. Stasheff, *Characteristic Classes* (Princeton Univ. Press, Princeton, 1974).

[15] F.P. Peterson, 'Some remarks on Chern classes', *Ann. of Math. (2)* **60** (1959), 414–420.

[16] L.G. Roberts, 'Comparison of algebraic and topological K-theory', *Algebraic K-Theory II. Lecture Notes in Math.* **342**, pp. 74–78 (Springer-Verlag, Berlin and New York, 1973).

[17] R.G. Swan, 'Vector bundles and projective modules', *Trans. Amer. Math. Soc.* **105** (1962), 264–277.

[18] R.G. Swan, 'Topological examples of projective modules', *Trans. Amer. Math. Soc.* **230** (1977), 201–234.

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